# Low temperature transport in integrable quantum systems

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## Abstract

The spin-1/2 XXZ chain is an integrable lattice model and exhibits an infinite number of conserved charges some protecting its spin current for anisotropies $|\Delta| < 1$. Temperature asymptotics are presented for zero external magnetic field and anisotropies $\Delta = \cos(n\pi/m)$ with $n, m$ coprime by use of thermodynamic Bethe ansatz (TBA).

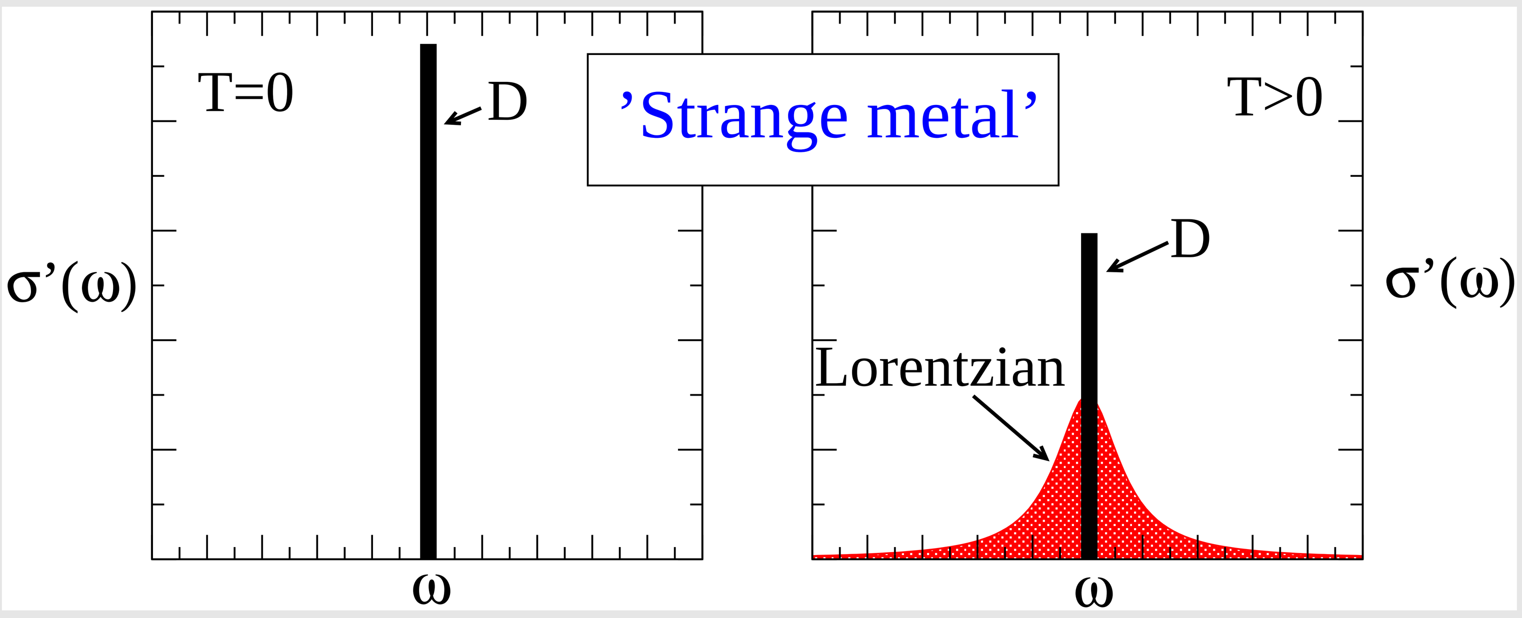


Figure: At $T = 0$ the spin sum rule is satisfied by the Drude weight, however at higher temperatures it melts and forms a Lorentzian that the Drude weight sits atop.

## The XXZ model and the Drude weight

The Hamiltonian of the one-dimensional Heisenberg XXZ model on $N$ sites with periodic boundary conditions is given by

$$H = J\sum_{l=1}^{N}\left(e^{-i\phi}\sigma_{l+1}^{+}\sigma_{l}^{-} + e^{i\phi}\sigma_{l}^{+}\sigma_{l+1}^{-} + \Delta\sigma_{l}^{z}\sigma_{l+1}^{z}\right) - 2h\sum_{l=1}^{N}\sigma_{l}^{z}, \tag{1}$$

where $\sigma^{+,-,z}$ are Pauli matrices, $\Delta = \cos(\gamma)$ is the anisotropy, and $h$ the applied magnetic field. A non-zero spin Drude weight is determined by the spin current-spin current correlator evaluated at infinite times

$$D(T) = \lim_{t\to\infty}\lim_{N\to\infty}\frac{1}{2NT}\langle J_0(0)J_0(t)\rangle \geq \lim_{N\to\infty}\frac{1}{2NT}\sum_{A}\frac{|\langle J_0 Q_A\rangle|^2}{\langle Q_A^{\dagger}Q_A\rangle}. \tag{2}$$

Alternatively, the Drude weight may be determined by examining the energy curvature from Kohn's formula (Zotos (1999))

$$D(T) = \frac{1}{2NZ}\sum_{n} e^{-\epsilon_n\beta}\frac{\partial^2\epsilon_n(\Phi)}{\partial\Phi^2}\Big|_{\Phi=0}. \tag{3}$$

## Thermodynamic Bethe Ansatz Formalism

The TBA formalism describes the behaviour of $L$ 'Bethe strings' and their holes in terms of rapidities. The particle and hole densities $\rho_j$ and $\rho_j^h$, respectively, of the $j$-th Bethe string can be used to determine thermodynamic quantities. The equilibrium state $\eta_j = \frac{\rho_j^h}{\rho_j}$ are characterized by the integral equation

$$\log\eta_j(\theta) = -\beta e_0(\theta) + \sum_{\ell}\left(K_{jk}\star\log|1+\eta_k|\right)(\theta), \tag{4}$$

where the kernel $K_{jk}$ is characterized by the scattering of the Bethe strings. By combining this formalism with Kohn's formula (3) Zotos' formula [Zotos (1999)] yields the Drude weight in terms of the second last Bethe string $\eta = \eta_{L-1}$ as

$$D = -\frac{J\sin\gamma}{2\pi\gamma\beta}\sigma\int_{-\infty}^{\infty}d\theta\frac{(\partial_\theta\log\eta)^2}{\partial_\beta\log\eta}\frac{\eta\,(\partial_{\beta\phi}\log\eta)^2}{(1+\eta)^2}, \tag{5}$$

the subscripts being suppressed for the $\eta$ functions and the $\sigma$, which introduces a $\pm 1$ depending on the anisotropy.

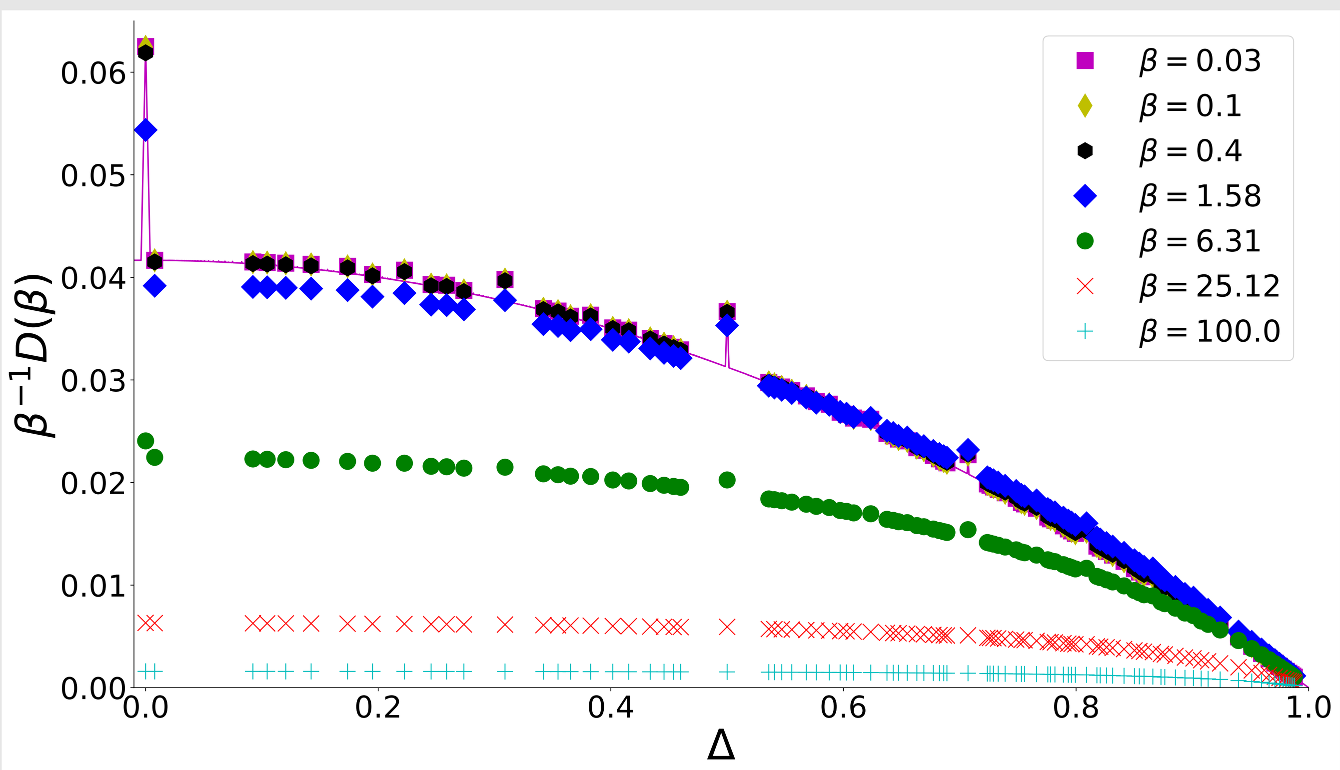


Figure: Drude weight coefficient $\beta^{-1}D(\gamma,\beta)$ for various anisotropies $\gamma = \pi n/m$ and temperatures. Note that the high temperature results ($\beta = 0.4, 0.1, 0.03$) are partly on top of each other on this scale. $D(\beta)$ is a nowhere continuous function except for at $\beta^{-1} = 0$.

## Zero-temperature result

By use of the Wiener Hopf method it's straightforward to obtain leading order corrections to the energy in terms of the scalar flux $\phi$. The result [Shastry (1990)] is a smooth function that is symmetric around the anisotropy

$$D(\infty) = \frac{\pi\sin(\gamma)}{8\gamma(\pi-\gamma)} = \frac{v_0 K}{4\pi}. \tag{6}$$

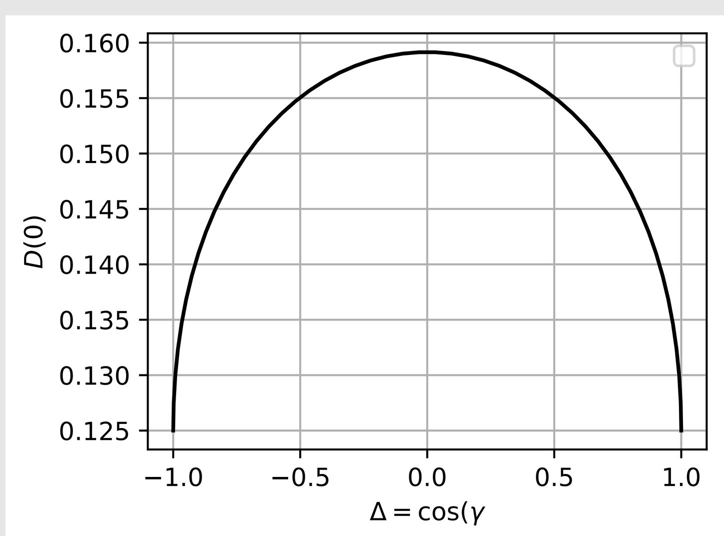


Figure: Drude weight coefficient at $T = 0$.

## Low-temperature Drude weight

At finite temperatures the Drude weight exhibits a fractal dependence on the anisotropy, which differs substantially from the zero temperature behaviour. To understand this higher subleading temperature corrections must be obtained.

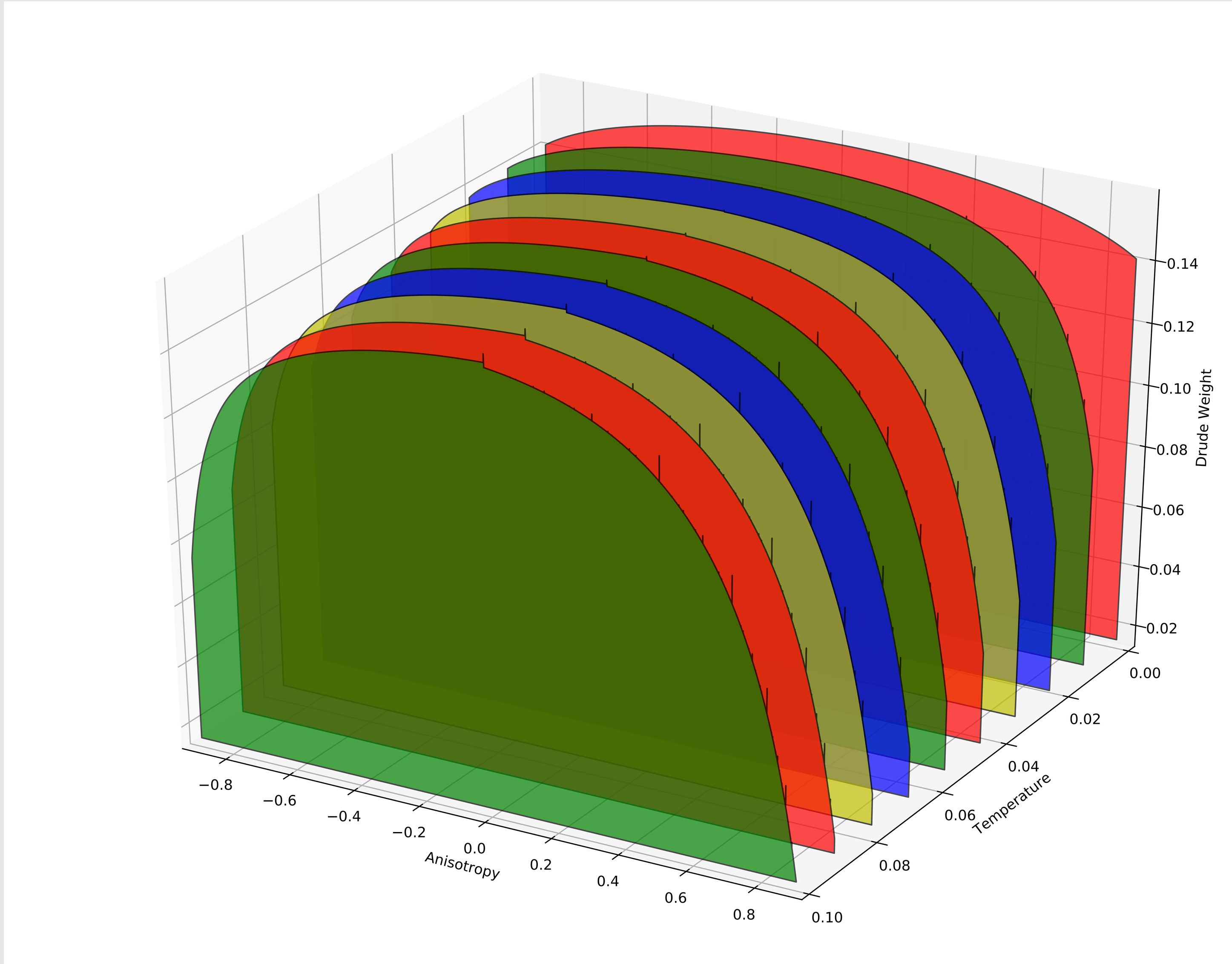


Figure: First order temperature corrections at low temperatures, the uneven melting of the Drude peak at different anisotropies is immediately apparent.

## Sub-leading Low Temperature Asymptotics

The subleading temperature is characterized by one of several prefactors depending on the anisotropy being considered. By grouping the prefactors we note that there are curvature contributions ($T^{2n}$) and Umklapp scattering terms ($T^{n(2K-2)}$) as

$$D(T) = D(0) - aT^{2K-2} - b_1T^2 - b_2T^{4K-4}. \tag{7}$$

With the lowest order Umklapp scattering term being responsible for the fractal structure at low temperatures. By use of the TBA structure the prefactor was determined to be [AU (2021)]

$$a(\Delta) = \frac{\sin\left(\frac{\pi K}{m}\right)}{\tan(\pi K)}\frac{v_0 mK}{2}\left[\frac{\sqrt{\pi}T}{v_0}\right]^{2K-2}\frac{\Gamma^2(K)\,\Gamma(1-2K)}{\Gamma^2(1-K)}\left[\frac{\Gamma\left(1+\frac{1}{2K-2}\right)}{\Gamma\left(1+\frac{K}{2K-2}\right)}\right]^{2K-2}, \tag{8}$$

The energy level curvature contribution is known from field theory [Sirker (2011)], however smoothly depends on the anisotropy

$$Y_4 = \frac{\pi^2}{6v^2}(\lambda_+ + 6\lambda_-), \qquad \lambda_+ = \frac{1}{2\pi}\tan\frac{\pi K}{2K-2}, \qquad \lambda_- = \frac{1}{12\pi K}\frac{\Gamma\left(\frac{3K}{2K-2}\right)\Gamma^3\left(\frac{1}{2K-2}\right)}{\Gamma\left(\frac{3}{2K-2}\right)\Gamma^3\left(\frac{K}{2K-2}\right)}.$$

$$b_1 = D(0)Y_4. \tag{9}$$

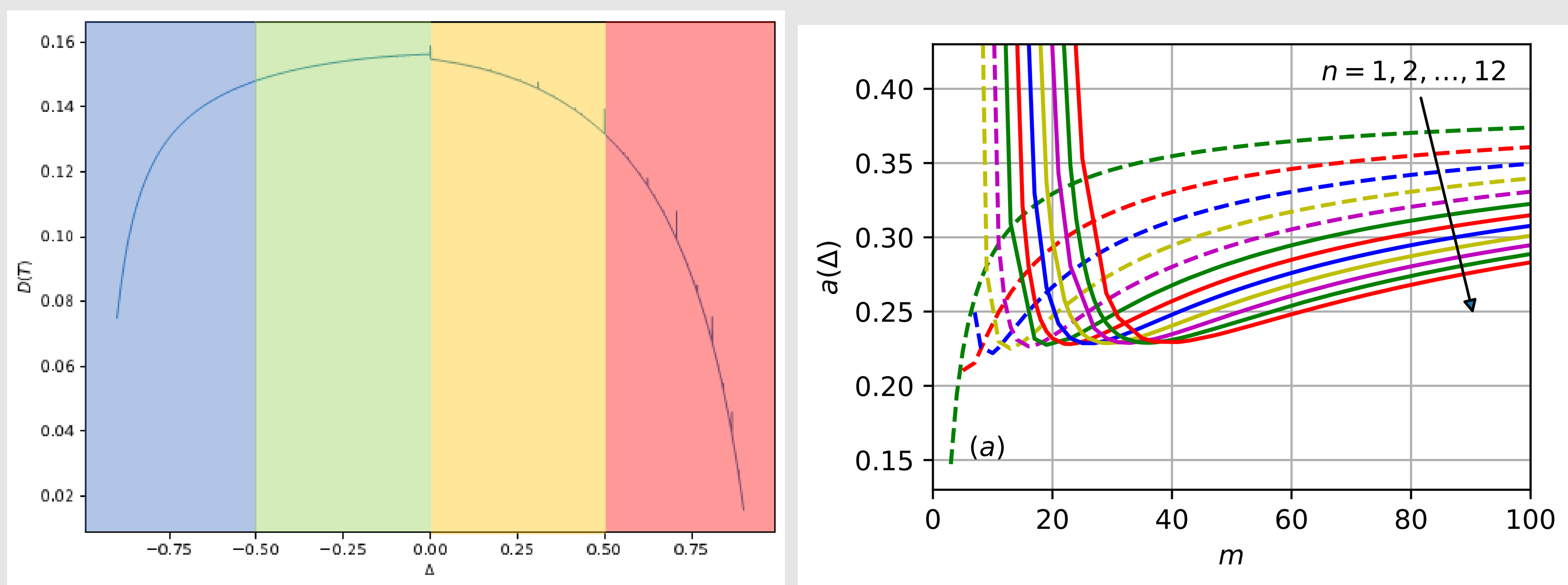


Figure: The different ranges of subleading and subsubleading temperature correction and demonstrating the different curves that the prefactor $a(\Delta)$ for varying values of $\gamma = \frac{n\pi}{m}$.

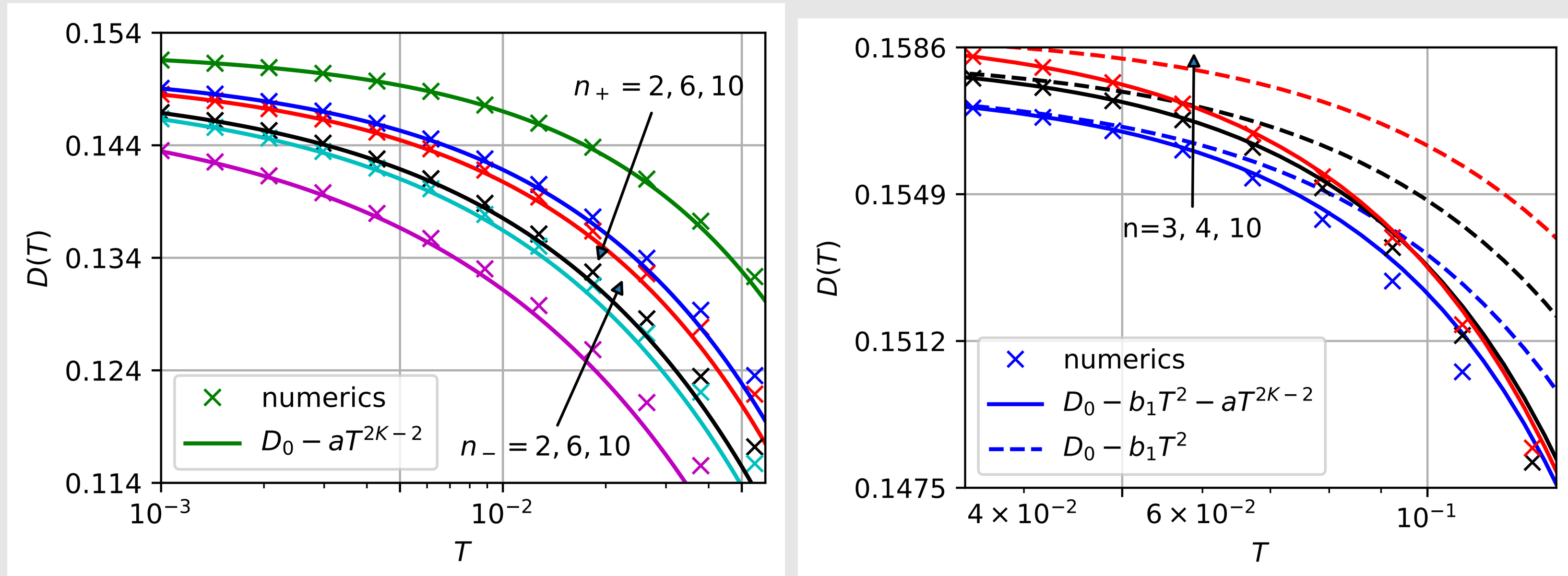


Figure: Demonstration of the consistency of the asymptotics with numerical results up to $T \approx 1/10$ for $\gamma = \pi\frac{n_\pm}{4n_\pm \pm 1}$ and $\gamma = \pi\frac{n}{2n-1}$ respectively.

## Conclusions and future directions

- The method used to determine these low temperature asymptotics should be applicable to the analysis of other thermodynamical quantities.
- The picture of the Drude peak melting into a Lorentzian seems to be naive based on preliminary diffusion results at low temperatures.
- Low temperature results from field theory seem to only capture the temperature corrections that depend smoothly on the anisotropy. We would like to understand how these correction might be accounted for in the non-linear Luttinger liquid framework.