Fermion propagator in presence of a strong external magnetic field

Arghya Mukherjee
Physics & Astronomy, Brandon University
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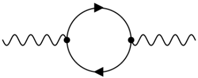
Winnipeg Institute for Theoretical Physics

Context:

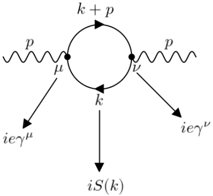
- Numerical simulations as well as theoretical predictions suggest extremely large generation of magnetic fields ($\sim 10^{18} - 10^{19} G$) in non-central HIC (arXiv:1509.04073).
- Such a strong magnetic field can have significant influences on the quark gluon plasma properties like transport and thermodynamic properties and may affect the QCD phase diagram.
- A typical perturbative approach to study the influence of the background field in the formalism of QFT requires evaluation of Feynman diagrams.

- Algorithm:
  - Theory with Interaction (coupling) : defined by $\mathcal{L}_{\text{QFT}}$
  - Order by order in coupling strength :

$\mathcal{L}_{\text{QED}}$ & $\mathcal{O}(e^2)$



  - Feynman Rules:



$$-e^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^\mu S_F(k) \gamma^\nu S_F(k+p)\right]$$

- Background magnetic field modifies propagators!

Propagators:

- E.O.M: 3+1 D analog of $\frac{d}{dt}\frac{\partial L(q_j,\dot{q}_j)}{\partial \dot{q}_j} - \frac{\partial L(q_j,\dot{q}_j)}{\partial q_j} = 0$
- Scalars: $(\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2 + m^2)\phi(x) = 0$
- Position space propagator: $G(x,x') = G(x-x')$:

$$(\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2 + m^2)G(x-x') = -\delta^4(x-x')$$

- Fourier Transform:

$$\delta^4(x-x') = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')}$$

$$G(x-x') = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')} G(p)$$

- Momentum space propagator:

$$iG(p) = \frac{i}{p^2 - m^2}$$

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- Fermion propagator with constant B along $\hat{z}$:

$$\tilde{S}(\omega, p_z; \mathbf{p}_\perp) = 2ie^{-p_\perp^2 \ell^2} \sum_{n=0}^{\infty} \frac{(-1)^n \mathcal{D}_n(\omega, p_z; \mathbf{p}_\perp)}{\omega^2 - p_z^2 - m^2 - 2n|qB|}$$

$$\begin{aligned}
\mathcal{D}_n(\omega, p_z; \mathbf{p}_\perp) &= (\gamma^0\omega - \gamma^3 p_z + m)\big[L_n(2\ell^2 p_\perp^2)\mathcal{P}_+ \\
&\quad - L_{n-1}(2\ell^2 p_\perp^2)\mathcal{P}_-\big] + 2(\gamma_\perp \cdot \mathbf{p}_\perp) L_{n-1}^1(2\ell^2 p_\perp^2) \\
\mathcal{P}_\pm &= \frac{1}{2}\big(1 \pm i\ \mathrm{sgn}(qB)\gamma^1\gamma^2\big) \\
\ell &= \frac{1}{\sqrt{|qB|}}, \quad L_n(x) = \textit{Laguerre polynomial}
\end{aligned}$$

The original work:

On Gauge Invariance and Vacuum Polarization

Julian Schwinger
Phys. Rev. **82**, 664 – Published 1 June 1951

Article | References | Citing Articles (5,043) | PDF | Export Citation

- Let us just compare the denominator of the propagator with and without eB.

$$S_F(p) = \frac{\gamma^\mu p_\mu + m\mathbb{1}_{4\times 4}}{p^2 - m^2}$$

$$\tilde{S}(\omega, p_z; \mathbf{p}_\perp) = 2ie^{-p_\perp^2 \ell^2} \sum_{n=0}^{\infty} \frac{(-1)^n \mathcal{D}_n(\omega, p_z; \mathbf{p}_\perp)}{\omega^2 - p_z^2 - m^2 - 2n|qB|}$$

- Pole of the propagator gives dispersion: $eB = 0$ is isotropic: $p^2 = m^2 \to E^2 - |\mathbf{p}|^2 = m^2$.

Eigenvalue Problem:

- Solving Dirac Equation with $eB \neq 0$:

$$i\frac{\partial \Psi}{\partial t} = H_B \Psi \tag{1}$$

$$H_B = \vec{\alpha} \cdot \vec{\Pi} + \beta m \ , \ \vec{\alpha} = \gamma_0 \vec{\gamma} \ , \ \beta = \gamma_0 \tag{2}$$

$$\alpha_i = \gamma_0 \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}_{4\times 4} \tag{3}$$

$$\beta = \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{4\times 4} \tag{4}$$

- Only change with $\vec{A} = (0, Bx, 0)$:

$$\vec{p} \rightarrow \vec{p} - q\vec{A} \rightarrow \vec{\Pi} = -i\vec{\nabla} - q\vec{A}$$

- We can write plane wave solution as:

$$\Psi = e^{-iEt} \begin{pmatrix} \phi \\ \chi \end{pmatrix}_{4\times 1} \tag{5}$$

- Explicitly we have

$$E \begin{pmatrix} \phi \\ \chi \end{pmatrix} e^{-iEt} = \begin{pmatrix} m & \vec{\sigma} \cdot \vec{\Pi} \\ \vec{\sigma} \cdot \vec{\Pi} & -m \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} e^{-iEt} \quad (6)$$

$$\begin{pmatrix} E\phi \\ E\chi \end{pmatrix} = \begin{pmatrix} m\,\phi + \vec{\sigma} \cdot \vec{\Pi}\,\chi \\ \vec{\sigma} \cdot \vec{\Pi}\,\phi - m\,\chi \end{pmatrix} \quad (7)$$

- So we get two coupled equations

$$(E - m)\phi = \vec{\sigma} \cdot \vec{\Pi}\,\chi \quad (8)$$

$$(E + m)\chi = \vec{\sigma} \cdot \vec{\Pi}\,\phi \quad (9)$$

- But they can be easily decoupled!

$$(E + m)(E - m)\phi = (\vec{\sigma} \cdot \vec{\Pi})^2 \phi \quad (10)$$

◇ We only need to evaluate $(\vec{\sigma}\cdot\vec{\Pi})^2$. Let us remind

$$\Pi^i = -i\frac{\partial}{\partial x^i} - qA^i \tag{11}$$

$$\Pi^1 = -i\frac{\partial}{\partial x^1}\,,\ \Pi^2 = -i\frac{\partial}{\partial x^2} - qBx^1\,,\ \Pi^3 = -i\frac{\partial}{\partial x^3} \tag{12}$$

◇

$$\left(\vec{\sigma}\cdot\vec{\Pi}\right)^2 \;=\; (\Pi)^2 - qB\sigma_3 \tag{13}$$

$$((\Pi)^2 - qB\sigma_3)\phi \;=\; (E^2 - m^2)\phi \tag{14}$$

$$\begin{aligned} \text{LHS,} \qquad & \left[-\frac{\partial^2}{\partial x^2} + (i\frac{\partial}{\partial y} + qBx)^2 - \frac{\partial^2}{\partial z^2} - qB\sigma_3\right]\phi \\ = \; & \left[-\nabla^2 + 2iqBx\frac{\partial}{\partial y} + q^2B^2x^2 - qB\sigma_3\right]\phi \end{aligned} \tag{15}$$

◇ The trial solution:

$$\phi = e^{i(p_y y + p_z z)} f(x) \tag{16}$$

σ_3 is 2×2 matrix. So $f(x)$ must be 2 component vector. Now we write $f(x)$ in eigen basis of σ_3.

$$f(x) = F_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + F_-(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{17}$$

$$-\frac{d^2 F_s}{dx^2} + (p_y - qBx)^2 F_s - (E^2 - m^2 - p_z^2 + sqB) F_s = 0$$

◇ Now we change the variable.

$$\xi = \sqrt{|qB|} \left(\frac{p_y}{qB} - x \right) \tag{18}$$

- The final equation:

$$\left[\frac{d^2}{d\xi^2} - \xi^2 + a_s\right] F_s = 0 \tag{19}$$

where $a_s = \frac{1}{|qB|}(E^2 - m^2 - p_z^2 + sqB)$.

- Textbook by David J. Griffiths

We return now to the Schrödinger equation for the harmonic oscillator (Equation 2.39):

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi = E\psi.$$

Things look a little cleaner if we introduce the dimensionless variable

$$\xi \equiv \sqrt{\frac{m\omega}{\hbar}}x; \tag{2.55}$$

in terms of ξ, the Schrödinger equation reads

$$\frac{d^2\psi}{d\xi^2} = (\xi^2 - K)\psi, \tag{2.56}$$

- Textbook by David J. Griffiths

h_{even} *or* the series h_{odd}; the *other* one must be zero from the start). For physically acceptable solutions, then, we must have

$$K = 2n + 1,$$

for some positive integer n, which is to say (referring to Equation 2.57) that the *energy* must be of the form

$$E_n = (n + \frac{1}{2})\hbar\omega, \quad \text{for } n = 0, 1, 2, \ldots . \qquad [2.67]$$

- $a_s = \frac{1}{|qB|}(E^2 - m^2 - p_z^2 + sqB) = 2l + 1, \ \ l \geq 0.$

$$\begin{aligned} E^2 &= m^2 + p_z^2 - sqB + (2l+1)|qB| \\ &= p_z^2 + m^2 + [(2l+1) - s\,\text{sgn}(qB)]\,|qB| \\ &= p_z^2 + m^2 + 2n|qB| \end{aligned} \qquad (20)$$

- Landau Levels:

$$n = l + \frac{1}{2} - \frac{s}{2}\text{sgn}(qB)$$

◇ Fermion propagator with constant B along $\hat{z}$:

$$\tilde{S}(\omega, p_z; \mathbf{p}_\perp) = 2ie^{-p_\perp^2\ell^2} \sum_{n=0}^{\infty} \frac{(-1)^n \mathcal{D}_n(\omega, p_z; \mathbf{p}_\perp)}{\omega^2 - p_z^2 - m^2 - 2n|qB|} \tag{21}$$

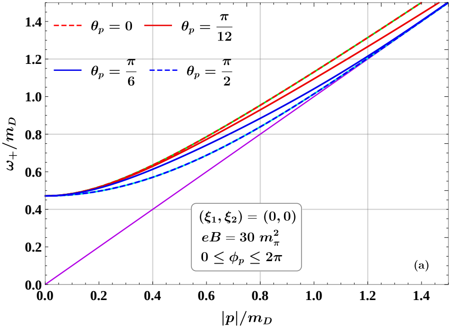
$$\begin{aligned}
\mathcal{D}_n(\omega, p_z; \mathbf{p}_\perp) &= (\gamma^0\omega - \gamma^3 p_z + m)\big[L_n(2\ell^2 p_\perp^2)\mathcal{P}_+ \\
&\quad - L_{n-1}(2\ell^2 p_\perp^2)\mathcal{P}_-\big] + 2(\gamma_\perp \cdot \mathbf{p}_\perp)L_{n-1}^1(2\ell^2 p_\perp^2) \\
\mathcal{P}_\pm &= \frac{1}{2}\big(1 \pm i\ \mathrm{sgn}(qB)\gamma^1\gamma^2\big) \\
\ell &= \frac{1}{\sqrt{|qB|}}
\end{aligned} \tag{22}$$

◇ $eB \neq 0 \rightarrow$ Landau Quantization!

Take Home

- For $eB \neq 0$, the Fermion propagator is complicated with sum over infinite Landau levels.
- eB brings anisotropy.
- Dispersion shows angular dependence:

$$\omega^2 - p^2 - \left[f_{g+gh}(\omega, |\mathbf{p}|, T) + f_q(\omega, p_z, |\mathbf{p}_\perp^2)|, eB, T)\right] = 0$$



- arXiv: 2204.09646 (one loop HTL+LLL)

Thank You