State transfer on weighted graphs with twin vertices

Hermie Monterde (she/her)
University of Manitoba
Joint work with Stephen Kirkland and Sarah Plosker

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Notation

- G: connected weighted undirected graph with possible loops but no multiple edges
- M: adjacency (XY model) or Laplacian matrix (XYZ model) of G
- $\sum_j \lambda_j E_j$: spectral decomposition of M.
- $\mathbf{e}_u$: characteristic vector of vertex u
- $\phi(M, t)$: characteristic polynomial of M

Continuous-time quantum walks

Quantum spin network: modelled by an undirected graph with vertices and edges representing the qubits (spin-half quantum particles) and their interactions (couplings), respectively.

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The *continuous-time quantum walk* on G is given by $U(t) = e^{itM}$, where

$$\sum_{j=1}^{n} |U(t)_{j,u}|^2 = 1 \text{ for each } u \in V(G).$$

Note: $|U(t)_{u,v}|^2$ is the probability (aka *fidelity*) that the initial state at u is found in v at time t.

Types of state transfer

Definition

We say that vertices u and v exhibit (α, β)-**fractional revival** (with respect to M) if there exists $\tau > 0$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$ such that

$$U(\tau)\mathbf{e}_u = \alpha\mathbf{e}_u + \beta\mathbf{e}_v.$$

If $\beta = 0$, we say that u is **periodic**. If $\beta \neq 0$, we say that the fractional revival is *proper*, and if $\alpha = 0$, then we say **perfect state transfer** (PST) occurs between u and v.

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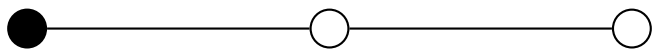
Definition

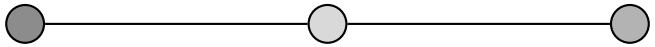
1. **Local (instantaneous) uniform mixing** occurs at u at time τ if $|U(\tau)_{v,u}| = \frac{1}{\sqrt{|V(G)|}}$ for any $v \in V(G)$.
2. **Instantaneous uniform mixing** occurs in G at time τ if local uniform mixing occurs at every vertex u of G at time τ.

Quantum walk on P_3



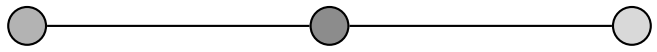
at $t = 0$

Quantum walk on P_3



at $0 < t < \frac{\pi}{\sqrt{4+2\sqrt{2}}}$

Quantum walk on P_3



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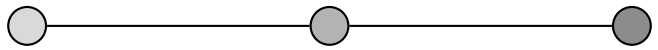

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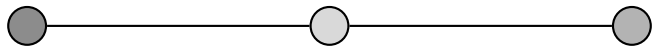

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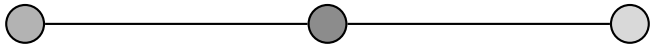

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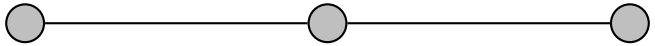

at $t = \frac{1}{2} \arctan\left(\sqrt{2}\right)$

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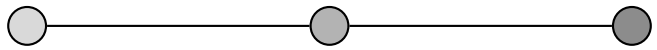

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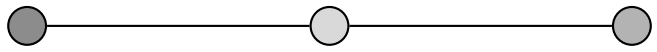

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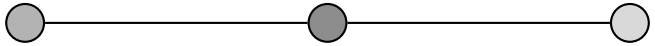

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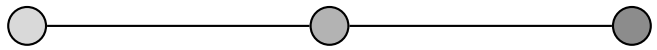

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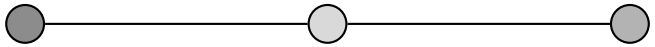

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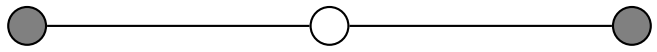

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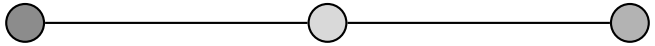

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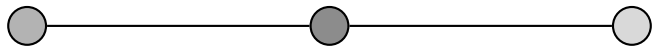

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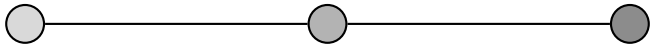

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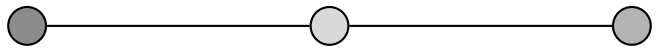

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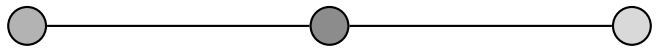

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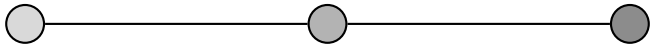

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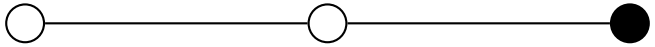

at $t = \frac{\pi}{\sqrt{2}}$

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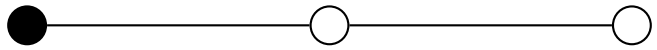

at $t = \frac{2\pi}{\sqrt{2}}$

Twins

Definition

Vertices u and v are *twins* if the following conditions hold.

1. $N_G(u) \backslash \{u, v\} = N_G(v) \backslash \{u, v\}$.
2. The edges (u, w) and (v, w) have the same weight for each $w \in N_G(u) \backslash \{u, v\}$.
3. If there are loops on u and v, then they have the same weight.

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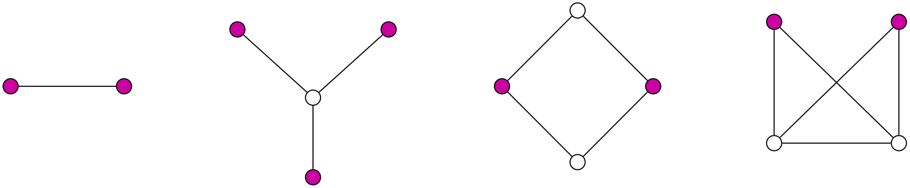

Spectral Characterization of Twins

We say that $T = T(\omega, \eta)$ is a set of twins if every pair of vertices in T are twins with loops of weight ω that are connected by an edge of weight η.

Lemma

Vertices u and v are in T if and only if an eigenvector for M corresponding to the eigenvalue

$$\theta = \begin{cases} \omega - \eta, & \text{if } M = A \\ deg(u) - \omega + \eta, & \text{if } M = L. \end{cases}$$

Uniform Mixing

Lemma

Let T be a set of twins in G. For any $u, v \in T$, $|U(t)_{u,u}| + |U(t)_{u,v}| \geq 1$ for all $t \in \mathbb{R}$.

Theorem

Let T be a set of twins in G. If G has $n \geq 5$ vertices, then local uniform mixing does not occur at any vertex in T.

Some facts

- If FR occurs between u and v at time τ, then u and v are parallel, i.e., $E_j\mathbf{e}_u$ and $\pm E_j\mathbf{e}_v$ are parallel vectors for each j.
- If PST occurs between u and v at time τ, then
  - u and v are strongly cospectral, i.e., $E_j\mathbf{e}_u = \pm E_j\mathbf{e}_v$ for each j
  - u and v are periodic at time 2τ.

Definition

The eigenvalue support of a vertex u (with respect to M) is

$$\sigma_u(M) = \{\lambda_j : E_j\mathbf{e}_u \neq 0\}.$$

Parallel twins

Lemma

Let T be a set of twins in G. If $|T| \geq 3$, then each vertex $u \in T$ is not parallel with any vertex $v \neq u$.

Theorem

Let $T = \{u, v\}$ be a set of twins. If Ω is an orthogonal set of eigenvectors for θ such that $\mathbf{e}_u - \mathbf{e}_v \in \Omega$, then u and v are strongly cospectral if and only if

- $|\Omega| = 1$; *or*
- $\mathbf{w}^T \mathbf{e}_u = \mathbf{w}^T \mathbf{e}_v = 0$ *for all* $\mathbf{w} \in \Omega \backslash \{\mathbf{e}_u - \mathbf{e}_v\}$.

Moreover, if u and v are strongly cospectral, then u and v cannot be strongly cospectral to any other vertex in G

Periodicity

Theorem

A vertex is periodic if $\sigma_u(M)$ satisfies the ratio condition, i.e.,

$$\frac{\lambda_j - \lambda_k}{\lambda_r - \lambda_s} \in \mathbb{Q}$$

for all $\lambda_j, \lambda_k, \lambda_r, \lambda_s \in \sigma_u(M)$ with $\lambda_r \neq \lambda_s$.

Theorem

Let $\phi(M, t) \in \mathbb{Z}[x]$, and T be a set of twins in G. Then T is periodic if and only if the eigenvalues in the support of a vertex in T are all the form $\theta + b_j\sqrt{\Delta}$, where b_j is an integer, and either $\Delta = 1$ or $\Delta > 1$ is a square-free integer.

Perfect state transfer

Theorem

Let $\phi(M, t) \in \mathbb{Z}[x]$, and $T = \{u, v\}$ be a set of twins. Let $\sigma_u(M) = \{\theta, \lambda_1, \ldots, \lambda_r\}$. Then perfect state transfer occurs between u and v if and only if the following conditions hold.

1. *u and v are strongly cospectral.*
2. *For each j, $\lambda_j = \theta + b_j\sqrt{\Delta}$, where b_j is an integer, and $\Delta = 1$ or $\Delta > 1$ is square-free.*
3. *For each j, $\nu_2(b_j) = q$, where q is a nonnegative integer.*

Fractional Revival

Theorem

Let $T = \{u, v\}$ *be a set of twins with* $\sigma_u(M) = \{\theta, \lambda_1, \ldots, \lambda_r\}$. *Then* $(e^{i\zeta}\cos\gamma, ie^{i\zeta}\sin\gamma)$*-fractional revival occurs from* u *to* v *at time* τ *if and only if*

1. u *and* v *are strongly cospectral, and*
2. *For each* j, $\tau(\lambda_j - \theta) \equiv 2\gamma \pmod{2\pi}$ *and* $\tau\theta \equiv \zeta - \gamma \pmod{2\pi}$.