Holographic Complexity and Extra Dimensions

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Outline

- What is Holographic Complexity?
- String theory and AdS/CFT correspondence.
- Extra Dimensional Spacetimes.
- Complexity by Volume - Definition, Calculations and Results.
- Complexity by Action - Definition and Discussion.

Complexity

- *Quantum Relative Complexity (QRC or RC)*: The minimal number of simple operations needed to go from $|A\rangle$ to $|B\rangle$. It is an information theory quantity.
- We want to define the same for Quantum Fields, but it is non-trivial
- Hence, we try to use the dictionary of AdS/CFT to give a holographic definition of Complexity for Quantum Fields, calling it Holographic Complexity (HC).
- There are parallels between gravitational physics (specially Black Hole Physics) and nature of complexity in Quantum Circuits that motivate us to define HC.

Some remarks about String Theory

- Superstring Theory: theory of gravity and quantum field theories as some aspect of vibrating strings of energy and D-Branes in 10 dimensional spacetime.
- Strings are present in two forms: Closed loops/strings or open strings.
- D-Branes or Dirichlet branes are extended objects of different dimensions on which open strings can end on.

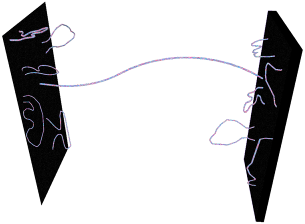

Figure: Illustration of open strings on D-branes. Source: Wikipedia

AdS/CFT Correspondence

- $SU(N)$ Yang-Mills Theory appears at low energies on N coincident D3-branes.
- The spacetime background in Type IIB superstring theory is produced by a coincident stack of N D3-branes, same N as in $SU(N)$.
- The $SU(N)$ Yang-Mills theory exists on the conformal boundary of this spacetime which is $AdS_5 \times S^5$ where as the Type IIB superstring theory lives in the bulk of the $AdS_5 \times S^5$.
- $SU(N)$ Yang-Mills Theory $\sim$ Type IIB Superstring theory in $AdS_5 \times S^5$.
- These seemingly unrelated theories for the same physical scenario prompts us to believe that they are equivalent.
- The first motivation came from matching symmetries on both side of the duality.

AdS/*CFT* (Contd...)

- There relation via the correspondence is hence called Holographic Principle as the information in the bulk is available at the boundary that is one less in dimension.
- Another example of a holographic theory is Black Hole thermodynamics where the bulk thermal information of the black hole is extracted by the variables of the boundary i.e. event horizon.
- It has been around for 23 years, has stood multiple tests over time, has been generalized and also provided heuristic arguments to explain some of experiments in QCD.
- Using the correspondence, we can calculate a given quantity on either side using variables on the other side. Many properties of CFTs come out from the *AdS* spacetime.

Motivation for looking at Full 10D

- HC has been studied for spacetimes that are just *AdS*. These are toy model scenarios as compared to the actual picture.
- We are using the prescriptions of AdS/CFT without studying the effects of Extra Dimensions.
- We would like to look at HC in the full 10D picture of String Theory.
- We would be looking at two spacetimes, $AdS_5 \times S^5$ and Multicentered *AdS*, the second of which has one of the extra dimensions more explicitly involved.
- We would like to find out how does HC depends on the parameters of the Extra Dimensions, and look at its behaviour.

$AdS_5 \times S^5$

- Good base case to test HC calculations.
- It is a product space of AdS_5 (Anti-de Sitter Spacetime) and S^5 (Five Sphere).
- In Type IIB String Theory, a coincident stack of N D3 Branes generate such a spacetime. The value of N determined the curvature of the AdS and Sphere part.
- It is the natural spacetime in which AdS/CFT correspondence is realized.
- The metric for spacetimes produces by these stacks of $D3$ branes has a general form

$$ds^2 = H_3^{-1/2}(\eta_{\mu\nu}x^\mu dx^\nu) + H_3^{1/2}(dr^2 + r^2 d\theta^2 + d\Omega_4^2)$$

where the H_3 function has information about the nature of the stacks of D3 Branes that produce the spacetime.

Multi-centered *AdS*

- $AdS_5 \times S^5$, we will see, would be trivial for HC calculations, so we study Multi-centered AdS which is created by more than one coincident stack of $D3$-branes.
- The stacks of $D3$-branes or centers are multiple, may or may not be symmetric.
- We are currently trying to solve for 2 center asymmetric case where one of the center is added as a perturbation.

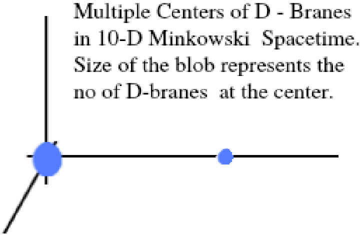

Figure: Schematic of the asymmetric case.

Multi-centered AdS (Contd...)

- The stacks of D3-branes is represented by a function H_3, which has a term for each stack i.e.

$$H_3 = \sum_{i=0}^{n} \frac{q_n}{|\vec{r} - \vec{r_i}|^4}$$

  where n is the number of stacks and q_n is proportional to the number of D3-branes in the stack.
- For a single stack, we just have $H_3 = 1/r^4$, because of which the radial direction with Minkowski directions makes the AdS Spacetime.
- In multi-centered case, 4-dimensions (3+1) remain parallel to the D-branes and the other 6 are perpendicular. Out of these 6, 4 have angular symmetry, and the rest two together form a half-plane where differences occur from the $AdS_5 \times S^5$.
- This lack of symmetry forces an extra dimension to explicitly appear in the calculations.

Complexity by Volume - CV Proposal

- HC of a CFT at boundary time τ is defined as the volume of the extremal co-dimension-one bulk hyper-surface which meets the asymptotic boundary at time τ divided by Newton's constant and a length ℓ, i.e.

$$C_v(\tau) = \max_{\partial B(t=\tau)} \left[\frac{V(B)}{G_N \ell} \right]$$

where B is the family of hyper-surfaces with the same boundary time τ.

- For our case, the time coordinate is boundary time. So

$$V = V_x S_4(1) \int_{r=0}^{r_{max}} \int_{\theta=0}^{\theta=\pi} dr d\theta \; r^5 \sin^4\theta H_3^{3/4}$$

where V_x is the volume along the Minkowskian directions, $S_4(1)$ is the volume of 4 Sphere with radius 1.

CV - $AdS_5 \times S^5$

- For the case of $AdS_5 \times S^5$, we have

$$H_3 = \frac{1}{r^4}$$

- This gives us

$$V(AdS_5 \times S^5) \to V_X S_5(1) \frac{r_{max}^3}{3}$$

- This tells us that the HC for $AdS_5 \times S^5$ is the same as that for AdS_5 since G_{10} cancels the extra factor of $S_5(1)$ and becomes G_5.
- Notice the divergent dependence on r_{max}.

CV - *Muticenterd AdS*

- For the case of Multicentered AdS, we have

$$H_3 = \frac{1}{r^4} + \frac{\epsilon^4}{(r^2 + d^2 - 2rd\cos\theta)^2}$$

  where d is the distance between the two centers and ϵ is the perturbation parameter.

- So the integral looks like

$$V = V_x S_4(1) \int_0^{r_{max}} \int_0^{\pi} drd\theta \; r^5 \sin^4\theta \left(\frac{1}{r^4} + \frac{\epsilon^4}{(r^2 + d^2 - 2rd\cos\theta)^2} \right)^{3/4}$$

- In order to remove the divergence with r_{max}, a more natural quantity to compute is Complexity of Formation. It is defined as the (Complexity of spacetime - Complexity of plain $AdS_5 \times S^5$). This gives

$$V_{\text{formation}} = -\frac{\pi d^3}{14}\frac{3\epsilon^4}{4}$$

- The first non-trivial order is ϵ^4, the cubic dependence on d comes from dimensional dependence of *AdS* radius.
- The negative sign is non trivial. There are results in [1] that show HC should always be positive (though we expect something different for when we involve Extra Dimensions).

- Another hypothesis to calculate Complexity:

$$C_A = \frac{S}{\pi \hbar}$$

  where S is the action.
- Action S is the gravitational action of the spacetime of a black hole integrated over a Wheeler-De Witt Patch. It is the region inside two light sheets emitted from the boundary of spacetime, at a constant time, in both the future and past directions.

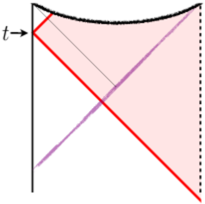

FIGURE: WDW patch(red), boundary time t. Source: [2]

CA Proposal Contd...

- The CA proposal empowers us to calculate the complexity for any spacetime without solving a maximizing problem.
- The PDE for the WDW patch (in the half plane) turns out to be (Eikonal equation) of the form

$$\left(\frac{\partial t}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial t}{\partial \theta}\right)^2 = H_3$$

- Tough to solve analytically, even in a perturbative manner.

CA Proposal contd...

- The action is given by

$$S_{\mathcal{V}} = \frac{1}{16\pi G_N}\left[\int_{\mathcal{W}} R\sqrt{-g}\,\mathrm{d}V - 2\int_F \kappa \mathrm{d}S\mathrm{d}\lambda + 2\int_P \kappa \mathrm{d}S\mathrm{d}\lambda \right.$$
$$\left. - 2\oint_{\Sigma} a\mathrm{d}S - 2\int_F \Theta \ln|l\Theta|\mathrm{d}S\mathrm{d}\lambda + 2\int_P \Theta \ln|l\Theta|\mathrm{d}S\mathrm{d}\lambda\right]$$

  where the first term is the Einstein-Hilbert action and the rest of the terms are boundary terms introduced to get sensible EOMs within finite boundaries.

- The bulk term CA calculation for spacetime we are dealing has an additional term

$$\frac{1}{2}\int_{\mathcal{W}}(-|\widetilde{F}_5|^2)d^{10}x$$

  where $\widetilde{F}_5$ is called the Ramond Self dual five form. A five form very simply can be treated as 5-index tensor, and in string theory, $\widetilde{F}_5$ is responsible for electromagnetism, analogous to the Maxwell tensor $F_{\mu\nu}$ in classical field theory.

Future Work and Conclusion

- In these calculation we have calculated HC that mix the extra dimensions of String Theory with the AdS directions in a novel way.
- For Multicentered AdS, we get interesting results for CV definition and are expecting the same soon for CA definition too (even though the calculation is highly challenging). The $AdS_5 \times S^5$ turns out to be trivial for CA too.
- We are able to study the dependence of complexity on the distance d between the centers.
- We would like to inquire more about the negative value of HC. We would like to explore if fixing the reference state has some effect on the Complexity of Formation.

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Thank You For Your Time And Attention!