PHYS 4010/7010: General Relativity
Assignment 2

Due: Wednesday, September 24, 2025 by 10:20 am

1. Lorentz Transformation for Arbitrary Velocity

The Lorentz Transformation (LT) for the case of two-dimensional velocity $\vec{v} = v_x \vec{e}_x + v_y \vec{e}_y$ is given by

$$\Lambda\left(\vec{v}\right) = \begin{pmatrix} \gamma & -\gamma v_x/c & -\gamma v_y/c \\ -\gamma v_x/c & \Lambda_{11} & \Lambda_{12} \\ -\gamma v_y/c & \Lambda_{21} & \Lambda_{22} \end{pmatrix} \tag{1}$$

where we have used

$$\Lambda_{ij} = \delta_{ij} + \frac{v_i v_j}{v^2}\left(\gamma - 1\right).$$

Derive Eq. (1) by calculating $\Lambda(\vec{v}) = \Lambda(\alpha^T)\Lambda(v)\Lambda(\alpha)$ as discussed in class. Here we have used the usual one-dimensional LT $\Lambda(v)$ and the rotation

$$\Lambda\left(\alpha\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\alpha\right) & \sin\left(\alpha\right) \\ 0 & -\sin\left(\alpha\right) & \cos\left(\alpha\right) \end{pmatrix}. \tag{2}$$

2. General Form of the Additionstheorem

In Special Relativity, the additionstheorem for arbitrary velocity vectors $\vec{v}_1$ and $\vec{v}_2$ is

$$\vec{V} = \frac{\vec{v}_1 + \vec{v}_2 + \left(\vec{v}_2 - \vec{v}_1 \frac{\vec{v}_1 \cdot \vec{v}_2}{\vec{v}_1^2}\right)\left(\sqrt{1 - \frac{\vec{v}_1^2}{c^2}} - 1\right)}{1 + \frac{\vec{v}_1 \cdot \vec{v}_2}{c^2}}. \tag{3}$$

a) Proof Eq. (3) by evaluating $\Lambda^0_\alpha(\vec{V}) = \Lambda^0_\beta(\vec{v}_2)\Lambda^\beta_\alpha(\vec{v}_1)$ for $\alpha = 0$ and $\alpha = j$ with $j = 1, 2, 3$.

b) Simplify Eq. (3) by assuming that the two velocities are parallel $\vec{v}_1 \parallel \vec{v}_2$.

c) Simplify Eq. (3) by considering the limit $v_1 \ll c$ and $v_2 \ll c$.

Hint:
For the Lorentz-transformations $\Lambda^0_\beta(\vec{v}_2)$ and $\Lambda^\beta_\alpha(\vec{v}_1)$ you can use the expression discussed in class!