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New results on integration on the Levi-Civita field

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Abstract

New results for integration of functions on the Levi-Civita field $\mathcal{R}$ are presented in this paper which is a continuation of the work done in Shamseddine and Berz (2003) [13] and complements it. For example, we show that if f and g are bounded on a measurable set A and $f = g$ almost everywhere on A then f is measurable on A if and only if g is measurable on A in which case the integrals of f and g over A are equal. We also show that if $A \subset \mathcal{R}$ is measurable and if (f_n) is a sequence of measurable functions that converge uniformly on A to f, then f itself is measurable on A and its integral over A is given by $\int_A f = \lim_{n\to\infty} \int_A f_n$.

Keywords: Levi-Civita field; Non-Archimedean analysis; Measurable sets; Measurable functions; Integration

1. Introduction

New results for measure theory and integration on the Levi-Civita field $\mathcal{R}$ [4,5,13] are presented. We recall that the elements of $\mathcal{R}$ are functions from $\mathbb{Q}$ to $\mathbb{R}$ with left-finite support (denoted by supp). That is, below every rational number q, there are only finitely many points where the given function does not vanish. For the further discussion, it is convenient to introduce the following terminology.

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Definition 1.1 ($\lambda, \sim, \approx$). For $x \neq 0$ in $\mathcal{R}$, we let $\lambda(x) = \min(\text{supp}(x))$, which exists because of the left-finiteness of $\text{supp}(x)$; and we let $\lambda(0) = +\infty$.

Given $x, y \neq 0$ in $\mathcal{R}$, we say $x \sim y$ if $\lambda(x) = \lambda(y)$; and we say $x \approx y$ if $\lambda(x) = \lambda(y)$ and $x[\lambda(x)] = y[\lambda(y)]$.

At this point, these definitions may feel somewhat arbitrary; but after having introduced an order on $\mathcal{R}$, we will see that λ describes orders of magnitude, the relation $\approx$ corresponds to agreement up to infinitely small relative error, while $\sim$ corresponds to agreement of order of magnitude.

The set $\mathcal{R}$ is endowed with formal power series multiplication and componentwise addition, which make it into a field [2] in which we can isomorphically embed $\mathbb{R}$ as a subfield via the map $\Pi : \mathbb{R} \to \mathcal{R}$ defined by

$$\Pi(x)[q] = \begin{cases} x & \text{if } q = 0 \\ 0 & \text{else.} \end{cases} \tag{1.1}$$

Definition 1.2 (*Order in* $\mathcal{R}$). Let $x, y \in \mathcal{R}$ be given. Then we say $x \geq y$ if $x = y$ or $[x \neq y$ and $(x - y)[\lambda(x - y)] > 0]$.

It is easy to check that the relation $\geq$ is a total order and $(\mathcal{R}, +, \cdot, \geq)$ is an ordered field (which will be denoted henceforth, simply, by $\mathcal{R}$). Moreover, the embedding Π in Eq. (1.1) of $\mathbb{R}$ into $\mathcal{R}$ is compatible with the order. The order induces an absolute value on $\mathcal{R}$ in the natural way: $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. We also note here that λ, as defined above, is a valuation; moreover, the relation $\sim$ is an equivalence relation, and the set of equivalence classes (the value group) is (isomorphic to) $\mathbb{Q}$.

Besides the usual order relations, some other notations are convenient.

Definition 1.3 ($\ll, \gg$). Let $x, y \in \mathcal{R}$ be non-negative. We say x is infinitely smaller than y (and write $x \ll y$) if $nx < y$ for all $n \in \mathbb{N}$; we say x is infinitely larger than y (and write $x \gg y$) if $y \ll x$. If $x \ll 1$, we say x is infinitely small; if $x \gg 1$, we say x is infinitely large. Infinitely small numbers are also called infinitesimals or differentials. Infinitely large numbers are also called infinite. Non-negative numbers that are neither infinitely small nor infinitely large are also called finite.

Definition 1.4 (*The Number d*). Let d be the element of $\mathcal{R}$ given by $d[1] = 1$ and $d[q] = 0$ for $q \neq 1$.

It is easy to check that $d^q \ll 1$ if $q > 0$ and $d^q \gg 1$ if $q < 0$. Moreover, for all $x \in \mathcal{R}$, the elements of $\text{supp}(x)$ can be arranged in ascending order, say $\text{supp}(x) = \{q_1, q_2, \ldots\}$ with $q_j < q_{j+1}$ for all j; and x can be written as $x = \sum_{j=1}^{\infty} x[q_j]d^{q_j}$, where the series converges in the topology induced by the absolute value [2].

Altogether, it follows that $\mathcal{R}$ is a non-Archimedean field extension of $\mathbb{R}$. For a detailed study of this field, we refer the reader to [9,15] and the references therein. In particular, it is shown that $\mathcal{R}$ is complete with respect to the topology induced by the absolute value; that is, every Cauchy sequence of elements of $\mathcal{R}$ converges to an element of $\mathcal{R}$. In the wider context of valuation theory, it is interesting to note that the topology induced by the absolute value is the same as that introduced via the valuation λ, as was shown in [14].

It follows therefore that the field $\mathcal{R}$ is just a special case of the class of fields discussed in [8]. For a general overview of the algebraic properties of formal power series fields in general, we refer the reader to the comprehensive overview by Ribenboim [7], and for an overview of the related valuation theory to the books by Krull [3], Schikhof [8] and Alling [1]. A thorough and complete treatment of ordered structures can also be found in [6].

Besides being the smallest ordered non-Archimedean field extension of the real numbers that is both complete in the order topology and real closed, the Levi-Civita field $\mathcal{R}$ is of particular interest because of its practical usefulness. Since the supports of the elements of $\mathcal{R}$ are left-finite, it is possible to represent these numbers on a computer [2]. Having infinitely small numbers, the errors in classical numerical methods can be made infinitely small and hence irrelevant in all practical applications. One such application is the computation of derivatives of real functions representable on a computer [11], where both the accuracy of formula manipulators and the speed of classical numerical methods are achieved.

In [13] we introduce a measure that proves to be a natural generalization of the Lebesgue measure on the field of the real numbers and have similar properties. Then we introduce a family of simple functions from which we obtain a larger family of measurable functions and derive a simple characterization of such functions. We study the properties of measurable functions, we show how to integrate them over measurable sets of $\mathcal{R}$, and we show that the resulting integral satisfies similar properties to those of the Lebesgue integral of real calculus. In Section 2 we review the definitions and key results proved in [13] then in Section 3 we present new results that complement the work done in [13].

2. Review of measure theory and integration on $\mathcal{R}$

In this section we review key definitions and results that will be needed in Section 3; for proofs and other details we refer the reader to [13]. Before we define a measure on $\mathcal{R}$, we introduce the following notations which will be adopted throughout this paper: $I(a,b)$ will be used to denote any one of the intervals $[a,b]$, $(a,b]$, $[a,b)$ or (a,b), unless we explicitly specify a particular choice of one of the four intervals. Also, to denote the length of a given interval I, we will use the notation $l(I)$.

Definition 2.1. Let $A \subset \mathcal{R}$ be given. Then we say that A is measurable if for every $\epsilon > 0$ in $\mathcal{R}$, there exist a sequence of mutually disjoint intervals (I_n) and a sequence of mutually disjoint intervals (J_n) such that $\cup_{n=1}^{\infty} I_n \subset A \subset \cup_{n=1}^{\infty} J_n$, $\sum_{n=1}^{\infty} l(I_n)$ and $\sum_{n=1}^{\infty} l(J_n)$ converge in $\mathcal{R}$, and $\sum_{n=1}^{\infty} l(J_n) - \sum_{n=1}^{\infty} l(I_n) \le \epsilon$.

Given a measurable set A, then for every $k \in \mathbb{N}$, we can select a sequence of mutually disjoint intervals $\left(I_n^k\right)$ and a sequence of mutually disjoint intervals $\left(J_n^k\right)$ such that $\sum_{n=1}^{\infty} l\left(I_n^k\right)$ and $\sum_{n=1}^{\infty} l\left(J_n^k\right)$ converge in $\mathcal{R}$ for all k,

$$\cup_{n=1}^{\infty} I_n^k \subset \cup_{n=1}^{\infty} I_n^{k+1} \subset A \subset \cup_{n=1}^{\infty} J_n^{k+1} \subset \cup_{n=1}^{\infty} J_n^k,$$

and

$$\sum_{n=1}^{\infty} l\left(J_n^k\right) - \sum_{n=1}^{\infty} l\left(I_n^k\right) \le d^k$$

for all $k \in \mathbb{N}$. Since $\mathcal{R}$ is Cauchy-complete in the order topology, it follows that $\lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(I_n^k\right)$ and $\lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(J_n^k\right)$ both exist and they are equal. We call the common value of the limits the measure of A and we denote it by $m(A)$. Thus,

$$m(A) = \lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(I_n^k\right) = \lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(J_n^k\right). \tag{2.1}$$

Moreover, since the sequence $\left(\sum_{n=1}^{\infty} l\left(I_n^k\right)\right)_{k\in\mathbb{N}}$ is nondecreasing and since the sequence $\left(\sum_{n=1}^{\infty} l\left(J_n^k\right)\right)_{k\in\mathbb{N}}$ is nonincreasing, we have that

$$\sum_{n=1}^{\infty} l\left(I_n^k\right) \le m(A) \le \sum_{n=1}^{\infty} l\left(J_n^k\right) \quad \text{for all } k \in \mathbb{N}. \tag{2.2}$$

Contrary to the real case,

$$\sup\left\{\sum_{n=1}^{\infty} l(I_n) : I_n\text{'s are mutually disjoint intervals and } \cup_{n=1}^{\infty} I_n \subset A\right\}$$

and

$$\inf\left\{\sum_{n=1}^{\infty} l(J_n) : A \subset \cup_{n=1}^{\infty} J_n\right\}$$

need not exist for a given set $A \subset \mathcal{R}$. However, as shown in [13], if A is measurable then both the supremum and infimum exist and they are equal to $m(A)$. This shows that the definition of measurable sets in Definition 2.1 is a natural generalization of the Lebesgue measure of real analysis that corrects for the lack of suprema and infima in non-Archimedean ordered fields.

It follows directly from Definition 2.1 that $m(A) \ge 0$ for any measurable set $A \subset \mathcal{R}$ and that any interval $I(a, b)$ is measurable with measure $m(I(a, b)) = b - a$. It also follows that if A is a countable union of mutually disjoint intervals $(I_n(a_n, b_n))$ such that $\sum_{n=1}^{\infty}(b_n - a_n)$ converges then A is measurable with $m(A) = \sum_{n=1}^{\infty}(b_n - a_n)$. Moreover, if $B \subset A \subset \mathcal{R}$ and if A and B are measurable, then $m(B) \le m(A)$.

In [13] we show that the measure defined on $\mathcal{R}$ above has similar properties to those of the Lebesgue measure on $\mathbb{R}$. For example, we show that any subset of a measurable set of measure 0 is itself measurable and has measure 0. We also show that any countable unions of measurable sets whose measures form a null sequence is measurable and the measure of the union is less than or equal to the sum of the measures of the original sets; moreover, the measure of the union is equal to the sum of the measures of the original sets if the latter are mutually disjoint. Furthermore, we show that any finite intersection of measurable sets is also measurable and that the sum of the measures of two measurable sets is equal to the sum of the measures of their union and intersection.

It is worth noting that the complement of a measurable set in a measurable set need not be measurable. For example, $[0, 1]$ and $[0, 1] \cap \mathbb{Q}$ are both measurable with measures 1 and 0, respectively. However, the complement of $[0, 1] \cap \mathbb{Q}$ in $[0, 1]$ is not measurable. On the other hand, if $B \subset A \subset \mathcal{R}$ and if A, B and $A \setminus B$ are all measurable, then $m(A) = m(B) + m(A \setminus B)$.

The example of $[0, 1] \setminus [0, 1] \cap \mathbb{Q}$ above shows that the axiom of choice is not needed here to construct a nonmeasurable set, as there are many simple examples of nonmeasurable sets. Indeed, any uncountable real subset of $\mathcal{R}$, like $[0, 1] \cap \mathbb{R}$ for example, is not measurable.

Like in $\mathbb{R}$, we first introduce a family of simple functions on $\mathcal{R}$ from which we obtain a larger family of measurable functions. In the Lebesgue measure theory on $\mathbb{R}$, the simple functions consist only of step functions (piece-wise constant functions); and all measurable functions including all monomials, polynomials and power series are obtained as uniform limits of simple functions. It can be easily shown that in $\mathcal{R}$ the order topology is too strong and none of the monomials can be obtained as a uniform limit of polynomials of lower degrees. So using the step functions as our simple functions would yield a too small class of functions that we can integrate. So we introduce a larger family of simple functions. In [13] we define such a family of simple functions in an abstract way, as follows.

Definition 2.2. Let $a < b$ in $\mathcal{R}$ be given and $\mathcal{S}(a, b)$ a family of functions from $I(a, b)$ to $\mathcal{R}$. Then we say that $\mathcal{S}(a, b)$ is a family of simple functions on $I(a, b)$ if the following are true:

(1) $\mathcal{S}(a, b)$ is an algebra that contains the identity function;
(2) for all $f \in \mathcal{S}(a, b)$, f is Lipschitz on $I(a, b)$ and there exists an anti-derivative F of f in $\mathcal{S}(a, b)$;
(3) for all differentiable $f \in \mathcal{S}(a, b)$, if $f' = 0$ on $I(a, b)$ then f is constant on $I(a, b)$; moreover, if $f' \le 0$ on $I(a, b)$ then f is nonincreasing on $I(a, b)$.

If $f \in \mathcal{S}(a, b)$, we say that f is simple on $I(a, b)$.

It follows from the first condition in Definition 2.2 that any constant function on $I(a, b)$ is in $\mathcal{S}(a, b)$; moreover, if $f, g \in \mathcal{S}(a, b)$ and if $\alpha \in \mathcal{R}$, then $f + \alpha g \in \mathcal{S}(a, b)$. Also, it follows from the third condition that the anti-derivative in the second condition is unique up to a constant. A close look at Definition 2.2 reveals that the polynomials algebra on $I(a, b)$ is the smallest family of simple functions on $I(a, b)$. Another example (which we will assume for the rest of this paper) is the family of power series that converge for all $x \in I(a, b)$ (where convergence is in the weak topology of [2,12,10]).

While the third condition in Definition 2.2 is automatically satisfied in real analysis, this is not the case in $\mathcal{R}$, as the following example shows.

Example 2.3. Let $g : (0, 1) \to \mathcal{R}$ be given by $g(x)[q] = x[q/3]$ for all $q \in \mathbb{Q}$. Then g is differentiable on $(0, 1)$ with $g'(x) = 0$ for all $x \in (0, 1)$. We first observe that $g(x + y) = g(x) + g(y)$ for all $x, y \in (0, 1)$. Now let $x \in (0, 1)$ and $\epsilon > 0$ in $\mathcal{R}$ be given. Let $\delta = \min\{\epsilon, d\}$, and let $y \in (0, 1)$ be such that $0 < |y - x| < \delta$. Then

$$\left|\frac{g(y) - g(x)}{y - x}\right| = \left|\frac{g(y - x)}{y - x}\right| \sim (y - x)^2 \quad \text{since } g(y - x) \sim (y - x)^3.$$

Since $|y - x| < \min\{\epsilon, d\}$, we obtain that $(y - x)^2 \ll \epsilon$. Hence

$$\left|\frac{g(y) - g(x)}{y - x}\right| < \epsilon \quad \text{for all } y \in (0, 1) \text{ satisfying } 0 < |y - x| < \delta;$$

which shows that g is differentiable at x and $g'(x) = 0$.

Now let $f : (0, 1) \to \mathcal{R}$ be given by $f(x) = g(x) - x^4$. Then f is differentiable on $(0, 1)$ with $f'(x) = -4x^3 < 0$ for all $x \in (0, 1)$. However, we have that $d > d^2$ and $f(d) = d^3 - d^4 > f(d^2) = d^6 - d^8$. Thus, even though $f' < 0$ everywhere on $(0, 1)$, f is not nonincreasing on $(0, 1)$.

Starting with the family of simple functions, we will show below how to obtain a larger family of measurable functions. Then we define the integral of a simple function over an interval; and, based on that, we show how to integrate any measurable function over a measurable set.

Definition 2.4. Let $A \subset \mathcal{R}$ be a measurable subset of $\mathcal{R}$ and let $f : A \to \mathcal{R}$ be bounded on A. Then we say that f is measurable on A if for all $\epsilon > 0$ in $\mathcal{R}$, there exists a sequence of mutually disjoint intervals (I_n) such that $I_n \subset A$ for all n, $\sum_{n=1}^{\infty} l(I_n)$ converges in $\mathcal{R}$, $m(A) - \sum_{n=1}^{\infty} l(I_n) \leq \epsilon$ and f is simple on I_n for all n.

Proposition 2.5 (*Characterization of Measurable Functions*)**.** *Let $A \subset \mathcal{R}$ be measurable and let $f : A \to \mathcal{R}$ be measurable. Then f is locally a simple function almost everywhere on A.*

The following example shows that the converse of Proposition 2.5 need not be true.

Example 2.6. Let $f : [0, 1] \to \mathcal{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ x & \text{otherwise.} \end{cases}$$

Then f is locally simple almost everywhere on $[0, 1]$; but f is not measurable on $[0, 1]$.

As an immediate result of Proposition 2.5 and the properties of simple functions, we obtain the following result which will prove very useful in defining the integral of a measurable function.

Corollary 2.7. *Let $a < b$ in $\mathcal{R}$ and let $f : I(a, b) \to \mathcal{R}$ be measurable. Then f is continuous almost everywhere on $I(a, b)$. Moreover, if f is differentiable on $I(a, b)$ and if f' vanishes everywhere, then f is constant on $I(a, b)$.*

Corollary 2.8. *Let $a < b$ in $\mathcal{R}$ and let $f, g : [a, b] \to \mathcal{R}$ be measurable. Assume that f and g are both differentiable with $f' = g'$ on $[a, b]$. Then there exists a constant c such that $f(x) = g(x) + c$ for all $x \in [a, b]$; and hence $f(b) - f(a) = g(b) - g(a)$.*

Now assume that $f : I(a, b) \to \mathcal{R}$ is simple and let F be a simple anti-derivative of f on $I(a, b)$. Then we define the integral of f over $I(a, b)$ as the $\mathcal{R}$ number

$$\int_{I(a,b)} f = \lim_{x \to b} F(x) - \lim_{x \to a} F(x).$$

The limits are needed for the case when the interval $I(a, b)$ is not closed, in which case the limits do exist because simple functions are Lipschitz and in which case the extended function $\bar{F} : [a, b] \to \mathcal{R}$, given by

$$\bar{F} = \begin{cases} F(x) & \text{if } x \in I(a,b) \\ \lim\limits_{x \to a} F(x) & \text{if } x = a \\ \lim\limits_{x \to b} F(x) & \text{if } x = b \end{cases}$$

is simple on $[a, b]$ and it is an antiderivative of the extension $\bar{f}$ of f to $[a, b]$. Moreover, because of Corollary 2.8, the integral is well-defined.

Next, let $A \subset \mathcal{R}$ be measurable, let $f : A \to \mathcal{R}$ be measurable and let M be a bound for $|f|$ on A. Then for every $k \in \mathbb{N}$, there exists a sequence of mutually disjoint intervals $\left(I_n^k\right)_{n \in \mathbb{N}}$ such that $\cup_{n=1}^{\infty} I_n^k \subset A$, $\sum_{n=1}^{\infty} l\left(I_n^k\right)$ converges, $m(A) - \sum_{n=1}^{\infty} l\left(I_n^k\right) \leq d^k$, and f is simple on I_n^k for all $n \in \mathbb{N}$. Without loss of generality, we may assume that $I_n^k \subset I_n^{k+1}$ for all $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$. Since $\lim_{n \to \infty} l\left(I_n^k\right) = 0$, and since $\left|\int_{I_n^k} f\right| \leq Ml\left(I_n^k\right)$ by Corollary 4.6 in [13], it follows that

$$\lim_{n\to\infty}\int_{I_n^k} f = 0 \quad \text{for all } k \in \mathbb{N}.$$

Thus, $\sum_{n=1}^{\infty}\int_{I_n^k} f$ converges in $\mathcal{R}$ for all $k \in \mathbb{N}$ [12].

Moreover, the sequence $\left(\sum_{n=1}^{\infty}\int_{I_n^k} f\right)_{k\in\mathbb{N}}$ converges in $\mathcal{R}$ [13]. We define the unique limit as the integral of f over A.

Definition 2.9. Let $A \subset \mathcal{R}$ be measurable and let $f : A \to \mathcal{R}$ be measurable. Then the integral of f over A, denoted by $\int_A f$, is given by

$$\int_A f = \lim_{\substack{\sum_{n=1}^{\infty} l(I_n)\to m(A) \\ \cup_{n=1}^{\infty} I_n \subset A \\ (I_n)\text{ are mutually disjoint} \\ f \text{ is simple on } I_n\ \forall\ n}} \sum_{n=1}^{\infty}\int_{I_n} f.$$

We show in [13] that the integral defined in Definition 2.9 satisfies similar properties to those of the Lebesgue and Riemann integrals on $\mathbb{R}$. In particular, we prove the linearity property of the integral and that if $|f| \le M$ on A then $\left|\int_A f\right| \le Mm(A)$, where $m(A)$ is the measure of A. We also show that a function f measurable on two measurable subsets A and B of $\mathcal{R}$ is also measurable on their union and intersection; moreover, in this case,

$$\int_A f + \int_B f = \int_{A\cup B} f + \int_{A\cap B} f.$$

3. New results

Definition 3.1. Let $A \subset \mathcal{R}$ be measurable and let $f, g : A \to \mathcal{R}$. Then we say that $f = g$ almost everywhere and write $f = g$ a.e. on A if the set $B := \{x \in A : f(x) = g(x)\}$ is measurable with $m(B) = m(A)$.

Lemma 3.2. *Let $S \subset \mathcal{R}$ be measurable with $m(S) = 0$, and let $f : S \to \mathcal{R}$ be bounded. Then f is measurable on S and $\int_S f = 0$.*

Proof. That f is measurable on S follows from Definition 2.4 with the "intervals" I_n taken to be all the empty set. Now let M be a bound of $|f|$ on S; then using Corollary 4.12 in [13], it follows that

$$0 \le \left|\int_S f\right| \le \int_S |f| \le Mm(S) = 0.$$

Thus, $\left|\int_S f\right| = 0$ and hence $\int_S f = 0$. $\square$

Lemma 3.3. *Let $A, B \subset \mathcal{R}$ be measurable such that $B \subset A$ and $m(B) = m(A)$. Then $A \setminus B$ is measurable with $m(A \setminus B) = 0$.*

Proof. Let $\epsilon > 0$ in $\mathcal{R}$ be given. Then there exist a sequence of mutually disjoint intervals (I_n) and a sequence of mutually disjoint intervals (J_n) such that $\cup_{n=1}^{\infty} I_n \subset A \subset B \subset$

$\cup_{n=1}^{\infty} J_n$, $\sum_{n=1}^{\infty} l(I_n)$ and $\sum_{n=1}^{\infty} l(J_n)$ converge in $\mathcal{R}$, and

$$\sum_{n=1}^{\infty} l(J_n) - m(B) < \frac{\epsilon}{2} \quad \text{and} \quad m(A) - \sum_{n=1}^{\infty} l(I_n) < \frac{\epsilon}{2}.$$

It follows that $B \setminus A \subset \cup_{n=1}^{\infty} J_n \setminus \cup_{n=1}^{\infty} I_n$. Since $\cup_{n=1}^{\infty} I_n \subset \cup_{n=1}^{\infty} J_n$ and since the I_n's are mutually disjoint and so are the J_n's, each of the I_n's is exclusively contained in one of the J_n's. It follows that we can write $\cup_{n=1}^{\infty} J_n \setminus \cup_{n=1}^{\infty} I_n$ as a union of mutually disjoint intervals, say $\cup_{n=1}^{\infty} K_n$, such that $\sum_{n=1}^{\infty} l(K_n)$ converges and

$$\begin{aligned}
\sum_{n=1}^{\infty} l(K_n) &= \sum_{n=1}^{\infty} l(J_n) - \sum_{n=1}^{\infty} l(I_n) \\
&= \left(\sum_{n=1}^{\infty} l(J_n) - m(A)\right) + \left(m(A) - \sum_{n=1}^{\infty} l(I_n)\right) \\
&= \left(\sum_{n=1}^{\infty} l(J_n) - m(B)\right) + \left(m(A) - \sum_{n=1}^{\infty} l(I_n)\right) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus, given $\epsilon > 0$ in $\mathcal{R}$, we were able to find a sequence of mutually disjoint intervals (K_n) such that $A \setminus B \subset \cup_{n=1}^{\infty} K_n$, $\sum_{n=1}^{\infty} l(K_n)$ converges, and

$$\sum_{n=1}^{\infty} l(K_n) < \epsilon.$$

This shows that $A \setminus B$ is measurable with $m(A \setminus B) = 0$. $\square$

Using Definition 3.1 and Lemma 3.3, we readily obtain the following corollary.

Corollary 3.4. *Let $A \subset \mathcal{R}$ be measurable and let $f, g : A \to \mathcal{R}$ be such that $f = g$ a.e. on A. Then the set $S := \{x \in A : f(x) \neq g(x)\}$ is measurable and $m(S) = 0$.*

Remark 3.5. In $\mathbb{R}$, the conditions

(1) $m(\{x \in A : f(x) = g(x)\}) = m(A)$
(2) $m(\{x \in A : f(x) \neq g(x)\}) = 0$

are equivalent. However, in $\mathcal{R}$ Condition (1) entails Condition (2) (Corollary 3.4), but the converse is not true as the complement of a set of measure 0 in a measurable set need not be measurable (see Section 2 above). Hence Definition 3.1.

Lemma 3.6. *Let $A \subset \mathcal{R}$ be measurable, let $f : A \to \mathcal{R}$ be measurable on A, and let $B \subset A$ be measurable with $m(B) = m(A)$. Then f is measurable on B with*

$$\int_B f = \int_A f.$$

Proof. Let $S = A \setminus B$. Then S is measurable with $m(S) = 0$ by Lemma 3.3. Let $\epsilon > 0$ in $\mathcal{R}$ be given. Then there exists a sequence of mutually disjoint intervals (I_n) such that $\cup_{n=1}^{\infty} I_n \subset A$, $\sum_{n=1}^{\infty} l(I_n)$ converges, $m(A) - \sum_{n=1}^{\infty} l(I_n) < \epsilon/2$, and f is simple on I_n for each n.

Moreover, since $m(S) = 0$ there exists a sequence of mutually disjoint intervals (J_n) such that $S \subset \cup_{n=1}^{\infty} J_n$, $\sum_{n=1}^{\infty} l(J_n)$ converges and $\sum_{n=1}^{\infty} l(J_n) < \epsilon/2$. We can write $\cup_{n=1}^{\infty} I_n \setminus \cup_{n=1}^{\infty} J_n$ as a union of mutually disjoint intervals, say $\cup_{n=1}^{\infty} K_n$ such that $\sum_{n=1}^{\infty} l(K_n)$ converges. It follows that

$$\cup_{n=1}^{\infty} K_n = \cup_{n=1}^{\infty} I_n \setminus \cup_{n=1}^{\infty} J_n \subset A \setminus S = B,$$

f is simple on K_n for each n, and

$$\begin{aligned} m(B) - \sum_{n=1}^{\infty} l(K_n) &= m(A) - \sum_{n=1}^{\infty} l(K_n) \\ &\le m(A) - \sum_{n=1}^{\infty} l(I_n) + \sum_{n=1}^{\infty} l(J_n) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This shows that f is measurable on B. Moreover, f is measurable on S with $\int_S f = 0$ by Lemma 3.2. It follows, using Proposition 4.13 in [13], that

$$\int_A f = \int_B f + \int_S f = \int_B f. \quad \square$$

Theorem 3.7. *Let $A \subset \mathcal{R}$ be measurable and let $f, g : A \to \mathcal{R}$ be bounded such that $f = g$ a.e. on A. Then f is measurable on A if and only if g is measurable on A in which case we have*

$$\int_A f = \int_A g.$$

Proof. It suffices to show that if f is measurable on A then g is measurable on A and $\int_A g = \int_A f$. So assume f is measurable on A, and let

$$B = \{x \in A : f(x) = g(x)\} \quad \text{and} \quad S = A \setminus B = \{x \in A : f(x) \neq g(x)\}.$$

Then B and S are both measurable by Definition 3.1 and Corollary 3.4 with $m(B) = m(A)$ and $m(S) = 0$. By Lemma 3.6, f is measurable on B with

$$\int_A f = \int_B f.$$

But $g = f$ on B; therefore g is measurable on B and

$$\int_B g = \int_B f.$$

Moreover, since g is bounded on S and since $m(S) = 0$, it follows by Lemma 3.2 that g is measurable on S with

$$\int_S g = 0.$$

Altogether, it follows that g is measurable on $A = B \cup S$ and

$$\int_A g = \int_B g + \int_S g = \int_B g = \int_B f = \int_A f. \quad \square$$

Lemma 3.8. *Let $a < b$ in $\mathcal{R}$ be given, for each $k \in \mathbb{N}$ let $f_k : I(a,b) \to \mathcal{R}$ be simple. Suppose that (f_k) converges uniformly to f on $I(a,b)$. Then f is simple on $I(a,b)$.*

Proof. Without loss of generality we may assume that the interval is closed, that is $I(a,b) = [a,b]$. If we define $F_k, F : [0,1] \to \mathcal{R}$ by

$$F_k(x) = f_k(a + (b-a)x) \quad \text{for } k \in \mathbb{N};\ F(x) = f(a + (b-a)x),$$

then F_k is simple on $[0,1]$ [14]; and f is simple on $[a,b]$ if and only if F is simple on $[0,1]$. So without loss of generality we may assume that $a = 0$ and $b = 1$.

Since (f_k) converges uniformly (to f) on $[0,1]$ it follows that (f_k) is uniformly Cauchy on $[0,1]$. Thus, given $q \in \mathbb{Q}$, we can find $N \in \mathbb{N}$ such that

$$|f_k(x) - f_m(x)| < d^{q+1} \quad \text{for all } k, m \geq N \text{ and for all } x \in [0,1].$$

In particular,

$$f_k(X)[t] = f_N(X)[t] \quad \forall k \geq N,\ \forall X \in [0,1] \cap \mathbb{R},\ \text{and } \forall t \leq q. \tag{3.1}$$

This shows that $\cup_{k=1}^{\infty} \cup_{X \in [0,1]\cap\mathbb{R}} \operatorname{supp}(f_k(X))$ is left-finite, say $\{q_1, q_2, q_3, \ldots\}$ with $q_1 < q_2 < \cdots$.

Also, given $q \in \mathbb{Q}$ there exists $M \in \mathbb{N}$ such that

$$|f_k(X) - f(X)| < d^{q+1} \quad \text{for all } k \geq M \text{ and for all } X \in [0,1] \cap \mathbb{R}. \tag{3.2}$$

This shows that

$$f(X)[q] = f_M(X)[q],$$

which is a real power series. Moreover,

$$\cup_{X \in [0,1]\cap\mathbb{R}} \operatorname{supp}(f(X)) = \cup_{k=1}^{\infty} \cup_{X \in [0,1]\cap\mathbb{R}} \operatorname{supp}(f_k(X)) = \{q_1, q_2, q_3, \ldots\}.$$

Thus it follows that, for all $X \in [0,1] \cap \mathbb{R}$,

$$f(X) = \sum_{j=1}^{\infty} d^{q_j} F_j(X), \tag{3.3}$$

where $F_j : [0,1] \cap \mathbb{R} \to \mathbb{R}$ is a converging real power series, given by $F_j(X) = f(X)[q_j]$. It follows that $f(X)$ is a power series that converges weakly in $\mathcal{R}$ for all $X \in [0,1] \cap \mathbb{R}$.

Similarly, for each $k \in \mathbb{N}$, we can write

$$f_k(X) = \sum_{j=1}^{\infty} d^{q_j} f_{k_j}(X), \tag{3.4}$$

where $f_{k_j} : [0,1] \cap \mathbb{R} \to \mathbb{R}$ is a converging real power series, given by $f_{k_j}(X) = f_k(X)[q_j]$. From Eqs. (3.1)–(3.4), it follows that $(f_k^{(m)}(X))$ converges uniformly to $f^{(m)}(X)$ and the convergence is uniform for all $X \in [0,1] \cap \mathbb{R}$ and for all $m \in \mathbb{N}$.

Now let $x \in [0,1]$ be given, and let $X = x[0]$ (i.e. X is the real part of x and hence $|x - X| \ll 1$). Then, for each $k \in \mathbb{N}$, we have [14]

$$f_k(x) = \sum_{m=0}^{\infty} \frac{f_k^{(m)}(X)}{m!}(x - X)^m.$$

Let $\epsilon > 0$ in $\mathcal{R}$ be given. Then there exists $N \in \mathbb{N}$ such that

$$|f_k^{(m)}(X) - f^{(m)}(X)| < \epsilon d \quad \text{for } k \geq N, \text{ for all } X \in [0,1] \cap \mathbb{R} \text{ and for all } m \in \mathbb{N}.$$

Thus, for $k \geq N$, we have that

$$\begin{aligned} &\left| f_k(X) - \sum_{m=0}^{\infty} \frac{f^{(m)}(X)}{m!}(x-X)^m \right| \\ &= \left| \sum_{m=0}^{\infty} \frac{f_k^{(m)}(X)}{m!}(x-X)^m - \sum_{m=0}^{\infty} \frac{f^{(m)}(X)}{m!}(x-X)^m \right| \\ &\leq \sum_{m=0}^{\infty} \frac{\left| f_k^{(m)}(X) - f^{(m)}(X) \right|}{m!} |x-X|^m \\ &< \epsilon d \sum_{m=0}^{\infty} \frac{|x-X|^m}{m!} = \epsilon d \exp(|x-X|) \\ &< 2\epsilon d < \epsilon. \end{aligned}$$

This shows that $(f_k(x))$ converges to $\sum_{m=0}^{\infty} \frac{f^{(m)}(X)}{m!}(x-X)^m$; therefore, by uniqueness of limits it follows that

$$f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(X)}{m!}(x-X)^m. \tag{3.5}$$

Eqs. (3.3) and (3.5) together show that f is simple on $[0, 1]$. $\square$

The next theorem is an improved restatement of Theorem 4.14 in [13]. In that earlier theorem, we required the uniform limit of a sequence of measurable functions on a measurable set to be measurable. But in Theorem 3.9, we drop that requirement and show that the limit will in fact be measurable as a result of being the uniform limit of measurable functions. The proof of the latter statement consists of the entire proof of Lemma 3.8 and part of the proof of Theorem 3.9; and it is a central result in this new paper.

Theorem 3.9. *Let $A \subset \mathcal{R}$ be measurable, let $f : A \to \mathcal{R}$, for each $k \in \mathbb{N}$ let $f_k : A \to \mathcal{R}$ be measurable on A, and let the sequence (f_k) converge uniformly to f on A. Then f is measurable on A, $\lim_{k\to\infty} \int_A f_k$ exists, and*

$$\lim_{k\to\infty} \int_A f_k = \int_A f.$$

Proof. First note that since f is the uniform limit of bounded functions on A it follows that f itself is bounded on A. If $m(A) = 0$, there is nothing to prove by Lemma 3.2; so without loss of generality, we may assume that $m(A) > 0$. Let $\epsilon > 0$ in $\mathcal{R}$ be given. Then for each $k \in \mathbb{N}$, we can find a sequence $(I_{k_n})_{n\in\mathbb{N}}$ of mutually disjoint intervals such that $\cup_{n=1}^{\infty} I_{k_n} \subset A$, $\sum_{n=1}^{\infty} l\left(I_{k_n}\right)$ converges in $\mathcal{R}$,

$$m(A) - \sum_{n=1}^{\infty} l\left(I_{k_n}\right) < d^k \epsilon,$$

and f_k is simple on I_{k_n} for each $n \in \mathbb{N}$. Then it follows that we can write $\cap_{k=1}^{\infty}\left(\cup_{n=1}^{\infty} I_{k_n}\right)$ as a countable union of mutually disjoint intervals $(J_j)_{j\in\mathbb{N}}$ such that $\cup_{j=1}^{\infty} J_j \subset A$, and for each $j \in \mathbb{N} f_k$ is simple on I_j for all $k \in \mathbb{N}$. Moreover, $\sum_{j=1}^{\infty} l(J_j)$ converges in $\mathcal{R}$, and

$$\begin{aligned} m(A) - \sum_{j=1}^{\infty} l(J_j) &= m(A) - l\left(\cup_{j=1}^{\infty} J_j\right) \\ &= m(A) - l\left(\cap_{k=1}^{\infty}\left(\cup_{n=1}^{\infty} I_{k_n}\right)\right) \\ &\leq \sum_{k=1}^{\infty}\left[m(A) - l\left(\cup_{n=1}^{\infty} I_{k_n}\right)\right] \\ &= \sum_{k=1}^{\infty}\left[m(A) - \sum_{n=1}^{\infty} l\left(I_{k_n}\right)\right] \\ &< \sum_{k=1}^{\infty} d^k \epsilon = \frac{d}{1-d}\epsilon \\ &< \epsilon. \end{aligned}$$

To show that f is measurable on A it remains to show that f is simple on I_j for each $j \in \mathbb{N}$. So let $j \in \mathbb{N}$ be given. Then f_k is simple on I_j for all $k \in \mathbb{N}$; and (f_k) converges uniformly to f on I_j. It follows from Lemma 3.8 that f is simple on I_j.

Next we show that $\lim_{k\to\infty} \int_A f_k$ exists. Let $\epsilon > 0$ in $\mathcal{R}$ be given and let

$$\epsilon_1 = \epsilon / m(A).$$

Then $\epsilon_1 > 0$ and there exists $K \in \mathbb{N}$ such that $|f_k(x) - f_j(x)| \leq \epsilon_1$ for all $k, j \geq K$ and for all $x \in A$. It follows that

$$\left|\int_A f_k - \int_A f_j\right| = \left|\int_A (f_k - f_j)\right| \leq \epsilon_1 m(A) = \epsilon \quad \text{for all } k, j \geq K.$$

Thus, the sequence $\left(\int_A f_k\right)$ is Cauchy. Since $\mathcal{R}$ is Cauchy complete in the order topology, the sequence $\left(\int_A f_k\right)$ converges in $\mathcal{R}$; that is, $\lim_{k\to\infty} \int_A f_k$ exists in $\mathcal{R}$.

Finally to show that $\lim_{k\to\infty} \int_A f_k = \int_A f$, we follow the same steps as in the previous paragraph, replacing f_j by f. □

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