Contemporary Mathematics
Volume **596**, 2013

A brief survey of the study of power series and analytic functions on the Levi-Civita fields

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Abstract. In this survey paper, we will review the convergence and analytical properties of power series on the Levi-Civita field $\mathcal{R}$ (resp. $\mathcal{C} := \mathcal{R} \oplus i\mathcal{R}$) as well as the properties of the so-called $\mathcal{R}$-analytic functions on an interval $[a, b]$ of $\mathcal{R}$. In particular, we will show that these have the same smoothness properties as real (resp. complex) power series and real analytic functions on an interval of $\mathbb{R}$, respectively.

1. Introduction

A brief survey of our work on power series and analytic functions on the Levi-Civita fields $\mathcal{R}$ and $\mathcal{C}$ [**14**, **17**, **19**–**21**] will be presented. We recall that the elements of $\mathcal{R}$ and its complex counterpart $\mathcal{C}$ are functions from $\mathbb{Q}$ to $\mathbb{R}$ and $\mathbb{C}$, respectively, with left-finite support (denoted by supp). That is, below every rational number q, there are only finitely many points where the given function does not vanish. For the further discussion, it is convenient to introduce the following terminology.

Definition 1.1. (λ, $\sim$, $\approx$) For $x \neq 0$ in $\mathcal{R}$ or $\mathcal{C}$, we let $\lambda(x) = \min(\mathrm{supp}(x))$, which exists because of the left-finiteness of $\mathrm{supp}(x)$; and we let $\lambda(0) = +\infty$.

Given $x, y \neq 0$ in $\mathcal{R}$ or $\mathcal{C}$, we say $x \sim y$ if $\lambda(x) = \lambda(y)$; and we say $x \approx y$ if $\lambda(x) = \lambda(y)$ and $x[\lambda(x)] = y[\lambda(y)]$.

At this point, these definitions may feel somewhat arbitrary; but after having introduced an order on $\mathcal{R}$, we will see that λ describes orders of magnitude, the relation $\approx$ corresponds to agreement up to infinitely small relative error, while $\sim$ corresponds to agreement of order of magnitude.

The sets $\mathcal{R}$ and $\mathcal{C}$ are endowed with formal power series multiplication and componentwise addition, which make them into fields [**3**] in which we can isomorphically embed $\mathbb{R}$ and $\mathbb{C}$ (respectively) as subfields via the map $\Pi : \mathbb{R}, \mathbb{C} \to \mathcal{R}, \mathcal{C}$ defined by

$$\Pi(x)[q] = \begin{cases} x & \text{if } q = 0 \\ 0 & \text{else} \end{cases} . \tag{1.1}$$

Definition 1.2. (Order in $\mathcal{R}$) Let $x, y \in \mathcal{R}$ be given. Then we say $x \geq y$ (or $y \leq x$) if $x = y$ or $[x \neq y$ and $(x - y)[\lambda(x - y)] > 0]$.

2010 *Mathematics Subject Classification.* Primary 26E30, 30G06, 11D88, 12J25.
Key words and phrases. Levi-Civita fields, non-Archimedean analysis, power series, analytic functions.

With this definition of the order relation, $\mathcal{R}$ is an ordered field. Moreover, the embedding Π in Equation (1.1) of $\mathbb{R}$ into $\mathcal{R}$ is compatible with the order. We also note here that λ, as defined above, is a valuation; moreover, the relation $\sim$ is an equivalence relation, and the set of equivalence classes (the value group) is (isomorphic to) $\mathbb{Q}$.

The order induces an ordinary absolute value on $\mathcal{R}$:

$$|x|_o = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0; \end{cases}$$

which induces the same topology on $\mathcal{R}$ (called the order topology or valuation topology) as that induced by the ultrametric absolute value:

$$|x| = e^{-\lambda(x)},$$

as was shown in [**19**]. Moreover, two corresponding absolute values are defined on $\mathcal{C}$ in the natural way:

$$|x+iy|_o = \sqrt{x^2+y^2}; \text{ and } |x+iy| = e^{-\lambda(x+iy)} = \max\{|x|, |y|\}.$$

Thus, $\mathcal{C}$ is topologically isomorphic to $\mathcal{R}^2$ provided with the product topology induced by $|\cdot|_o$ (or $|\cdot|$) in $\mathcal{R}$.

Besides the usual order relations, some other notations are convenient.

Definition 1.3. ($\ll, \gg$) Let $x, y \in \mathcal{R}$ be non-negative. We say x is infinitely smaller than y (and write $x \ll y$) if $nx < y$ for all $n \in \mathbb{N}$; we say x is infinitely larger than y (and write $x \gg y$) if $y \ll x$. If $x \ll 1$, we say x is infinitely small; if $x \gg 1$, we say x is infinitely large. Infinitely small numbers are also called infinitesimals or differentials. Infinitely large numbers are also called infinite. Non-negative numbers that are neither infinitely small nor infinitely large are also called finite.

Definition 1.4. (The Number d) Let d be the element of $\mathcal{R}$ given by $d[1] = 1$ and $d[q] = 0$ for $q \neq 1$.

It is easy to check that $d^q \ll 1$ if $q > 0$ and $d^q \gg 1$ if $q < 0$. Moreover, for all $x \in \mathcal{R}$ (resp. $\mathcal{C}$), the elements of $\text{supp}(x)$ can be arranged in ascending order, say $\text{supp}(x) = \{q_1, q_2, \ldots\}$ with $q_j < q_{j+1}$ for all j; and x can be written as $x = \sum_{j=1}^{\infty} x[q_j]d^{q_j}$, where the series converges in the valuation topology [**3**].

Altogether, it follows that $\mathcal{R}$ (resp. $\mathcal{C}$) is a non-Archimedean field extension of $\mathbb{R}$ (resp. $\mathbb{C}$). For a detailed study of these fields, we refer the reader to [**12, 22**] and references therein. In particular, it is shown that $\mathcal{R}$ and $\mathcal{C}$ are complete with respect to the natural (valuation) topology.

It follows therefore that the fields $\mathcal{R}$ and $\mathcal{C}$ are just special cases of the class of fields discussed in [**11**]. For a general overview of the algebraic properties of formal power series fields in general, we refer the reader to the comprehensive overview by Ribenboim [**10**], and for an overview of the related valuation theory to the books by Krull [**4**], Schikhof [**11**] and Alling [**1**]. A thorough and complete treatment of ordered structures can also be found in [**9**].

Besides being the smallest ordered non-Archimedean field extension of the real numbers that is both complete in the order topology and real closed, the Levi-Civita field $\mathcal{R}$ is of particular interest because of its practical usefulness. Since the supports of the elements of $\mathcal{R}$ are left-finite, it is possible to represent these numbers on a computer [**3**]. Having infinitely small numbers, the errors in classical numerical methods can be made infinitely small and hence irrelevant in all practical applications. One such application is the computation of derivatives of real functions representable on a computer [**16**], where both the accuracy of formula manipulators and the speed of classical numerical methods are achieved.

In [**17**, **19**, **21**], we study the convergence and analytical properties of power series in a topology weaker than the valuation topology used in [**11**], and thus allow for a much larger class of power series to be included in the study. Previous work on power series on the Levi-Civita fields $\mathcal{R}$ and $\mathcal{C}$ had been mostly restricted to power series with real or complex coefficients. In [**5**–**8**], they could be studied for infinitely small arguments only, while in [**3**], using the newly introduced weak topology, also finite arguments were possible. Moreover, power series over complete valued fields in general have been studied by Schikhof [**11**], Alling [**1**] and others in valuation theory, but always in the valuation topology.

In [**17**], we study the general case when the coefficients in the power series are Levi-Civita numbers, using the weak convergence of [**3**]. We derive convergence criteria for power series which allow us to define a radius of convergence η such that the power series converges weakly for all points whose distance from the center is smaller than η by a finite amount and it converges strongly for all points whose distance from the center is infinitely smaller than η. Then, in [**19**], we study the analytical properties of power series on $\mathcal{R}$ and $\mathcal{C}$ within their domain of convergence. We show that, within their radius of convergence, power series are infinitely often differentiable and the derivatives to any order are obtained by differentiating the power series term by term. Also, power series can be re-expanded around any point in their domain of convergence and the radius of convergence of the new series is equal to the difference between the radius of convergence of the original series and the distance between the original and new centers of the series. We then study a class of functions that are given locally by power series (which we call $\mathcal{R}$-analytic functions) and show that they are closed under arithmetic operations and compositions and they are infinitely often differentiable with the derivative functions of all orders being $\mathcal{R}$-analytic themselves.

In [**21**], we focus on the proof of the intermediate value theorem for the $\mathcal{R}$-analytic functions. Given a function f that is $\mathcal{R}$-analytic on an interval $[a, b]$ and a value S between $f(a)$ and $f(b)$, we use iteration to construct a sequence of numbers in $[a, b]$ that converges strongly to a point $c \in [a, b]$ such that $f(c) = S$. The proof is quite involved, making use of many of the results proved in [**17**, **19**] as well as some results from Real Analysis.

In [**20**] we generalize the results in [**17**, **19**, **21**] to power series with rational exponents over $\mathcal{R}$, where the exponents occurring in the series form a left-finite subset of $\mathbb{Q}$.

Finally, in [**14**], we state and prove necessary and sufficient conditions for the existence of relative extrema. Then we use that as well as the intermediate value theorem and its proof to prove the extreme value theorem, the mean value theorem, and the inverse function theorem for functions that are $\mathcal{R}$-analytic on an interval $[a, b]$, thus showing that such functions behave as nicely as real analytic functions.

The convergence properties of power series in $\mathcal{R}$ (resp. $\mathcal{C}$) are used to extend real (resp. complex) power series to $\mathcal{R}$ (resp. $\mathcal{C}$); then that, together with the existence of infinitely small numbers (such as d) in $\mathcal{R}$ and the fact that the $\mathcal{R}$ numbers and calculus can be implemented on a computer, can be used in computational applications such as the fast and accurate (up to machine precision) of the derivatives of real-valued functions representable on a computer, whenever the derivatives exist [**16**].

Furthermore, the nice smoothness properties of power series on $\mathcal{R}$ and $\mathcal{C}$ allow for their use as the building blocks (simple functions) for measurable functions in a Lebesgue-like measure and integration theory on $\mathcal{R}$ [**15**, **18**], which can be easily extended to $\mathcal{C}$. It can be shown that power series form the smallest algebra of functions needed for that theory in order to get all the results of [**15**, **18**].

2. Review of Key Results about Power Series and $\mathcal{R}$-Analytic Functions

We start this section with a brief review of the convergence of sequences in two different topologies; and we refer the reader to [**17**] for a more detailed study.

Definition 2.1. A sequence (s_n) in $\mathcal{R}$ or $\mathcal{C}$ is called regular if the union of the supports of all members of the sequence is a left-finite subset of $\mathbb{Q}$.

Definition 2.2. We say that a sequence (s_n) converges strongly in $\mathcal{R}$ or $\mathcal{C}$ if it converges in the valuation topology.

It is shown in [**2**] that the fields $\mathcal{R}$ and $\mathcal{C}$ are complete with respect to the valuation topology; and a detailed study of strong convergence can be found in [**12**, **17**].

Since power series with real (complex) coefficients do not converge strongly for any nonzero real (complex) argument, it is advantageous to study a new kind of convergence. We do that by defining a family of semi-norms on $\mathcal{R}$ or $\mathcal{C}$, which induces a topology weaker than the topology induced by the absolute value and called weak topology [**3**, **12**, **13**, **17**].

Definition 2.3. Given $r \in \mathbb{R}$, we define a mapping $\|\cdot\|_r : \mathcal{R}$ or $\mathcal{C} \to \mathbb{R}$ as follows: $\|x\|_r = \max\{|x[q]| : q \in \mathbb{Q} \text{ and } q \leq r\}$.

The maximum in Definition 2.3 exists in $\mathbb{R}$ since, for any $r \in \mathbb{R}$, only finitely many of the $x[q]$'s considered do not vanish.

Definition 2.4. A sequence (s_n) in $\mathcal{R}$ (resp. $\mathcal{C}$) is said to be weakly convergent if there exists $s \in \mathcal{R}$ (resp. $\mathcal{C}$), called the weak limit of the sequence (s_n), such that for all $\epsilon > 0$ in $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $\|s_m - s\|_{1/\epsilon} < \epsilon$ for all $m \geq N$.

It is shown [**3**] that $\mathcal{R}$ and $\mathcal{C}$ are not Cauchy complete with respect to the weak topology and that strong convergence implies weak convergence to the same limit.

2.1. Power Series.

In the following, we review strong and weak convergence criteria for power series, Theorem 2.5 and Theorem 2.6, the proofs of which are given in [**17**]. We also note that Theorem 2.5 is a special case of the result on page 59 of [**11**].

Theorem 2.5. *(Strong Convergence Criterion for Power Series) Let (a_n) be a sequence in $\mathcal{R}$ (resp. $\mathcal{C}$), and let*

$$\lambda_0 = \limsup_{n\to\infty} \left(\frac{-\lambda(a_n)}{n} \right) \quad \text{in } \mathbb{R} \cup \{-\infty, \infty\}.$$

Let $x_0 \in \mathcal{R}$ (resp. $\mathcal{C}$) be fixed and let $x \in \mathcal{R}$ (resp. $\mathcal{C}$) be given. Then the power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges strongly if $\lambda(x-x_0) > \lambda_0$ and is strongly divergent if $\lambda(x-x_0) < \lambda_0$ or if $\lambda(x-x_0) = \lambda_0$ and $-\lambda(a_n)/n > \lambda_0$ for infinitely many n.

THEOREM 2.6. *(Weak Convergence Criterion for Power Series) Let (a_n) be a sequence in $\mathcal{R}$ (resp. $\mathcal{C}$), and let $\lambda_0 = \limsup_{n\to\infty}(-\lambda(a_n)/n) \in \mathbb{Q}$. Let $x_0 \in \mathcal{R}$ (resp. $\mathcal{C}$) be fixed, and let $x \in \mathcal{R}$ (resp. $\mathcal{C}$) be such that $\lambda(x-x_0) = \lambda_0$. For each $n \geq 0$, let $b_n = a_n d^{n\lambda_0}$. Suppose that the sequence (b_n) is regular and write $\bigcup_{n=0}^{\infty} supp(b_n) = \{q_1, q_2, \ldots\}$; with $q_{j_1} < q_{j_2}$ if $j_1 < j_2$. For each n, write $b_n = \sum_{j=1}^{\infty} b_{n_j} d^{q_j}$, where $b_{n_j} = b_n[q_j]$. Let*

$$\eta = \frac{1}{\sup\left\{\limsup_{n\to\infty} |b_{n_j}|^{1/n} : j \geq 1\right\}} \text{ in } \mathbb{R} \cup \{\infty\}, \tag{2.1}$$

with the conventions $1/0 = \infty$ and $1/\infty = 0$. Then $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges absolutely weakly if $|(x-x_0)[\lambda_0]| < \eta$ and is weakly divergent if $|(x-x_0)[\lambda_0]| > \eta$.

REMARK 2.7. The number η in Equation (2.1) is referred to as the radius of weak convergence of the power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$.

As an immediate consequence of Theorem 2.6, we obtain the following result which allows us to extend real and complex functions representable by power series to the Levi-Civita fields $\mathcal{R}$ and $\mathcal{C}$. This result is of particular interest for the application [**16**] mentioned in the Introduction above.

COROLLARY 2.8. *(Power Series with Purely Real or Complex Coefficients) Let $\sum_{n=0}^{\infty} a_n X^n$ be a power series with purely real (resp. complex) coefficients and with classical radius of convergence equal to η. Let $x \in \mathcal{R}$ (resp. $\mathcal{C}$), and let $A_n(x) = \sum_{j=0}^{n} a_j x^j \in \mathcal{R}$ (resp. $\mathcal{C}$). Then, for $|x|_o < \eta$ and $|x|_o \not\approx \eta$, the sequence $(A_n(x))$ converges absolutely weakly. We define the limit to be the continuation of the power series to $\mathcal{R}$ (resp. $\mathcal{C}$).*

DEFINITION 2.9 (The Functions Exp, Cos, Sin, Cosh, and Sinh). By Corollary 2.8, the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}, \ \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \ \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \ \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \text{ and } \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

converge absolutely weakly in $\mathcal{R}$ (resp. $\mathcal{C}$) for any $x \in \mathcal{R}$ (resp. $\mathcal{C}$), at most finite in (ordinary) absolute value (that is, for $\lambda(x) \geq 0$). For any such x, define

$$\begin{aligned} \exp(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n!}; \\ \cos(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}; \\ \sin(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}; \\ \cosh(x) &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}; \\ \sinh(x) &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}. \end{aligned}$$

A detailed study of the transcendental functions introduced on $\mathcal{R}$ in Definition 2.9 can be found in [**12**]. In particular, we show that addition theorems similar to the real ones hold, which is essential for the implementation of these functions on a computer (see Section 1.5 in [**12**]).

2.2. $\mathcal{R}$-Analytic Functions. In this section, we review the algebraic and analytical properties of a class of functions that are given locally by power series and we refer the reader to [**14**, **19**, **21**] for a more detailed study.

DEFINITION 2.10. Let $a < b$ in $\mathcal{R}$ be given and let $f : [a, b] \to \mathcal{R}$. Then we say that f is expandable or $\mathcal{R}$-analytic on $[a, b]$ if for all $x \in [a, b]$ there exists a positive $\delta \sim b - a$ in $\mathcal{R}$, and there exists a regular sequence $(a_n(x))$ in $\mathcal{R}$ such that, under weak convergence, $f(y) = \sum_{n=0}^{\infty} a_n(x)(y - x)^n$ for all $y \in (x - \delta, x + \delta) \cap [a, b]$.

It is shown in [**19**] that if f is $\mathcal{R}$-analytic on $[a, b]$ then f is bounded on $[a, b]$; also, if g is $\mathcal{R}$-analytic on $[a, b]$ and $\alpha \in \mathcal{R}$ then $f + \alpha g$ and $f \cdot g$ are $\mathcal{R}$-analytic on $[a, b]$. Moreover, the composition of $\mathcal{R}$-analytic functions is $\mathcal{R}$-analytic. Furthermore, using the fact that power series on $\mathcal{R}$ are infinitely often differentiable within their domain of convergence and the derivatives to any order are obtained by differentiating the power series term by term [**19**], we obtain the following result.

THEOREM 2.11. *Let $a < b$ in $\mathcal{R}$ be given, and let $f : [a, b] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[a, b]$. Then f is infinitely often differentiable on $[a, b]$, and for any positive integer m, we have that $f^{(m)}$ is $\mathcal{R}$-analytic on $[a, b]$. Moreover, if f is given locally around $x_0 \in [a, b]$ by $f(x) = \sum_{n=0}^{\infty} a_n(x_0)(x - x_0)^n$, then $f^{(m)}$ is given by*

$$f^{(m)}(x) = \sum_{n=m}^{\infty} n(n-1)\cdots(n-m+1)\, a_n(x_0)(x - x_0)^{n-m}.$$

In particular, we have that $a_m(x_0) = f^{(m)}(x_0)/m!$ for all $m = 0, 1, 2, \ldots$.

In [**21**], we prove the intermediate value theorem for $\mathcal{R}$-analytic functions on an interval $[a, b]$.

THEOREM 2.12. *(Intermediate Value Theorem) Let $a < b$ in $\mathcal{R}$ be given and let $f : [a, b] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[a, b]$. Then f assumes on $[a, b]$ every intermediate value between $f(a)$ and $f(b)$.*

Since Theorem 2.12 is a central result in the study of power series and $\mathcal{R}$-analytic functions, we present in the following the key steps of the proof and refer the reader to [**21**] for the detailed (lengthy) proof.

- Without loss of generality, we may assume that f is not constant on $[a, b]$. Let $F : [0, 1] \to \mathcal{R}$ be given by
$$F(x) = f((b-a)x + a) - \frac{f(a) + f(b)}{2}.$$
Then F is $\mathcal{R}$-analytic on $[0, 1]$; and f assumes on $[a, b]$ every intermediate value between $f(a)$ and $f(b)$ if and only if F assumes on $[0, 1]$ every intermediate value between $F(0) = (f(a) - f(b))/2$ and $F(1) = (f(b) - f(a))/2 = -F(0)$. So without loss of generality, we may assume that $a = 0$, $b = 1$, and $f = F$. Also, since scaling the function by a constant factor does not affect the existence of intermediate values, we may assume that
$$i(f) := \min \{\mathrm{supp}(f(x)) : x \in [0, 1]\} = 0.$$
- We define $f_R : [0, 1] \cap \mathbb{R} \to \mathbb{R}$ by $f_R(X) = f(X)[0]$. Then f_R is a real-valued analytic function on the real interval $[0, 1] \cap \mathbb{R}$. Let S be between $f(a) = f(0)$ and $f(b) = f(1)$; and let $S_R = S[0]$. Then S_R is a real value between $f_R(0)$ and $f_R(1)$. We use the classical intermediate value theorem to find a real point $X_0 \in [0, 1]$ such that $f_R(X_0) = S_R$.
- We use iteration to construct a convergent sequence (x_n) such that $\lambda(x_n) > 0$ and $\lambda(x_{n+2} - x_{n+1}) > \lambda(x_{n+1} - x_n)$ for all $n \in \mathbb{N}$. Let $x = \lim_{n\to\infty} x_n$; then $\lambda(x) > 0$, and we show that
$$X_0 + x \in [0, 1] \text{ and } f(X_0 + x) = S.$$

A close look at that proof shows that if f is not constant on $[a, b]$ and S is between $f(a)$ and $f(b)$ then there are only finitely many points c in $[a, b]$ such that $f(c) = S$. This is crucial for the proof of the extreme value theorem for the $\mathcal{R}$-analytic functions in [**14**].

In [**14**], we complete the study of $\mathcal{R}$-analytic functions: we state and prove necessary and sufficient conditions for the existence of relative extrema; then we prove the extreme value theorem, the mean value theorem and the inverse function theorem for these functions, thus showing that $\mathcal{R}$-analytic functions have all the nice properties of real analytic functions.

THEOREM 2.13. *Let $a < b$ in $\mathcal{R}$ be given; let $f : [a, b] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[a, b]$; let $x_0 \in (a, b)$ and let $m \in \mathbb{N}$ be the order of the first nonvanishing derivative of f at x_0. Then f has a relative extremum at x_0 if and only if m is even. In that case (m is even), the extremum is a minimum if $f^{(m)}(x_0) > 0$ and a maximum if $f^{(m)}(x_0) < 0$.*

THEOREM 2.14. *(Extreme Value Theorem) Let $a < b$ in $\mathcal{R}$ be given and let $f : [a, b] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[a, b]$. Then f assumes a maximum and a minimum on $[a, b]$.*

Using the intermediate value theorem and the extreme value theorem, then the following results become easy to prove.

Corollary 2.15. *Let $a < b$ in $\mathcal{R}$ be given and let $f : [a, b] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[a, b]$. Then there exist $m, M \in \mathcal{R}$ such that $f([a, b]) = [m, M]$.*

Corollary 2.16. *(Mean Value Theorem) Let $a < b$ in $\mathcal{R}$ be given and let $f : [a, b] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[a, b]$. Then there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Corollary 2.17. *Let $a < b$ in $\mathcal{R}$ be given, and let $f : [a, b] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[a, b]$. Then the following are true.*

(i) *If $f'(x) \neq 0$ for all $x \in (a, b)$ then either $f'(x) > 0$ for all $x \in (a, b)$ and f is strictly increasing on $[a, b]$, or $f'(x) < 0$ for all $x \in (a, b)$ and f is strictly decreasing on $[a, b]$.*

(ii) *If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.*

Corollary 2.18. *(Inverse Function Theorem) Let $a < b$ in $\mathcal{R}$ be given, let $f : [a, b] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[a, b]$, and let $x_0 \in (a, b)$ be such that $f'(x_0) > 0$ (resp. $f'(x_0) < 0$). Then there exists $\delta > 0$ in $\mathcal{R}$ such that*

(i) *$f' > 0$ and f is strictly increasing (resp. $f' < 0$ and f is strictly decreasing) on $[x_0 - \delta, x_0 + \delta]$.*

(ii) *$f([x_0 - \delta, x_0 + \delta]) = [m, M]$ where $m = f(x_0 - \delta)$ and $M = f(x_0 + \delta)$ (resp. $m = f(x_0 + \delta)$ and $M = f(x_0 - \delta)$).*

(iii) *$\exists g : [m, M] \to [x_0 - \delta, x_0 + \delta]$, strictly increasing (resp. strictly decreasing) on $[m, M]$, such that*
- *g is the inverse of f on $[x_0 - \delta, x_0 + \delta]$;*
- *g is differentiable on $[m, M]$; and for all $y \in [m, M]$,*

$$g'(y) = \frac{1}{f'(g(y))}.$$

Remark 2.19. Since power series over $\mathcal{R}$ are $\mathcal{R}$-analytic on any interval within their domain of convergence, all the results of Section 2.2 hold as well for power series on any interval in which the series converges.

2.3. Lebesgue Integration on $\mathcal{R}$ using Power Series. Using the nice smoothness properties of power series summarized above, we developed a Lebesgue-like measure and integration theory on $\mathcal{R}$ in [**15, 18**] that uses the power series as the family of simple functions instead of step functions as in the real case. This was possible in particular because the family $\mathcal{S}(a, b)$ of power series (that converge weakly) on a given interval $I(a, b) \subset \mathcal{R}$ (where $I(a, b)$ denotes any one of the intervals $[a, b]$, $(a, b]$, $[a, b)$ or (a, b)) satisfies the following crucial properties.

(1) $\mathcal{S}(a, b)$ is an algebra that contains the identity function;
(2) for all $f \in \mathcal{S}(a, b)$, f is Lipschitz on $I(a, b)$ and there exists an anti-derivative F of f in $\mathcal{S}(a, b)$, which is unique up to a constant;
(3) for all differentiable $f \in \mathcal{S}(a, b)$, if $f' = 0$ on (a, b) then f is constant on $I(a, b)$; moreover, if $f' \geq 0$ on (a, b) then f is increasing on $I(a, b)$.

Definition 2.20. Let $A \subset \mathcal{R}$ be given. Then we say that A is measurable if for every $\epsilon > 0$ in $\mathcal{R}$, there exist a sequence of mutually disjoint intervals (I_n) and a sequence of mutually disjoint intervals (J_n) such that $\cup_{n=1}^{\infty} I_n \subset A \subset \cup_{n=1}^{\infty} J_n$, $\sum_{n=1}^{\infty} l(I_n)$ and $\sum_{n=1}^{\infty} l(J_n)$ converge in $\mathcal{R}$, and $\sum_{n=1}^{\infty} l(J_n) - \sum_{n=1}^{\infty} l(I_n) \le \epsilon$.

Given a measurable set A, then for every $k \in \mathbb{N}$, we can select a sequence of mutually disjoint intervals $\left(I_n^k\right)$ and a sequence of mutually disjoint intervals $\left(J_n^k\right)$ such that $\sum_{n=1}^{\infty} l\left(I_n^k\right)$ and $\sum_{n=1}^{\infty} l\left(J_n^k\right)$ converge in $\mathcal{R}$ for all k,

$$\cup_{n=1}^{\infty} I_n^k \subset \cup_{n=1}^{\infty} I_n^{k+1} \subset A \subset \cup_{n=1}^{\infty} J_n^{k+1} \subset \cup_{n=1}^{\infty} J_n^k \text{ and } \sum_{n=1}^{\infty} l\left(J_n^k\right) - \sum_{n=1}^{\infty} l\left(I_n^k\right) \le d^k$$

for all $k \in \mathbb{N}$. Since $\mathcal{R}$ is Cauchy-complete in the order topology, it follows that $\lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(I_n^k\right)$ and $\lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(J_n^k\right)$ both exist and they are equal. We call the common value of the limits the measure of A and we denote it by $m(A)$. Thus,

$$m(A) = \lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(I_n^k\right) = \lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(J_n^k\right).$$

We prove in [**18**] that the measure defined above has similar properties to those of the Lebesgue measure of Real Analysis. Then we define a measurable function on a measurable set $A \subset \mathcal{R}$ using Definition 2.20 and simple functions (convergent power series).

Definition 2.21. Let $A \subset \mathcal{R}$ be a measurable subset of $\mathcal{R}$ and let $f : A \to \mathcal{R}$ be bounded on A. Then we say that f is measurable on A if for all $\epsilon > 0$ in $\mathcal{R}$, there exists a sequence of mutually disjoint intervals (I_n) such that $I_n \subset A$ for all n, $\sum_{n=1}^{\infty} l\left(I_n\right)$ converges in $\mathcal{R}$, $m(A) - \sum_{n=1}^{\infty} l(I_n) \le \epsilon$ and f is simple on I_n for all n.

In [**18**], we derive a simple characterization of measurable functions and we show that they form an algebra. Then we show that a measurable function is differentiable almost everywhere and that a function measurable on two measurable subsets of $\mathcal{R}$ is also measurable on their union and intersection.

We define the integral of a simple function over an interval $I(a,b)$ and we use that to define the integral of a measurable function f over a measurable set A.

Definition 2.22. Let $a < b$ in $\mathcal{R}$, let $f : I(a,b) \to \mathcal{R}$ be simple on $I(a,b)$, and let F be a simple anti-derivative of f on $I(a,b)$. Then the integral of f over $I(a,b)$ is the $\mathcal{R}$ number

$$\int_{I(a,b)} f = \lim_{x\to b} F(x) - \lim_{x\to a} F(x).$$

The limits in Definition 2.22 account for the case when the interval $I(a,b)$ does not include one or both of the end points; and these limits exist since F is Lipschitz on $I(a,b)$.

Now let $A \subset \mathcal{R}$ be measurable, let $f : A \to \mathcal{R}$ be measurable and let M be a bound for $|f|_0$ on A. Then for every $k \in \mathbb{N}$, there exists a sequence of mutually disjoint intervals $\left(I_n^k\right)_{n\in\mathbb{N}}$ such that $\cup_{n=1}^{\infty} I_n^k \subset A$, $\sum_{n=1}^{\infty} l\left(I_n^k\right)$ converges, $m(A) - \sum_{n=1}^{\infty} l\left(I_n^k\right) \le d^k$, and f is simple on I_n^k for all $n \in \mathbb{N}$. Without loss of generality, we may assume that $I_n^k \subset I_n^{k+1}$ for all $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$. Since $\lim_{n\to\infty} l\left(I_n^k\right) = 0$, and since $\left|\int_{I_n^k} f\right|_o \le Ml\left(I_n^k\right)$ (proved in [**18**] for simple functions), it follows that

$$\lim_{n\to\infty}\int_{I_n^k} f = 0 \text{ for all } k \in \mathbb{N}.$$

Thus, $\sum_{n=1}^{\infty}\int_{I_n^k} f$ converges in $\mathcal{R}$ for all $k \in \mathbb{N}$ [**17**].

We show that the sequence $\left(\sum_{n=1}^{\infty}\int_{I_n^k} f\right)_{k\in\mathbb{N}}$ converges in $\mathcal{R}$; and we define the unique limit as the integral of f over A.

DEFINITION 2.23. Let $A \subset \mathcal{R}$ be measurable and let $f : A \to \mathcal{R}$ be measurable. Then the integral of f over A, denoted by $\int_A f$, is given by

$$\int_A f = \lim_{\substack{\sum_{n=1}^{\infty} l(I_n) \to m(A) \\ \cup_{n=1}^{\infty} I_n \subset A \\ (I_n) \text{ are mutually disjoint} \\ f \text{ is simple on } I_n \ \forall\ n}} \sum_{n=1}^{\infty}\int_{I_n} f.$$

It turns out that the integral in Definition 2.23 satisfies similar properties to those of the Lebesgue integral on $\mathbb{R}$ [**18**]. In particular, we prove the linearity property of the integral and that if $|f|_o \le M$ on A then $\left|\int_A f\right|_o \le Mm(A)$, where $m(A)$ is the measure of A. We also show that the sum of the integrals of a measurable function over two measurable sets is equal to the sum of its integrals over the union and the intersection of the two sets.

In [**15**], which is a continuation of the work done in [**18**] and complements it, we show, among other results, that the uniform limit of a sequence of convergent power series on an interval $I(a, b)$ is again a power series that converges on $I(a, b)$. Then we use that to prove the uniform convergence theorem in $\mathcal{R}$.

THEOREM 2.24. *Let $A \subset \mathcal{R}$ be measurable, let $f : A \to \mathcal{R}$, for each $k \in \mathbb{N}$ let $f_k : A \to \mathcal{R}$ be measurable on A, and let the sequence (f_k) converge uniformly to f on A. Then f is measurable on A, $\lim_{k\to\infty}\int_A f_k$ exists, and*

$$\lim_{k\to\infty}\int_A f_k = \int_A f.$$

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