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Inner product on B^*-algebras of operators on a free Banach space over the Levi-Civita field

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Abstract

Let $\mathcal{C}$ be the complex Levi-Civita field and let $c_0(\mathcal{C})$ or, simply, c_0 denote the space of all null sequences $z = (z_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{C}$. The natural inner product on c_0 induces the sup-norm of c_0. In a previous paper Aguayo et al. (2013), we presented characterizations of normal projections, adjoint operators and compact operators on c_0. In this paper, we work on some B^*-algebras of operators, including those mentioned above; then we define an inner product on such algebras and prove that this inner product induces the usual norm of operators. We finish the paper with a characterization of closed subspaces of the B^*-algebra of all adjoint and compact operators on c_0 which admit normal complements.

Keywords: Banach spaces over non-Archimedean fields; Inner products; Compact operators; Self-adjoint operators; Positive operators; $B*$-algebras

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1. Introduction

Two of the most useful and beautiful mathematical theories in real or complex functional analysis have been Hilbert spaces and continuous linear operators. These theories have exactly matched the needs of many branches of physics, biology, and other fields of science.

The importance of Hilbert spaces over the real or complex fields has led many researchers to try and extend the concept to non-Archimedean fields. One of the first attempts to define an appropriate non-Archimedean inner product was made by G. K. Kalisch [3]. Two of the most recent papers about non-Archimedean Hilbert spaces are those of L. Narici and E. Beckenstein [4] and the authors [1]. They define a non-Archimedean inner product on a vector space E over a complete non-Archimedean and non-trivially valued field $\mathbb{K}$ as a non-degenerated $\mathbb{K}$-function in $E \times E$, which is linear in the first variable and satisfies what they call the Cauchy–Schwarz type inequality. Recall that a vector space E is said to be orthomodular if for every closed subspace M of E, we have that E is the direct sum of M and its normal complement. The existence of infinite-dimensional non-classical orthomodular spaces was an open question until the following interesting theorem was proved by M. P. Solèr [8]: "Let X be an orthomodular space and suppose it contains an orthonormal sequence $e_1, e_2, \ldots$ (in the sense of the inner product). Then the base field is $\mathbb{R}$ or $\mathbb{C}$". Based on the result of Solèr, if $\mathbb{K}$ is a non-Archimedean, complete valued field and $\mathcal{L}(c_0)$ is the space of all continuous linear operators on c_0, then there exist $T \in \mathcal{L}(c_0)$ which does not have an adjoint. For example, $T(x) = \left(\sum_{i=1}^{\infty} x_i\right) e_1$ is such a linear operator; on the other hand, the normal projections (see the definition below) admit adjoints.

In this paper, we consider the complex Levi-Civita field $\mathcal{C}$ as $\mathbb{K}$; in $\mathcal{C}$, we take the natural involution $z \to \overline{z}$. Recall that a free Banach space E is a non-Archimedean Banach space for which there exists a family $(e_i)_{i \in I}$ in $E \setminus \{\theta\}$ such that any element $x \in E$ can be written in the form of a convergent sum $x = \sum_{i \in I} x_i e_i,\ x_i \in \mathbb{K}$, i.e., $\lim_{i \in I} x_i e_i = 0$ (the limit is with respect to the Fréchet filter on I) and $\|x\| = \sup_{i \in I} |x_i| \, \|e_i\|$. The family $(e_i)_{i \in I}$ is called an orthogonal basis. Now, if E is a free Banach space of countable type over $\mathcal{C}$, then it is known that E is isometrically isomorphic to

$$c_0(\mathbb{N}, \mathcal{C}, s) := \left\{ (x_n)_{n \in \mathbb{N}} : x_n \in \mathcal{C};\ \lim_{n \to \infty} |x_n| s(n) = 0 \right\},$$

where $s : \mathbb{N} \to (0, \infty)$. Of course, it could be that, for some $i \in \mathbb{N}, s(i) \notin |\mathcal{C} \setminus \{0\}|$. But, if the range of s is contained in $|\mathcal{C} \setminus \{0\}|$, it is enough to study $c_0(\mathbb{N}, \mathcal{C})$ (taking s to be the constant function 1), which will be denoted by $c_0(\mathcal{C})$ or, simply, c_0. We already know that c_0 is not orthomodular.

In a previous paper [1], we characterized closed subspaces of c_0 with a normal complement; that is, we characterized those non-trivial closed subspaces M which admit a non-trivial closed subspace N such that

a. $c_0 = M \oplus N$, and
b. for $x \in M$ and $y \in N$, $\langle x, y \rangle = 0$.

N is actually the subspace $M^p = \{y \in c_0 : \langle x, y \rangle = 0 \text{ for all } x \in M\}$ and then $c_0 = M \oplus M^p$. Such a subspace, together with its normal complement, defines a special kind of projection, the so-called normal projection; that is, a linear operator $P : c_0 \to c_0$ such that

i. P is continuous;
ii. $P^2 = P$;
iii. $\langle z, w \rangle = 0$, for all $z \in N(P)$ and for all $w \in R(P)$.

Actually these concepts are not exclusive to c_0; if E is a vector space with an inner product, then "normal complements" and "normal projections" have similar meaning.

In this paper we will study some B^*-subalgebras of $\mathcal{L}(c_0)$ and we will define an inner product such that the usual norm of linear operators will be induced by this inner product.

2. Preliminaries and notations

Throughout this paper $\mathcal{R}$ (resp. $\mathcal{C}$) will denote the real (resp. complex) Levi-Civita field; for a detailed study of $\mathcal{R}$ (and $\mathcal{C}$), we refer the reader to [6,7] and the references therein. Any $z \in \mathcal{C}$ (resp. $\mathcal{R}$) is a function from $\mathbb{Q}$ into $\mathbb{C}$ (resp. $\mathbb{R}$) with left-finite support. For $w \in \mathcal{R}$ (resp. $\mathcal{C}$), we will denote by $\lambda(w) = \min(\operatorname{supp}(w))$, for $w \neq 0$, and $\lambda(0) = +\infty$. On the other hand, since each $z \in \mathcal{C}$ can be written as $z = x + iy$, where $x, y \in \mathcal{R}$, we have that $\lambda(z) = \min\{\lambda(x), \lambda(y)\}$. If we define

$$|z| = \begin{cases} e^{-\lambda(z)} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0, \end{cases}$$

then $|\cdot|$ is a non-Archimedean absolute value in $\mathcal{C}$. It is not hard to prove that $(\mathcal{C}, \Delta)$, where Δ is the metric induced by $|\cdot|$, is a complete metric space. Now let $z = x + iy$ in $\mathcal{C}$ be given. If $x \neq 0 \neq y$ then

$$|z| = e^{-\lambda(z)} = e^{-\min\{\lambda(x),\lambda(y)\}} = \max\left\{e^{-\lambda(x)}, e^{-\lambda(y)}\right\} = \max\{|x|, |y|\}.$$

We can easily also check that $|z| = \max\{|x|, |y|\}$ when $x = 0$ or $y = 0$. Thus,

$$|z| = \max\{|x|, |y|\} \quad \text{for all } z = x + iy \in \mathcal{C}.$$

In other words, $\mathcal{C}$ is topologically isomorphic to $\mathcal{R}^2$ provided with the product topology induced by $|\cdot|$ in $\mathcal{R}$.

We denote by $c_0(\mathcal{C})$, or simply c_0, the space

$$c_0 = \left\{z = (z_n)_{n\in\mathbb{N}} : z_n \in \mathcal{C};\ \lim_{n\to\infty} z_n = 0\right\}.$$

A natural non-Archimedean norm on c_0 is $\|z\|_\infty = \sup\{|z_n| : n \in \mathbb{N}\}$. Writing $z_n = x_n + iy_n$ and $x = (x_n)_{n\in\mathbb{N}},\ y = (y_n)_{n\in\mathbb{N}}$, we also have the equality

$$\|z\|_\infty = \max\left\{\|x\|_\infty, \|y\|_\infty\right\}.$$

It follows that $(c_0, \|\cdot\|_\infty)$ is a Banach space. For a detailed study of non-Archimedean Banach spaces, in general, we refer the reader to [9].

Recall that a topological space is called separable if it has a countable dense subset. In the class of real or complex Hilbert spaces, we can distinguish two types: those spaces which are separable and those which are not separable. If E is a separable normed space over $\mathbb{K}$, then each one-dimensional subspace is homeomorphic to $\mathbb{K}$, so $\mathbb{K}$ must be separable too. Nevertheless, we know that there exist non-Archimedean fields which are not separable, for example, the Levi-Civita fields $\mathcal{R}$ and $\mathcal{C}$. Thus, for non-Archimedean normed spaces the concept of separability cannot be used if $\mathbb{K}$ is not separable. However, by linearizing the notion of separability, we obtain a generalization, useful for each non-Archimedean valued field $\mathbb{K}$. A normed space E over $\mathbb{K}$ is said to be *of countable type* if it contains a countable subset whose linear hull is dense in E. An example of a normed space of countable type is $(c_0(\mathbb{K}), \|\cdot\|_\infty)$, for any non-Archimedean valued field $\mathbb{K}$, in particular, when $\mathbb{K}$ is the complex Levi-Civita field $\mathcal{C}$.

Let us consider the following form:

$$\langle \cdot, \cdot \rangle : c_0 \times c_0 \to \mathcal{C}; \qquad \langle z, w \rangle = \sum_{n=1}^{\infty} z_n \overline{w_n}.$$

This form is well-defined since $\lim_{n\to\infty} z_n \overline{w_n} = 0$ and, at the same time, $\langle \cdot, \cdot \rangle$ satisfies Definition 2.4.1, p. 38, in [5].

Let

$$\|z\| := \sqrt{|\langle z, z \rangle|}.$$

Then, since $|2| = 1$, $\|\cdot\|$ is a non-Archimedean norm on c_0 (Theorem 2.4.2 (ii) in [5]).

It follows easily that

$$\langle x, y \rangle = 0, \quad \forall y \in c_0 \Rightarrow x = \theta$$

which is referred to as the non-degeneracy condition.

The next theorem was proved in [4] and tells us when the non-Archimedean norm in a Banach space is induced by an inner product.

Theorem 1. *Let $(E, \|\cdot\|)$ be a $\mathbb{K}$-Banach space. Then, if $\|E\| \subset |\mathbb{K}|^{1/2}$ and if every one-dimensional subspace of E admits a normal complement, then E has, at least, an inner product that induces the norm $\|\cdot\|$.*

If $E = c_0$ and $\mathbb{K} = \mathcal{C}$, then the conditions of the theorem above are satisfied. In fact, if $z \in c_0$, $z \neq \theta$, then $\lim_{n\to\infty} z_n = 0$, which implies that there exists $j_o \in \mathbb{N}$ such that

$$\|z\|_\infty = \max\left\{\left|z_j\right| : j \in \mathbb{N}\right\} = \left|z_{j_0}\right| \in |\mathcal{C}|.$$

Now, since $|\mathcal{C}| \subset |\mathcal{C}|^{1/2}$, $\|c_0\| \subset |\mathcal{C}|^{1/2}$. The other condition is guaranteed by Lemma 2.3.19, p. 34 in [5].

It was proved in [1] that $\langle \cdot, \cdot \rangle$ is one of the inner products that induce the $\|\cdot\|_\infty$ norm on c_0. Such a result was guaranteed thanks to the following lemma which will be useful also in this paper.

Lemma 1. *If $\{z_1, z_2, \ldots, z_n\} \subset \mathcal{C}$, then*

$$|z_1\overline{z_1} + z_2\overline{z_2} + \cdots + z_n\overline{z_n}| = \max\{|z_1\overline{z_1}|, |z_2\overline{z_2}|, \ldots, |z_n\overline{z_n}|\}.$$

Definition 1. A subset D of c_0 such that for all $x, y \in D$, $x \neq y \Rightarrow \langle x, y \rangle = 0$, is called a normal family. A countable normal family $\{x_n : n \in \mathbb{N}\}$ of unit vectors is called an orthonormal sequence.

If $A \subset c_0$, then $[A]$ and $cl\,[A]$ will denote the linear and the closed linear span of A, respectively. If M is a subspace of c_0, then M^p will denote the subspace of all $y \in c_0$ such that $\langle y, x \rangle = 0$, for all $x \in M$. Since the definition of the inner product given in [5, p. 38], coincides with the definition of inner product given here, the Gram–Schmidt procedure can be used.

Theorem 2. *If $(z_n)_{n\in\mathbb{N}}$ is a sequence of linearly independent vectors in c_0, then there exists an orthonormal sequence $(y_n)_{n\in\mathbb{N}}$ such that $[\{z_1, \dots, z_n\}] = [\{y_1, \dots, y_n\}]$ for every $n \in \mathbb{N}$.*

Definition 2. A sequence $(z_n)_{n\in\mathbb{N}}$ of non-null vectors of c_0 has the Riemann–Lebesgue Property (RLP) if for all $z \in c_0$,

$$\lim_{n\to\infty} \langle z_n, z\rangle = 0.$$

Obviously, any orthonormal basis of c_0 has this property. The following theorem was proved in [4].

Theorem 3. *If $S \subset c_0$, is a finite orthonormal subset, say $\{z_1, \dots, z_n\}$, or is an orthonormal sequence $(z_n)_{n\in\mathbb{N}}$ which satisfies the RLP, then S can be extended to an orthonormal basis for c_0; that is, there exists a countable orthonormal sequence $(w_n)_{n\in\mathbb{N}}$ (possibly finite) such that $S \cup \{w_n : n \in \mathbb{N}\}$ is an orthonormal basis for c_0.*

Lemma 2. *If $(z_n)_{n\in\mathbb{N}}$ is an orthonormal sequence in c_0, then $(z_n)_{n\in\mathbb{N}}$ is orthogonal in the van Rooij's sense (see [9, p. 57]).*

If E and F are normed spaces over $\mathbb{K}$, then $\mathcal{L}(E, F)$ will be the normed space consisting of all continuous linear maps from E into F. $\mathcal{L}(E, \mathbb{K})$ will be denoted by E' and $\mathcal{L}(E, E)$ will be denoted by $\mathcal{L}(E)$. For a $T \in \mathcal{L}(E, F)$, $N(T)$ and $R(T)$ will denote the kernel and the range of T, respectively. It is well-known that the dual of c_0 is $c_0' \cong l^\infty$.

A linear map T from E into F is said to be compact if, for each $\epsilon > 0$, there exists a continuous linear map of finite-dimensional range S such that $\|T - S\| \le \epsilon$.

Any continuous linear operator $u \in \mathcal{L}(c_0)$ can be identified with the following matrix whose columns converge to 0:

$$[u] = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1j} & \cdots \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots & \alpha_{2j} & \cdots \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \cdots & \alpha_{3j} & \cdots \\ \vdots & & & \ddots & & \\ \alpha_{i1} & \alpha_{i2} & \alpha_{i3} & \cdots & \alpha_{ij} & \cdots \\ \vdots & & & & & \ddots \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \cdots \\ 0 & 0 & 0 & & 0 & \end{pmatrix}.$$

Definition 3. A linear operator $v : c_0 \to c_0$ is said to be an adjoint of a given linear operator $u \in \mathcal{L}(c_0)$ if $\langle u(x), y\rangle = \langle x, v(y)\rangle$, for all $x, y \in c_0$. In that case, we will say that u admits an adjoint v. We will also say that u is self-adjoint if $v = u$.

In [1] we showed that if a continuous linear operator u has an adjoint, then the adjoint is unique and continuous.

Lemma 3. *Let $u \in \mathcal{L}(c_0)$ with associated matrix $\left(\alpha_{i,j}\right)_{i,j\in\mathbb{N}}$. Then, u admits an adjoint operator v if and only if $\lim_{j\to\infty} \alpha_{ij} = 0$, for each $i \in \mathbb{N}$. In terms of matrices, this means* *that*

$$[u] = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1j} & \cdots & \to 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots & \alpha_{2j} & \cdots & \to 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \cdots & \alpha_{3j} & \cdots & \to 0 \\ \vdots & & & \ddots & & & \\ \alpha_{i1} & \alpha_{i2} & \alpha_{i3} & \cdots & \alpha_{ij} & \cdots & \to 0 \\ \vdots & & & & & \ddots & \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \cdots & \\ 0 & 0 & 0 & & 0 & & \end{pmatrix}.$$

In the classical Hilbert space theory, any continuous linear operator admits an adjoint. This is not true in the non-Archimedean case. For example, the operator $u \in \mathcal{L}(c_0)$ given by the matrix:

$$\begin{pmatrix} b & b^2 & b^3 & \cdots & b^j & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & & & & & \ddots \end{pmatrix},$$

with $1 < |b|$, does not admit an adjoint, by Lemma 3.

The following two theorems provide characterizations for normal projections (see the proofs in [1]).

Theorem 4. *Let $P \in \mathcal{L}(c_0)$. If P is a normal projection, then P is self-adjoint. Conversely, if P is self-adjoint and $P^2 = P$ then it is a normal projection.*

Theorem 5. *If $P : c_0 \to c_0$ is a normal projection with $R(P) = \overline{[\{y_1, y_2, \ldots\}]}$, where $\{y_1, y_2, \ldots\}$ is an orthonormal finite subset of c_0 or an orthonormal sequence with the Riemann–Lebesgue Property, then $Px = \sum_{i=1}^{\infty} \frac{\langle x, y_i \rangle}{\langle y_i, y_i \rangle} y_i$.*

The following theorem (proved in [1]) provides a way to construct compact and self-adjoint operators starting from an orthonormal sequence.

Theorem 6. *Let $(y_i)_{i \in \mathbb{N}}$ be an orthonormal sequence in c_0. Then, for any $\lambda = (\lambda_i)_{i \in \mathbb{N}}$ in c_0 such that $\lambda_i \in \mathcal{R}$, the map $T : c_0 \to c_0$ defined by*

$$T(\cdot) = \sum_{i=1}^{\infty} \lambda_i P_i(\cdot),$$

where $P_i(\cdot) = \frac{\langle \cdot, y_i \rangle}{\langle y_i, y_i \rangle} y_i$, is a compact and self-adjoint operator.

The converse is also true, as the following theorem shows.

Theorem 7. *Let $T : c_0 \to c_0$ be a compact, self-adjoint linear operator of infinite dimensional range. Then there exists an element $\lambda = (\lambda_n)_{n \in \mathbb{N}} \in c_0(\mathcal{R})$ and an orthonormal sequence $(y_n)_{n \in \mathbb{N}}$*

in c_0 *such that*

$$T = \sum_{n=1}^{\infty} \lambda_n P_n,$$

where

$$P_n = \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n$$

is a normal projection defined by y_n*.*

The uniqueness of the element $(\lambda_n)_{n\in\mathbb{N}}$ of $c_0(\mathcal{R})$ in Theorem 7 is shown by the following proposition, also proved in [1].

Proposition 1. *Let* $T = \sum_{n=1}^{\infty} \lambda_n \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n$ *be a compact and self-adjoint operator and let* $\mu \neq 0$ *in* $\mathcal{C}$ *be an eigenvalue of* T*. Then* $\mu = \lambda_n$ *for some* n*.*

3. B^*-algebras

Some of the results of this section have already been developed in [2] and our goal here is to connect those results with linear compact operators and linear normal projections.

Recall that each $T \in \mathcal{L}(c_0)$ has an associated matrix $\left(\alpha_{ij}\right)_{i,j\in\mathbb{N}}$ such that

(1) $\sup_{i,j\in\mathbb{N}} \left|\alpha_{ij}\right| < \infty$;
(2) $\lim_{i\to\infty} \alpha_{ij} = 0$, for any $j \in \mathbb{N}$;
(3) $T = \sum_{i,j\in\mathbb{N}} \alpha_{ij} e'_j \otimes e_i$, where $e'_j \otimes e_i : c_0 \to c_0$ is given by $e'_j \otimes e_i(z) = \left\langle z, e_j \right\rangle e_i$;
(4) $\|T\| = \sup_{i,j\in\mathbb{N}} \left|\alpha_{ij}\right| = \sup_{n\in\mathbb{N}} \|T e_n\|$.

If we let

$$\mathcal{M}(c_0) = \left\{ \left(\alpha_{ij}\right)_{i,j\in\mathbb{N}} : \sup_{i,j\in\mathbb{N}} \left|\alpha_{ij}\right| < \infty;\ \forall j \in \mathbb{N}, \lim_{i\to\infty} \alpha_{ij} = 0 \right\},$$

equipped with the natural supremum norm, then $\mathcal{L}(c_0)$ and $\mathcal{M}(c_0)$ are isometrically isomorphic.

A Riesz functional is a continuous linear functional of the form $x \to \langle x, y \rangle$ for some $y \in c_0$. If we take $T \in \mathcal{L}(c_0)$ and $y \in c_0$, then the functional $x \to \langle Tx, y \rangle$ belongs to c'_0; but, in general, this functional is not necessarily a Riesz functional. As an example, take $Tx = \left(\sum_{i=1}^{\infty} x_i\right) e_1$. On the other hand, if $Tx = Id(x) = x$, then $x \to \langle Tx, y \rangle$ is of the Riesz type for any $y \in c_0$.

Lemma 4. *Let* $T \in \mathcal{L}(c_0)$*. Then, for a given* $y \in c_0$*, the following conditions are equivalent:*

(1) *There exists* $y^* \in c_0$ *such that* $\langle Tx, y \rangle = \langle x, y^* \rangle,\ \forall x \in c_0$*.*
(2) $\lim_{n\to\infty} \langle T e_n, y \rangle = 0$.

Proof. (1) ⇒ (2): Assume that there exists $y^* \in c_0$ such that $\langle Tx, y \rangle = \langle x, y^* \rangle,\ \forall x \in c_0$. Then

$$\langle T e_n, y \rangle = \left\langle e_n, y^* \right\rangle = \overline{y_n^*} \underset{n\to\infty}{\to} 0.$$

(2) ⇒ (1): Assume that $\lim_{n\to\infty} \langle T e_n, y \rangle = 0$. Then $(\langle T e_n, y \rangle) \in c_0$. If we define $y^* = \sum_{n=1}^{\infty} \overline{\langle T e_n, y \rangle} e_n$, then for $x \in c_0$

$$\left\langle x, y^* \right\rangle = \left\langle x, \sum_{n=1}^{\infty} \overline{\langle T e_n, y \rangle} e_n \right\rangle = \sum_{n=1}^{\infty} \langle T e_n, y \rangle \langle x, e_n \rangle$$

and

$$\langle Tx, y\rangle = \left\langle T\left(\sum_{n=1}^{\infty} \langle x, e_n\rangle e_n\right), y\right\rangle = \sum_{n=1}^{\infty} \langle x, e_n\rangle \langle Te_n, y\rangle . \quad \square$$

Remark 1. (1) Given $T \in \mathcal{L}(c_0)$, we denote by $D(T^*)$ the following set:

$$\begin{aligned} D\left(T^*\right) &= \left\{y \in c_0 : \lim_{n\to\infty} \langle Te_n, y\rangle = 0\right\} \\ &= \left\{y \in c_0 : \exists y^* \in c_0,\ \ \langle Tx, y\rangle = \left\langle x, y^*\right\rangle \text{ for all } x \in c_0\right\}. \end{aligned}$$

Clearly, $D(T^*)$ is a closed subspace of c_0. Moreover, y^*, the corresponding element associated to $y \in D(T^*)$, is unique.

(2) If T satisfies condition (1) (or (2)) of Lemma 4, then for any arbitrary fixed $y \in c_0$, the functional $x \to \langle Tx, y\rangle$ is a Riesz functional.

Let us denote by $\mathcal{A}_0(c_0)$, or simply $\mathcal{A}_0$, the collection of all $T \in \mathcal{L}(c_0)$ such that $D(T^*) = c_0$. Clearly, $Id \in \mathcal{A}_0$ and $T \notin \mathcal{A}_0$, where $Tx = \left(\sum_{i=1}^{\infty} x_i\right) e_1$; therefore $\mathcal{A}_0 \neq \mathcal{L}(c_0)$. Also, $\mathcal{A}_0$ is a non-commutative Banach algebra with unity. Since for $T \in \mathcal{A}_0$ and for any $y \in D(T^*) = c_0$, y^* is unique, we can define $T^* : c_0 \to c_0$ by $T^*(y) = y^*$. Note that $\langle Tx, y\rangle = \langle x, y^*\rangle = \langle x, T^*(y)\rangle$, for any $x, y \in c_0$. Even more, if $T^*(y) \neq \theta$, then

$$\begin{aligned} \left\|T^*(y)\right\|^2 &= \left|\left\langle T^*(y), T^*(y)\right\rangle\right| = \left|\left\langle T\left(T^*(y)\right), y\right\rangle\right| \\ &\leq \left\|T\left(T^*(y)\right)\right\| \|y\| \leq \|T\| \left\|T^*(y)\right\| \|y\| ; \end{aligned}$$

from which it follows that

$$\left\|T^*(y)\right\| \leq \|T\| \|y\| .$$

Therefore, $T^* \in \mathcal{L}(c_0)$ and it is called the adjoint operator of T.

It is not hard to show that $T \in \mathcal{A}_0$ if and only if $\lim_{j\to\infty} \alpha_{ij} = 0$, for all $i \in \mathbb{N}$, where $[T] = \left(\alpha_{ij}\right)_{i,j\in\mathbb{N}}$ is the associated matrix of $T \in \mathcal{L}(c_0)$ with respect to the canonical base $\{e_n : n \in \mathbb{N}\}$. In other words, $\mathcal{A}_0$ is the collection of all continuous linear operators which admit adjoints. In particular, $\mathcal{A}_0$ contains normal projections.

By Lemma 4, $\mathcal{A}_0$ can be rewritten as

$$\mathcal{A}_0 = \left\{T \in \mathcal{L}(c_0) : \forall y \in c_0,\ \ \lim_{j\to\infty} \left\langle Te_j, y\right\rangle = 0\right\}.$$

Now, if $T \in \mathcal{A}_0,\ T = \sum_{i,j\in\mathbb{N}} \alpha_{ij} e'_j \otimes e_i$, then the sequence $\left(\alpha_{ij}\right)_{j\in\mathbb{N}} = \left(\left\langle Te_j, e_i\right\rangle\right)_{j\in\mathbb{N}} \in c_0$, for each $i \in \mathbb{N}$. In terms of matrices, T admits an adjoint if and only if

$$[T] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & & \alpha_{1j} & \to & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & & \alpha_{2j} & \to & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & & \alpha_{3j} & \to & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \\ \alpha_{i1} & \alpha_{i2} & \alpha_{i3} & & \alpha_{ij} & \to & 0 \\ \downarrow & \downarrow & \downarrow & & \downarrow & \cdots & \\ 0 & 0 & 0 & & 0 & \cdots & \end{bmatrix};$$

in such a case the matrix associated with the adjoint T^* of T is the complex conjugate of the transpose of the matrix associated with T. Thus, if

$$\mathcal{M}_0(c_0) = \left\{ \left(\alpha_{ij}\right)_{i,j\in\mathbb{N}} \in \mathcal{M}(c_0) : \forall i \in \mathbb{N}, \ \lim_{j\to\infty} \alpha_{ij} = 0 \right\},$$

then $\mathcal{A}_0$ and $\mathcal{M}_0(c_0)$ are isometrically isomorphic.

Also,

$$\left\langle T^* x, y\right\rangle = \overline{\langle y, T^* x\rangle} = \overline{\langle Ty, x\rangle} = \langle x, Ty\rangle;$$

that is, $T^* \in \mathcal{A}_0$ and, by uniqueness, $(T^*)^* = T^{**} = T$. Therefore, the map $^* : \mathcal{A}_0 \to \mathcal{A}_0;\ T \to T^*$, is an involution on $\mathcal{A}_0$. Altogether, we say that $\mathcal{A}_0$ is a non-Archimedean B^*-algebra.

For each $a \in c_0$, the linear operator M_a, defined by $M_a x = \sum_{i=1}^{\infty} a_i \langle x, e_i\rangle e_i$, belongs to $\mathcal{A}_0$; moreover,

$$\lim_{n\to\infty} \|M_a e_n\| = \lim_{n\to\infty} \left\| \sum_{i=1}^{\infty} a_i \langle e_n, e_i\rangle e_i \right\| = \lim_{n\to\infty} |a_n| = 0,$$

meanwhile, Id is also an element of $\mathcal{A}_0$, but

$$\lim_{n\to\infty} \|Id(e_n)\| = \lim_{n\to\infty} \|e_n\| = 1.$$

Let us denote by $\mathcal{A}_1(c_0)$, or simply $\mathcal{A}_1$, the collection of all $T \in \mathcal{L}(c_0)$ such that $\lim_{n\to\infty} Te_n = \theta$, i.e.,

$$\mathcal{A}_1 = \left\{ T \in \mathcal{L}(c_0) : \lim_{n\to\infty} Te_n = \theta \right\}.$$

From the fact that

$$|\langle Te_n, y\rangle| \le \|Te_n\| \, \|y\|,$$

we have that $\mathcal{A}_1 \subset \mathcal{A}_0$; but $\mathcal{A}_1 \ne \mathcal{A}_0$ since $Id \notin \mathcal{A}_1$.

The next theorem will identify linear compact operators in $\mathcal{A}_0$. In order to do that, we will use the following result given in [2].

Proposition 2. *A continuous linear operator* $T = \sum_{i,j\in\mathbb{N}} \alpha_{ij} e'_j \otimes e_i$ *is compact if and only if* $\lim_{i\to\infty} \sup_{j\in\mathbb{N}} \left|\alpha_{ij}\right| = 0$.

Theorem 8. $T \in \mathcal{A}_1$ *if and only if* T *is compact and* $T \in \mathcal{A}_0$.

Proof. Suppose first $T \in \mathcal{A}_1$. Now, for each $j \in \mathbb{N}$, we define $T_j : c_0 \to c_0$ by

$$T_j x = \sum_{i=1}^{j} x_i Te_i.$$

Clearly, T_j is a continuous linear operator and its range is finite-dimensional; hence it is compact. Moreover, if $y \in c_0$, then

$$\left\langle T_j e_k, y\right\rangle = \begin{cases} \langle Te_k, y\rangle & \text{if } k \le j \\ 0 & \text{if } k \ge j+1, \end{cases}$$

therefore $T_j \in \mathcal{A}_0$.

Now, since

$$\|Tx - T_j x\| = \left\| \sum_{i=j+1}^{\infty} x_i T e_i \right\| \leq \max_{i \geq j+1} \|x_i T e_i\| = \max_{i \geq j+1} |x_i| \, \|T e_i\| \leq \|x\| \max_{i \geq j+1} \|T e_i\|$$

and since $\lim_{j\to\infty} Te_j = \theta$, we have that (T_j) converges uniformly to T which implies that T is compact.

Conversely, we already know that if $T \in \mathcal{A}_0$, then $T^* \in \mathcal{A}_0$. On the other hand, since $T = \sum_{i,j\in\mathbb{N}} \alpha_{ij} e'_j \otimes e_i$ and T is compact, we have that $\lim_{i\to\infty} \sup_{j\in\mathbb{N}} |\alpha_{ij}| = 0$. Now, since $\sup_{j\in\mathbb{N}} |\alpha_{ij}| = \|T^* e_i\|$, we get that $T^* \in \mathcal{A}_1$. Applying the first part of the proof, we conclude that T^* is also compact. Thus, $\lim_{i\to\infty} \sup_{j\in\mathbb{N}} |\beta_{ij}| = 0$, where $T^* = \sum_{i,j\in\mathbb{N}} \beta_{ij} e'_j \otimes e_i$, with $\beta_{ij} = \overline{\alpha_{ji}}$, which is equivalent to $\lim_{i\to\infty} \sup_{j\in\mathbb{N}} |\alpha_{ji}| = 0$. Using the fact that $\sup_{j\in\mathbb{N}} |\alpha_{ji}| = \|Te_i\|$, we prove that $T \in \mathcal{A}_1$. □

Remark 2. If $T = \sum_{i,j\in\mathbb{N}} \alpha_{ij} e'_j \otimes e_i \in \mathcal{A}_1$, then

(1) $\lim_{i\to\infty} \sup_{j\in\mathbb{N}} |\alpha_{ij}| = 0$, since T is compact.
(2) $\lim_{j\to\infty} |\alpha_{ij}| = 0, \ \forall i \in \mathbb{N}$, since $T \in \mathcal{A}_0$.

Therefore, combining (1) and (2), we have that the double limit of the sequence $(\alpha_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ is 0, that is,

$$\lim_{(i,j)\in\mathbb{N}\times\mathbb{N}} \alpha_{ij} = 0.$$

Conversely, if $\lim_{(i,j)\in\mathbb{N}\times\mathbb{N}} \alpha_{ij} = 0$, then $T = \sum_{i,j\in\mathbb{N}} \alpha_{ij} e'_j \otimes e_i \in \mathcal{A}_1$.

A linear operator $T = \sum_{i,j\in\mathbb{N}} \alpha_{ij} e'_j \otimes e_i \in \mathcal{A}_0$ is said to be self-adjoint if $T = T^*$ or, equivalently, if $\alpha_{ij} = \overline{\alpha_{ji}}$. As a corollary of the previous result we have that any compact and self-adjoint operator T belongs to $\mathcal{A}_1$.

Proposition 3. *$\mathcal{A}_1$ is a closed subalgebra of $\mathcal{A}_0$.*

Proof. Let $T \in \overline{\mathcal{A}_1}$ and let $\epsilon > 0$ in $\mathbb{R}$ be given. Then there exists $S \in \mathcal{A}_1$ such that $\|T - S\| < \epsilon$. Since $S \in \mathcal{A}_1$, there exists $N \in \mathbb{N}$ such that

$$\|Se_n\| < \epsilon \quad \text{for } n \geq N.$$

Therefore, for $n \geq N$, we have that

$$\|Te_n\| \leq \max\{\|(T - S)e_n\|, \|Se_n\|\} \leq \max\{\|T - S\|, \|Se_n\|\} < \epsilon. \quad \square$$

We will denote by $\mathcal{A}_2$ the collection of all self-adjoint and compact operators; that is,

$$\mathcal{A}_2 = \left\{T \in \mathcal{A}_1 : T = T^*\right\}.$$

Note that the operator

$$S(\cdot) = \sum_{i=1}^{\infty} a_i \langle \cdot, e_i \rangle e_i \in \mathcal{A}_2 \Leftrightarrow (a_i)_{i\in\mathbb{N}} \in c_0(\mathcal{R}).$$

Therefore, if $\alpha \in \mathcal{C} \setminus \mathcal{R}$, then $\alpha S \notin \mathcal{A}_2$, which implies $\mathcal{A}_2$ is a proper subset of $\mathcal{A}_1$. Nevertheless, we have the following:

Proposition 4. *$\mathcal{A}_2$ is a closed subset of $\mathcal{A}_1$.*

Proof. Let $T \in \overline{\mathcal{A}_2}$; hence there exists a sequence $\{T_n\}$ in $\mathcal{A}_2$ such that $\lim_{n\to\infty} T_n = T$. Since

$$\langle Tx, y\rangle = \lim_{n\to\infty} \langle T_n x, y\rangle = \lim_{n\to\infty} \langle x, T_n y\rangle = \langle x, Ty\rangle ,$$

T is self-adjoint. That T is compact follows from the fact that the space of compact operators is closed in $\mathcal{L}(c_0)$. □

4. Inner product in $\mathcal{A}_1$

Now, we are ready to define an inner product in $\mathcal{A}_1$. Since $\lim_{n\to\infty} Se_n = \theta$ and since $\lim_{n\to\infty} Te_n = \theta$ for $S, T \in \mathcal{A}_1$, the mapping

$$\langle\cdot,\cdot\rangle : \mathcal{A}_1 \times \mathcal{A}_1 \to \mathcal{C}; \qquad (S, T) \to \langle S, T\rangle = \sum_{i=1}^{\infty} \langle Se_i, Te_i\rangle , \tag{4.1}$$

is well-defined, linear in the first variable and linear conjugate in the second variable. Note that $\langle S, T\rangle = \overline{\langle T, S\rangle}$ for any $S, T \in \mathcal{A}_1$.

It is clear that if $w \in c_0$, $\langle w, w\rangle \in \mathcal{R}$ and then there exists a $z \in \mathcal{C}$ such that $\langle w, w\rangle = z\overline{z}$. Therefore, if $w_1, w_2, \ldots, w_n \in c_0$, then by Lemma 1

$$|\langle w_1, w_1\rangle + \langle w_2, w_2\rangle + \cdots + \langle w_n, w_n\rangle| = \max\{|\langle w_i, w_i\rangle| : i = 1, \ldots, n\} .$$

From this lemma, we can prove that if $\langle S, S\rangle = 0$, then $S \equiv \theta$ and then $\langle\cdot,\cdot\rangle$ is an inner product according to Definition 2.4.1, p. 38, [5]. On the other hand, since $|2| = 1$, we can conclude that $\sqrt{|\langle S, S\rangle|}$ is a norm on $\mathcal{A}_1$ and $|\langle S, T\rangle|^2 \leq |\langle S, S\rangle|\,|\langle T, T\rangle|$. The next proposition shows that the norm $\|\cdot\|$ in $\mathcal{A}_1$ is induced by the above inner product.

Proposition 5. *Let $T \in \mathcal{A}_1$, $T \neq \theta$. Then, $|\langle T, T\rangle| = \|T\|^2$.*

Proof. Since $T \in \mathcal{A}_1$, there exists $N \in \mathbb{N}$ such that

$$i \geq N \Rightarrow |\langle Te_i, Te_i\rangle| = \|Te_i\|^2 < \|T\|^2 .$$

It follows that

$$\begin{aligned}
\left|\sum_{i=N}^{\infty} \langle Te_i, Te_i\rangle\right| &\leq \max\{|\langle Te_i, Te_i\rangle| : i \geq N\} \\
&< \|T\|^2 = \max\{|\langle Te_i, Te_i\rangle| : i \in \mathbb{N}\} \\
&= \max\{|\langle Te_i, Te_i\rangle| : i = 1, \ldots, N-1\} \\
&= \left|\sum_{i=1}^{N-1} \langle Te_i, Te_i\rangle\right| .
\end{aligned}$$

Thus,

$$\begin{aligned}
|\langle T, T\rangle| &= \left|\sum_{i=1}^{\infty} \langle Te_i, Te_i\rangle\right| = \left|\sum_{i=1}^{N-1} \langle Te_i, Te_i\rangle + \sum_{i=N}^{\infty} \langle Te_i, Te_i\rangle\right| \\
&= \left|\sum_{i=1}^{N-1} \langle Te_i, Te_i\rangle\right| = \|T\|^2 . \quad \square
\end{aligned}$$

Theorem 9. *c_0 is isometrically isomorphic to a closed subspace of $\mathcal{A}_1$. Moreover, the restriction of the inner product in $\mathcal{A}_1$ to this closed subspace coincides with the inner product defined in c_0.*

Proof. Let $a = (a_i)_{i\in\mathbb{N}} \in c_0$; hence $M_a : c_0 \to c_0$ defined by $M_a x = \sum_{i=1}^{\infty} a_i \langle x, e_i\rangle e_i$ is an element of $\mathcal{A}_1$, whose corresponding matrix is given by

$$[M_a] = \begin{pmatrix} a_1 & 0 & 0 & 0 & \cdots \\ 0 & a_2 & 0 & 0 & \cdots \\ 0 & 0 & a_3 & 0 & \cdots \\ 0 & 0 & 0 & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is easyto verify that $\|M_a\| = \|a\|$ and $V = \{M_a : a \in c_0\}$ is a closed subspace in $\mathcal{A}_1$. If we define $\Psi : c_0 \to \mathcal{A}_1,\ \Psi(a) = M_a$, then Ψ is a linear isometry.

Finally, if $a = (a_i)_{i\in\mathbb{N}},\ b = (b_i)_{i\in\mathbb{N}} \in c_0$, then

$$\langle M_a, M_b\rangle = \sum_{i=1}^{\infty} \langle M_a e_i, M_b e_i\rangle = \sum_{i=1}^{\infty} a_i \overline{b_i} = \langle a, b\rangle . \quad \square$$

Remark 3. Observe that if $T \in \mathcal{A}_1$, then $\langle \cdot, T\rangle : \mathcal{A}_1 \to \mathcal{C},\ S \to \langle S, T\rangle$ is a continuous linear functional; that is, $\langle \cdot, T\rangle \in \mathcal{A}_1'$. Such functionals are called Riesz functionals. But not all elements of $\mathcal{A}_1'$ are Riesz functionals. For example, consider

$$f : \mathcal{A}_1 \to \mathcal{C}; \qquad T = (\alpha_{ij}) \to f(T) = \sum_{i,j\in\mathbb{N}} \alpha_{ij},$$

then f is well-defined, since $\lim_{j\to\infty} \alpha_{ij} = 0$ for each $i \in \mathbb{N}$ and $\lim_{i\to\infty} \alpha_{ij} = 0$ uniformly in $j \in \mathbb{N}$, (this is true because T is compact and then $\lim_{i\to\infty} \sup_{j\in\mathbb{N}} |\alpha_{ij}| = 0$). f is also linear and continuous, since

$$|f(T)| = \left| \sum_{i,j\in\mathbb{N}} \alpha_{ij} \right| \le \max_{i,j\in\mathbb{N}} |\alpha_{ij}| = \|T\| .$$

We show that $f \neq \langle \cdot, S\rangle$, for all $S \in \mathcal{A}_1$. Suppose there exists $S \in \mathcal{A}_1$ such that $f = \langle \cdot, S\rangle$; since $f\left(e_j' \otimes e_i\right) = 1$, for each $i, j \in \mathbb{N},\ S$ should be non-null.

Now, if $S = \sum_{i,j\in\mathbb{N}} \beta_{ij} e_j' \otimes e_i$, then, among other limits, $\lim_{j\to\infty} \beta_{ij} = 0$. But,

$$1 = f\left(e_j' \otimes e_i\right) = \left\langle e_j' \otimes e_i, S\right\rangle = \overline{\beta_{ij}},$$

implies that $\beta_{ij} = 1$, for any $i, j \in \mathbb{N}$, which is a contradiction, since $S \in \mathcal{A}_1$.

Proposition 6. *Let f be a continuous linear functional. Then f is a Riesz functional if and only if the double sequence $\left(f\left(e_j' \otimes e_i\right)\right)_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ is convergent to 0, that is,*

$$\lim_{(i,j)\in\mathbb{N}\times\mathbb{N}} f\left(e_j' \otimes e_i\right) = 0.$$

Proof. First assume that f is a Riesz functional. Then there exists an $S = \sum_{i,j\in\mathbb{N}} \beta_{ij} e'_j \otimes e_i \in \mathcal{A}_1$ such that $f = \langle \cdot, S\rangle$. Now, for a fixed $(i, j) \in \mathbb{N}\times\mathbb{N}$

$$f\left(e'_j \otimes e_i\right) = \left\langle e'_j \otimes e_i, S\right\rangle = \sum_{k=1}^{\infty} \left\langle e'_j \otimes e_i\,(e_k)\,, S\,(e_k)\right\rangle = \langle e_i, Se_j\rangle = \overline{\beta_{ij}}.$$

Since $S \in \mathcal{A}_1$, we have that $\lim_{(i,j)\in\mathbb{N}\times\mathbb{N}} f\left(e'_j \otimes e_i\right) = 0$.

Conversely, assume that $\lim_{(i,j)\in\mathbb{N}\times\mathbb{N}} f\left(e'_j \otimes e_i\right) = 0$. We set $f\left(e'_j \otimes e_i\right) = \overline{\beta_{ij}}$, then $\lim_{(i,j)\in\mathbb{N}\times\mathbb{N}} \beta_{ij} = \lim_{(i,j)\in\mathbb{N}\times\mathbb{N}} \overline{f\left(e'_j \otimes e_i\right)} = 0$. We define $S = \sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} \beta_{ij} e'_j \otimes e_i$ and we claim that $f = \langle \cdot, S\rangle$. In fact, if $T = \sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} \alpha_{ij} e'_j \otimes e_i \in \mathcal{A}_1$, then

$$f\,(T) = \sum_{i,j\in\mathbb{N}} \alpha_{ij} f\left(e'_j \otimes e_i\right) = \sum_{i,j\in\mathbb{N}} \alpha_{ij}\overline{\beta_{ij}} = \langle T, S\rangle\,. \quad \square$$

Definition 4. Let M be a closed subspace of $\mathcal{A}_1$. We shall say that M admits a normal complement if $\mathcal{A}_1 = M \oplus M^p$, where $M^p = \{S \in \mathcal{A}_1 : \langle S, T\rangle = 0,\ \text{for all } T \in M\}$.

Remark 4. (1) Suppose that f is a non-null Riesz functional and $S \in \mathcal{A}_1,\ S \neq \theta,$ are such that $f = \langle \cdot, S\rangle$. We affirm that if $T \in \mathcal{A}_1$, then $T - \frac{f(T)}{\langle S,S\rangle} S \in N\,(f)\,,\ R\,(f) = [\{S\}] = N\,(f)^p$, where $N\,(f)$ and $R\,(f)$ are the kernel and the range of f. In fact, $\langle V, S\rangle = 0,\ \forall V \in N\,(f)$; hence $R\,(f) = [\{S\}] \subseteq N\,(f)^p$. On the other hand, for $U \in N\,(f)^p$,

$$U = \underbrace{\left(U - \frac{\langle U, S\rangle}{\langle S, S\rangle} S\right)}_{=V} + \frac{\langle U, S\rangle}{\langle S, S\rangle} S \in N\,(f) \oplus N\,(f)^p = \mathcal{A}_1.$$

Now,

$$0 = \langle U, V\rangle = \langle V, V\rangle + \left\langle \frac{\langle U, S\rangle}{\langle S, S\rangle} S, V\right\rangle = \langle V, V\rangle$$

implies that $V = \theta$ and then $U = \frac{\langle U,S\rangle}{\langle S,S\rangle} S \in [\{S\}] = R\,(f)$. Summarizing,

$$\mathcal{A}_1 = N\,(f) \oplus R\,(f) = N\,(f) \oplus N\,(f)^p\,.$$

(2) As in c_0, the Riemann–Lebesgue Property (RLP) in $\mathcal{A}_1$ can be defined as follows: a sequence $(T_n)_{n\in\mathbb{N}}$ in $\mathcal{A}_1$, of nonzero elements, is said to have the Riemann–Lebesgue Property if for any $S \in \mathcal{A}_1$,

$$\langle S, T_n\rangle \to 0 \quad \text{as } n \to \infty.$$

(3) Obviously, $\left(e'_j \otimes e_i\right)_{i,j\in\mathbb{N}}$ has the Riemann–Lebesgue Property. Note that $\left\|e'_j \otimes e_i\right\| = 1$ and $\left\langle e'_j \otimes e_i, e'_t \otimes e_s\right\rangle = 0$ if $(i, j) \neq (s, t)$. A sequence like this is called an orthonormal sequence.

(4) The following results have similar proofs to those of the corresponding ones in [1]:

(a) If $\mathcal{S} \subset \mathcal{A}_1$ is a finite orthonormal set $\{T_1, T_2, \ldots, T_n\}$ or an orthonormal sequence $\{T_n : n \in \mathbb{N}\}$ with the Riemann–Lebesgue Property then $\mathcal{S}$ can be extended to an orthonormal basis for $\mathcal{A}_1$.

(b) If M is finite-dimensional or has an orthonormal basis $\{T_n : n \in \mathbb{N}\}$ with the Riemann–Lebesgue Property then M admits a normal complement; that is, $\mathcal{A}_1 = M \oplus M^p$.

(5) A linear operator $P : \mathcal{A}_1 \to \mathcal{A}_1$ is said to be a Normal Projection if:
(a) $P^2 = P$ (which entails that $\mathcal{A}_1 = N(P) \oplus R(P)$),
(b) P is continuous, and
(c) $\langle S, T \rangle = 0$ for $S \in N(P)$ and $T \in R(P)$.
As a consequence, we have the following results which are similar to those proved in [1] with c_0 instead of $\mathcal{A}_1$ there:
(i) Let M be a closed subspace of $\mathcal{A}_1$ which admits a normal complement. Then there exists a unique normal projection P such that $N(P) = M$.
(ii) If P is a normal projection and $\{S_n : n \in \mathbb{N}\}$ is an orthonormal basis of $N(P)$, then it has the Riemann–Lebesgue Property.

(6) Summarizing, if M be an infinite dimensional closed subspace of $\mathcal{A}_1$, then, the following statements are equivalent:
(a) M has a normal complement.
(b) M has an orthonormal basis with the Riemann–Lebesgue Property.
(c) There exists a normal projection P such that $N(P) = M$.

Remark 5. (1) By Remark 4, the kernel of each Riesz functional admits an orthonormal basis with the Riemann–Lebesgue Property.

(2) $(\mathcal{A}_1, \langle \cdot, \cdot \rangle)$ is not orthomodular. For example, for the closed subspace

$$\mathcal{V} = \left\{ T = \sum_{i,j \in \mathbb{N}} \alpha_{ij} e'_j \otimes e_i \in \mathcal{A}_1 : \sum_{i,j \in \mathbb{N}} \alpha_{ij} = 0 \right\}$$

$\mathcal{V}^p$ is the null subspace. In fact, suppose that $S = \sum_{i,j \in \mathbb{N}} \omega_{ij} e'_j \otimes e_i \in \mathcal{V}^p$, then

$$0 = \langle S, T \rangle = \sum_{i=1}^{\infty} \langle S e_i, T e_i \rangle, \quad \forall T \in \mathcal{V}.$$

In particular, for each $T_{ij} = e'_j \otimes e_i - e'_{j+i} \otimes e_i \in \mathcal{V}$, we have that

$$0 = \langle S, T_{ij} \rangle = \sum_{k=1}^{\infty} \langle S e_k, T_{ij} e_k \rangle = \omega_{ij} - \omega_{ij+i},$$

or equivalently, $\omega_{ij} = \omega_{ij+i}$. Since $S \in \mathcal{A}_1$, it then follows that $\omega_{ij} = 0 \, \forall i, j \in \mathbb{N}$. Therefore, $\mathcal{V}$ does not admit a normal complement.

(3) While $(\mathcal{A}_1, \langle \cdot, \cdot \rangle)$ is not orthomodular, there exist closed subspaces of $\mathcal{A}_1$ which admit normal complements. In the following, we shall give an example of such a closed subspace $\mathcal{S}$.

We have already shown above that c_0 can be embedded in $\mathcal{A}_1$ and its image is $\mathcal{S} = \{M_a : a \in c_0\}$. We claim that $\mathcal{S}$ has a normal complement. In fact, for each $a \in c_0$, we have that $M_a = \sum_{n=1}^{\infty} a_n \langle \cdot, e_n \rangle e_n$; in particular, $M_{e_n} = \langle \cdot, e_n \rangle e_n = e'_n \otimes e_n (\cdot)$; hence

$$M_a = \sum_{n=1}^{\infty} a_n e'_n \otimes e_n (\cdot).$$

Thus, $\mathcal{S} = cl \left[\left\{ e'_n \otimes e_n : n \in \mathbb{N} \right\} \right]$. Note that since $\left\| e'_n \otimes e_n \right\| = 1$ and since

$$\langle e'_n \otimes e_n, e'_m \otimes e_m \rangle = \langle M_{e_n}, M_{e_m} \rangle = \langle e_n, e_m \rangle = 0$$

for $m \neq n$, it follows that $\left\{ e'_n \otimes e_n : n \in \mathbb{N} \right\}$ is an orthonormal basis of $\mathcal{S}$.

Now, let $T \in \mathcal{A}_1$ be given; then $\lim_{n\to\infty} Te_n = \theta$ and hence

$$\left|\left\langle T, M_{e_n}\right\rangle\right| = |\langle Te_n, e_n\rangle| \leq \|Te_n\| \to 0.$$

Thus, $\mathcal{S}$ admits an orthonormal basis with the Riemann–Lebesgue Property. It follows that $\mathcal{A}_1 = \mathcal{S} \oplus \mathcal{S}^p$.

Next, we will explicitly find the subspace $\mathcal{S}^p$. Let

$$\mathcal{T} = \left\{ T = \sum_{i,j\in\mathbb{N}} \omega_{ij} e'_j \otimes e_i \in \mathcal{A}_1 : \omega_{ii} = 0,\ i \in \mathbb{N} \right\};$$

then $\mathcal{T}$ is clearly a subspace of $\mathcal{A}_1$ and it is closed, since if $T \in \overline{\mathcal{T}}$ and $(T_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{T}$ such that $T_n \to T$ then, for each $k \in \mathbb{N}$, $\langle T_n e_k, e_k\rangle \to \langle Te_k, e_k\rangle$ or, equivalently, $\omega^n_{kk} \to \omega_{kk}$ which implies that $\omega_{kk} = 0$.

On the other hand, for $M_a \in \mathcal{S}$ and $T \in \mathcal{T}$, we have that

$$\langle M_a, T\rangle = \sum_{k=1}^{\infty} \langle M_a e_k, Te_k\rangle = \sum_{k=1}^{\infty} a_k \langle e_k, Te_k\rangle = \sum_{k=1}^{\infty} a_k \overline{\omega_{kk}} = 0,$$

and therefore $\mathcal{T} \subseteq \mathcal{S}^p$. The other inclusion follows from the fact that $0 = \left\langle M_{e_k}, T\right\rangle = \overline{\omega_{kk}}$ for $k \in \mathbb{N}$.

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