Contemporary Mathematics
Volume **665**, 2016
http://dx.doi.org/10.1090/conm/665/13304

Measure theory and Lebesgue-like integration in two and three dimensions over the Levi-Civita field

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Abstract. In this paper, we develop the foundations for a Lebesgue-like measure and integration theory over the spaces $\mathcal{R}^2$ and $\mathcal{R}^3$, where $\mathcal{R}$ is the Levi-Civita field. First we review the one-dimensional theory then we extend the results to two and three dimensions. In particular, we introduce a measure on $\mathcal{R}^2$ (resp. on $\mathcal{R}^3$) that has similar properties to those of the Lebesgue measure of Real Analysis. Then we introduce a family of $\mathcal{R}$-valued analytic functions from which we obtain a larger family of measurable functions defined on measurable subsets of $\mathcal{R}^2$ (resp. $\mathcal{R}^3$). We study the properties of measurable functions, we show how to integrate them over measurable subsets of $\mathcal{R}^2$ (resp. $\mathcal{R}^3$), and we show that the resulting integral satisfies similar properties to those of the Lebesgue integral of Real Analysis.

1. Introduction

A Lebesgue-like measure and integration theory on the Levi-Civita spaces $\mathcal{R}^2$ and $\mathcal{R}^3$ will be presented. We recall that the elements of the Levi-Civita field $\mathcal{R}$ and its complex counterpart $\mathcal{C}$ are functions from $\mathbb{Q}$ to $\mathbb{R}$ and $\mathbb{C}$, respectively, with left-finite support (denoted by supp). That is, below every rational number q, there are only finitely many points where the given function does not vanish. For the further discussion, it is convenient to introduce the following terminology.

Definition 1.1. (λ, $\sim$, $\approx$) For $x \neq 0$ in $\mathcal{R}$ or $\mathcal{C}$, we let $\lambda(x) = \min(\text{supp}(x))$, which exists because of the left-finiteness of $\text{supp}(x)$; and we let $\lambda(0) = +\infty$. Moreover, we denote the value of x at $q \in \mathbb{Q}$ with brackets like $x[q]$.

Given $x, y \neq 0$ in $\mathcal{R}$ or $\mathcal{C}$, we say $x \sim y$ if $\lambda(x) = \lambda(y)$; and we say $x \approx y$ if $\lambda(x) = \lambda(y)$ and $x[\lambda(x)] = y[\lambda(y)]$.

At this point, these definitions may feel somewhat arbitrary; but after having introduced an order on $\mathcal{R}$, we will see that λ describes orders of magnitude, the relation $\approx$ corresponds to agreement up to infinitely small relative error, while $\sim$ corresponds to agreement of order of magnitude.

The sets $\mathcal{R}$ and $\mathcal{C}$ are endowed with formal power series multiplication and componentwise addition, which make them into fields [**2**] in which we can isomorphically embed $\mathbb{R}$ and $\mathbb{C}$ (respectively) as subfields via the map $\Pi : \mathbb{R}, \mathbb{C} \to \mathcal{R}, \mathcal{C}$

2010 *Mathematics Subject Classification.* 28E05, 26A42, 26E30, 12J25, 11D88, 46S10.

Key words and phrases. non-Archimedean analysis, Levi-Civita field, power series, measure theory and integration, double and triple integrals.

defined by

$$\Pi(x)[q] = \begin{cases} x & \text{if } q = 0 \\ 0 & \text{else} \end{cases} . \tag{1.1}$$

Definition 1.2. (Order in $\mathcal{R}$) Let $x, y \in \mathcal{R}$ be given. Then we say that $x > y$ (or $y < x$) if $x \neq y$ and $(x-y)[\lambda(x-y)] > 0$; and we say $x \geq y$ (or $y \leq x$) if $x = y$ or $x > y$.

It follows that the relation $\geq$ (or $\leq$) defines a total order on $\mathcal{R}$ which makes it into an ordered field. Note that, given $a < b$ in $\mathcal{R}$, we define the $\mathcal{R}$-interval $[a, b] = \{x \in \mathcal{R} : a \leq x \leq b\}$, with the obvious adjustments in the definitions of the intervals $[a, b[$, $]a, b]$, and $]a, b[$. Moreover, the embedding Π in Equation (1.1) of $\mathbb{R}$ into $\mathcal{R}$ is compatible with the order.

The order leads to the definition of an ordinary absolute value on $\mathcal{R}$:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0; \end{cases}$$

which induces the same topology on $\mathcal{R}$ (called the order topology or valuation topology) as that induced by the ultrametric absolute value:

$$|x|_u = e^{-\lambda(x)},$$

as was shown in [**17**]. Moreover, two corresponding absolute values are defined on $\mathcal{C}$ in the natural way:

$$|x + iy| = \sqrt{x^2 + y^2}; \text{ and } |x + iy|_u = e^{-\lambda(x+iy)} = \max\{|x|_u, |y|_u\}.$$

Thus, $\mathcal{C}$ is topologically isomorphic to $\mathcal{R}^2$ provided with the product topology induced by $|\cdot|$ (or $|\cdot|_u$) in $\mathcal{R}$.

We note in passing here that $|\cdot|_u$ is a non-Archimedean valuation on $\mathcal{R}$ (resp. $\mathcal{C}$); that is, it satisfies the following properties

(1) $|v|_u \geq 0$ for all $v \in \mathcal{R}$ (resp. $v \in \mathcal{C}$) and $|v|_u = 0$ if and only if $v = 0$;
(2) $|vw|_u = |v|_u |w|_u$ for all $v, w \in \mathcal{R}$ (resp. $v, w \in \mathcal{C}$); and
(3) $|v + w|_u \leq \max\{|v|_u, |w|_u\}$ for all $v, w \in \mathcal{R}$ (resp. $v, w \in \mathcal{C}$): the strong triangle inequality.

Thus, $(\mathcal{R}, |\cdot|)$ and $(\mathcal{C}, |\cdot|)$ are non-Archimedean valued fields.

Besides the usual order relations on $\mathcal{R}$, some other notations are convenient.

Definition 1.3. ($\ll, \gg$) Let $x, y \in \mathcal{R}$ be non-negative. We say x is infinitely smaller than y (and write $x \ll y$) if $nx < y$ for all $n \in \mathbb{N}$; we say x is infinitely larger than y (and write $x \gg y$) if $y \ll x$. If $x \ll 1$, we say x is infinitely small; if $x \gg 1$, we say x is infinitely large. Infinitely small numbers are also called infinitesimals or differentials. Infinitely large numbers are also called infinite. Non-negative numbers that are neither infinitely small nor infinitely large are also called finite.

Definition 1.4. (The Number d) Let d be the element of $\mathcal{R}$ given by $d[1] = 1$ and $d[t] = 0$ for $t \neq 1$.

REMARK 1.5. Given $m \in \mathbb{Z}$, then d^m is the positive $\mathcal{R}$-number given by

$$d^m = \begin{cases} \underbrace{dd\cdots d}_{m \text{ times}} & \text{if } m > 0 \\ 1 & \text{if } m = 0 \\ \frac{1}{d^{-m}} & \text{if } m < 0 \end{cases}.$$

Moreover, given a rational number $q = m/n$ (with $n \in \mathbb{N}$ and $m \in \mathbb{Z}$), then d^q is the positive nth root of d^m in $\mathcal{R}$ (that is, $(d^q)^n = d^m$) and it is given by

$$d^q[t] = \begin{cases} 1 & \text{if } t = q \\ 0 & \text{otherwise} \end{cases}.$$

It is easy to check that $d^q \ll 1$ if $q > 0$ and $d^q \gg 1$ if $q < 0$ in $\mathbb{Q}$. Moreover, for all $x \in \mathcal{R}$ (resp. $\mathcal{C}$), the elements of $\mathrm{supp}(x)$ can be arranged in ascending order, say $\mathrm{supp}(x) = \{q_1, q_2, \ldots\}$ with $q_j < q_{j+1}$ for all j; and x can be written as $x = \sum_{j=1}^{\infty} x[q_j]d^{q_j}$, where the series converges in the valuation topology [**2**].

Altogether, it follows that $\mathcal{R}$ (resp. $\mathcal{C}$) is a non-Archimedean field extension of $\mathbb{R}$ (resp. $\mathbb{C}$). For a detailed study of these fields, we refer the reader to the survey paper [**13**] and the references therein. In particular, it is shown that $\mathcal{R}$ and $\mathcal{C}$ are complete with respect to the natural (valuation) topology.

It follows therefore that the fields $\mathcal{R}$ and $\mathcal{C}$ are just special cases of the class of fields discussed in [**7**]. For a general overview of the algebraic properties of formal power series fields in general, we refer the reader to the comprehensive overview by Ribenboim [**6**], and for an overview of the related valuation theory to the books by Krull [**4**], Schikhof [**7**] and Alling [**1**]. A thorough and complete treatment of ordered structures can also be found in [**5**].

Besides being the smallest ordered non-Archimedean field extension of the real numbers that is both complete in the order topology and real closed, the Levi-Civita field $\mathcal{R}$ is of particular interest because of its practical usefulness. Since the supports of the elements of $\mathcal{R}$ are left-finite, it is possible to represent these numbers on a computer [**2**]; and having infinitely small numbers in the field allows for many computational applications. One such application is the computation of derivatives of real functions representable on a computer [**14**], where both the accuracy of formula manipulators and the speed of classical numerical methods are achieved.

2. Measure Theory and Integration on $\mathcal{R}$

Using the nice smoothness properties of power series (see [**10**] and the references therein), we developed a Lebesgue-like measure and integration theory on $\mathcal{R}$ in [**11, 16**] that uses the $\mathcal{R}$-analytic functions (functions given locally by power series-Definition 2.4) as the building blocks for measurable functions instead of the step functions used in the real case. This was possible in particular because the family $\mathcal{S}(a, b)$ of $\mathcal{R}$-analytic functions on a given interval $I(a, b) \subset \mathcal{R}$ (where $I(a, b)$ denotes any one of the intervals $[a, b]$, $]a, b]$, $[a, b[$ or $]a, b[$) satisfies the following crucial properties.

(1) $\mathcal{S}(a, b)$ is an algebra that contains the identity function;

(2) for all $f \in \mathcal{S}(a,b)$, f is Lipschitz on $I(a,b)$ and there exists an anti-derivative F of f in $\mathcal{S}(a,b)$, which is unique up to a constant;
(3) for all differentiable $f \in \mathcal{S}(a,b)$, if $f' = 0$ on $]a,b[$ then f is constant on $I(a,b)$; moreover, if $f' \geq 0$ on $]a,b[$ then f is nondecreasing on $I(a,b)$.

NOTATION 2.1. Let $a < b$ in $\mathcal{R}$ be given. Then by $l(I(a,b))$ we will denote the length of the interval $I(a,b)$, that is

$$l(I(a,b)) = \text{ length of } I(a,b) = b - a.$$

DEFINITION 2.2. Let $A \subset \mathcal{R}$ be given. Then we say that A is measurable if for every $\epsilon > 0$ in $\mathcal{R}$, there exist a sequence of mutually disjoint intervals (I_n) and a sequence of mutually disjoint intervals (J_n) such that $\bigcup\limits_{n=1}^{\infty} I_n \subset A \subset \bigcup\limits_{n=1}^{\infty} J_n$, $\sum\limits_{n=1}^{\infty} l(I_n)$ and $\sum_{n=1}^{\infty} l(J_n)$ converge in $\mathcal{R}$, and $\sum\limits_{n=1}^{\infty} l(J_n) - \sum\limits_{n=1}^{\infty} l(I_n) \leq \epsilon$.

Given a measurable set A, then for every $k \in \mathbb{N}$, we can select a sequence of mutually disjoint intervals $\left(I_n^k\right)$ and a sequence of mutually disjoint intervals $\left(J_n^k\right)$ such that $\sum\limits_{n=1}^{\infty} l\left(I_n^k\right)$ and $\sum\limits_{n=1}^{\infty} l\left(J_n^k\right)$ converge in $\mathcal{R}$ for all k,

$$\bigcup_{n=1}^{\infty} I_n^k \subset \bigcup_{n=1}^{\infty} I_n^{k+1} \subset A \subset \bigcup_{n=1}^{\infty} J_n^{k+1} \subset \bigcup_{n=1}^{\infty} J_n^k \text{ and } \sum_{n=1}^{\infty} l\left(J_n^k\right) - \sum_{n=1}^{\infty} l\left(I_n^k\right) \leq d^k$$

for all $k \in \mathbb{N}$. Since $\mathcal{R}$ is Cauchy-complete in the order topology, it follows that $\lim\limits_{k\to\infty} \sum_{n=1}^{\infty} l\left(I_n^k\right)$ and $\lim\limits_{k\to\infty} \sum\limits_{n=1}^{\infty} l\left(J_n^k\right)$ both exist and they are equal. We call the common value of the limits the measure of A and we denote it by $m(A)$. Thus,

$$m(A) = \lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(I_n^k\right) = \lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(J_n^k\right).$$

Contrary to the real case,

$$\sup\left\{\sum_{n=1}^{\infty} l(I_n) : I_n\text{'s are mutually disjoint intervals and } \bigcup_{n=1}^{\infty} I_n \subset A\right\}$$

and

$$\inf\left\{\sum_{n=1}^{\infty} l(J_n) : J_n\text{'s are mutually disjoint intervals and } A \subset \bigcup_{n=1}^{\infty} J_n\right\}$$

need not exist for a given set $A \subset \mathcal{R}$. However, as shown in [**16**], if A is measurable then both the supremum and infimum exist and they are equal to $m(A)$. This shows that the definition of measurable sets in Definition 2.2 is a natural generalization of that of the Lebesgue measurable sets of real analysis that corrects for the lack of suprema and infima in non-Archimedean ordered fields.

It follows directly from the definition that $m(A) \geq 0$ for any measurable set $A \subset \mathcal{R}$ and that any interval $I(a,b)$ is measurable with measure $m(I(a,b)) = l(I(a,b)) = b - a$. It also follows that if A is a countable union of mutually disjoint intervals $(I_n(a_n,b_n))$ such that $\sum\limits_{n=1}^{\infty} (b_n - a_n)$ converges then A is measurable with

$m(A) = \sum_{n=1}^{\infty}(b_n - a_n)$. Moreover, if $B \subset A \subset \mathcal{R}$ and if A and B are measurable, then $m(B) \leq m(A)$.

In [**16**] we show that the measure defined on $\mathcal{R}$ above has similar properties to those of the Lebesgue measure on $\mathbb{R}$. For example, we show that any subset of a measurable set of measure 0 is itself measurable and has measure 0. We also show that any countable unions of measurable sets whose measures form a null sequence is measurable and the measure of the union is less than or equal to the sum of the measures of the original sets; moreover, the measure of the union is equal to the sum of the measures of the original sets if the latter are mutually disjoint. Furthermore, we show that any finite intersection of measurable sets is also measurable and that the sum of the measures of two measurable sets is equal to the sum of the measures of their union and intersection.

It is worth noting that the complement of a measurable set in a measurable set need not be measurable. For example, $[0,1]$ and $[0,1]\cap\mathbb{Q}$ are both measurable with measures 1 and 0, respectively. However, the complement of $[0,1]\cap\mathbb{Q}$ in $[0,1]$ is not measurable. On the other hand, if $B \subset A \subset \mathcal{R}$ and if A, B and $A\setminus B$ are all measurable, then $m(A) = m(B) + m(A\setminus B)$.

The example of $[0,1]\setminus[0,1]\cap\mathbb{Q}$ above shows that the axiom of choice is not needed here to construct a nonmeasurable set, as there are many simple examples of nonmeasurable sets. Indeed, any uncountable real subset of $\mathcal{R}$, like $[0,1]\cap\mathbb{R}$ for example, is not measurable.

Then we define in [**16**] a measurable function on a measurable set $A \subset \mathcal{R}$ using Definition 2.2 and $\mathcal{R}$-analytic functions (Definition 2.4 below).

Definition 2.3. A sequence $(a_n)_{n=1}^{\infty}$ in $\mathcal{R}$ (or $\mathcal{C}$) is said to be regular if the union of the supports of all members of the sequence is a left-finite subset of $\mathbb{Q}$.

Definition 2.4. Let $a < b$ in $\mathcal{R}$ be given and let $f : I(a,b) \to \mathcal{R}$. Then we say that f is $\mathcal{R}$-analytic (or simply analytic) on $I(a,b)$ if for all $x \in I(a,b)$ there exists a positive $\delta \sim b-a$ in $\mathcal{R}$, and there exists a regular sequence $(a_n(x))_{n=1}^{\infty}$ in $\mathcal{R}$ such that, under weak convergence,

$$f(y) = \sum_{n=0}^{\infty} a_n(x)(y-x)^n \text{ for all } y \in \,]x-\delta, x+\delta[\, \cap I(a,b).$$

Definition 2.5. Let $A \subset \mathcal{R}$ be a measurable subset of $\mathcal{R}$ and let $f : A \to \mathcal{R}$ be bounded on A. Then we say that f is measurable on A if for all $\epsilon > 0$ in $\mathcal{R}$, there exists a sequence of mutually disjoint intervals (I_n) such that $I_n \subset A$ for all n, $\sum_{n=1}^{\infty} l(I_n)$ converges in $\mathcal{R}$, $m(A) - \sum_{n=1}^{\infty} l(I_n) \leq \epsilon$ and f is $\mathcal{R}$-analytic on I_n for all n.

In [**16**], we derive a simple characterization of measurable functions and we show that they form an algebra. Then we show that a measurable function is differentiable almost everywhere and that a function measurable on two measurable subsets of $\mathcal{R}$ is also measurable on their union and intersection.

We define the integral of an $\mathcal{R}$-analytic function over an interval $I(a,b)$ and we use that to define the integral of a measurable function f over a measurable set A.

Definition 2.6. Let $a < b$ in $\mathcal{R}$, let $f : I(a,b) \to \mathcal{R}$ be $\mathcal{R}$-analytic on $I(a,b)$, and let F be an $\mathcal{R}$-analytic anti-derivative of f on $I(a,b)$. Then the integral of f

over $I(a,b)$ is the $\mathcal{R}$ number

$$\int_{I(a,b)} f = \lim_{x\to b} F(x) - \lim_{x\to a} F(x).$$

The limits in Definition 2.6 account for the case when the interval $I(a,b)$ does not include one or both of the end points; and these limits exist since F is Lipschitz on $I(a,b)$ [**8**].

Now let $A \subset \mathcal{R}$ be measurable, let $f : A \to \mathcal{R}$ be measurable and let M be a bound for $|f|$ on A. Then for every $k \in \mathbb{N}$, there exists a sequence of mutually disjoint intervals $\left(I_n^k\right)_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} I_n^k \subset A$, $\sum_{n=1}^{\infty} l\left(I_n^k\right)$ converges, $m(A) - \sum_{n=1}^{\infty} l\left(I_n^k\right) \le d^k$, and f is $\mathcal{R}$-analytic on I_n^k for all $n \in \mathbb{N}$. Without loss of generality, we may assume that $I_n^k \subset I_n^{k+1}$ for all $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$. Since $\lim_{n\to\infty} l\left(I_n^k\right) = 0$, and since $\left|\int_{I_n^k} f\right| \le Ml\left(I_n^k\right)$ (proved in [**16**] for $\mathcal{R}$-analytic functions), it follows that

$$\lim_{n\to\infty} \int_{I_n^k} f = 0 \text{ for all } k \in \mathbb{N}.$$

Thus, $\sum_{n=1}^{\infty} \int_{I_n^k} f$ converges in $\mathcal{R}$ for all $k \in \mathbb{N}$ [**15**].

We show that the sequence $\left(\sum_{n=1}^{\infty} \int_{I_n^k} f\right)_{k=1}^{\infty}$ converges in $\mathcal{R}$; and we define the unique limit as the integral of f over A.

Definition 2.7. Let $A \subset \mathcal{R}$ be measurable and let $f : A \to \mathcal{R}$ be measurable. Then the integral of f over A, denoted by $\int_A f$, is given by

$$\int_A f = \lim_{\substack{\sum_{n=1}^{\infty} l(I_n) \to m(A) \\ \bigcup_{n=1}^{\infty} I_n \subset A \\ I_n'\text{s are mutually disjoint} \\ f \text{ is } \mathcal{R}\text{-analytic on } I_n \ \forall\ n}} \sum_{n=1}^{\infty} \int_{I_n} f.$$

It turns out that the integral in Definition 2.7 satisfies similar properties to those of the Lebesgue integral on $\mathbb{R}$ [**16**]. In particular, we prove the linearity property of the integral and that if $|f| \le M$ on A then $\left|\int_A f\right| \le Mm(A)$, where $m(A)$ is the measure of A. We also show that the sum of the integrals of a measurable function over two measurable sets is equal to the sum of its integrals over the union and the intersection of the two sets.

In [**11**], which is a continuation of the work done in [**16**] and complements it, we show, among other results, that the uniform limit of a sequence of convergent power series on an interval $I(a,b)$ is again a power series that converges on $I(a,b)$. Then we use that to prove the uniform convergence theorem in $\mathcal{R}$.

Theorem 2.8. *Let $A \subset \mathcal{R}$ be measurable, let $f : A \to \mathcal{R}$, for each $k \in \mathbb{N}$ let $f_k : A \to \mathcal{R}$ be measurable on A, and let the sequence (f_k) converge uniformly to f on A. Then f is measurable on A, $\lim_{k\to\infty} \int_A f_k$ exists, and*

$$\lim_{k\to\infty} \int_A f_k = \int_A f.$$

In this paper, we generalize the results of [**11**, **16**] to two and three dimensions. In particular, we define a Lebesgue-like measure on $\mathcal{R}^2$ (resp. $\mathcal{R}^3$). Then we define measurable functions on measurable sets using analytic functions in two (resp. three) variables and show how to integrate those measurable functions using iterated integration. The resulting double (resp. triple) integral satisfies similar properties to those of the single integral in [**11**, **16**] as well as those properties satisfied by the double and triple integrals of real calculus.

3. Measure Theory and Integration on $\mathcal{R}^2$

3.1. Simple regions and Measurable Sets.

Definition 3.1 (Simple Region). Let $G \subset \mathcal{R}^2$. Then we say that G is a simple region if there exist constants $a, b \in \mathcal{R}$ with $a \leq b$ and $\mathcal{R}$-analytic functions $g_1, g_2 : I(a,b) \to \mathcal{R}$ with $g_1 < g_2$ on $I(a,b)$ such that

$$G = \{(x,y) \in \mathcal{R}^2 : y \in I(g_1(x), g_2(x)), x \in I(a,b)\}$$

or

$$G = \{(x,y) \in \mathcal{R}^2 : x \in I(g_1(y), g_2(y)), y \in I(a,b)\}.$$

Definition 3.2 (Area of a Simple Region). Let $G \subset \mathcal{R}^2$ be a simple region. If G is of the form $G = \{(x,y) \in \mathcal{R}^2 : y \in I(g_1(x), g_2(x)), x \in I(a,b)\}$ then we define the area of G, denoted by $a(G)$, as

$$a(G) = \int\limits_{x \in I(a,b)} [g_2(x) - g_1(x)],$$

which is well-defined since $g_1(x)$ and $g_2(x)$ are both analytic on $I(a,b)$ and hence so is $g_2(x) - g_1(x)$. On the other hand, if G is of the form $G = \{(x,y) \in \mathcal{R}^2 : x \in I(g_1(y), g_2(y)), y \in I(a,b)\}$ then

$$a(G) = \int\limits_{y \in I(a,b)} [g_2(y) - g_1(y)].$$

It is a simple exercise to show that the intersection, union and difference of two simple regions in $\mathcal{R}^2$ can each be written as a finite union of mutually disjoint simple regions; we refer the interested reader to [**3**] for the proof of this statement.

Definition 3.3 (Measurable Set). Let $A \subset \mathcal{R}^2$. Then we say that A is measurable if for every $\epsilon > 0$ in $\mathcal{R}$ there exist a sequence of mutually disjoint simple regions $(G_n)_{n=1}^{\infty}$ and a sequence of (mutually disjoint) simple regions $(H_n)_{n=1}^{\infty}$ such that $\bigcup\limits_{n=1}^{\infty} G_n \subset A \subset \bigcup\limits_{n=1}^{\infty} H_n$, $\sum\limits_{n=1}^{\infty} a(G_n)$ and $\sum\limits_{n=1}^{\infty} a(H_n)$ both converge, and $\sum\limits_{n=1}^{\infty} a(H_n) - \sum\limits_{n=1}^{\infty} a(G_n) < \epsilon$.

Remark 3.4. Let $A \subset \mathcal{R}$ be a measurable set. Then for every $k \in \mathbb{N}$ there are two sequences of mutually disjoint simple regions $(G_n^k)_{n=1}^{\infty}$ and $(H_n^k)_{n=1}^{\infty}$ such that $\bigcup\limits_{n=1}^{\infty} G_n^k \subset A \subset \bigcup\limits_{n=1}^{\infty} H_n^k$, $\sum\limits_{n=1}^{\infty} G_n^k$ and $\sum\limits_{n=1}^{\infty} H_n^k$ both converge, and $\sum\limits_{n=1}^{\infty} a(H_n^k) - \sum\limits_{n=1}^{\infty} a(G_n^k) < d^k$.

We show that $\left(\sum_{n=1}^{\infty} a(G_n^k)\right)_{k=1}^{\infty}$ is a Cauchy sequence. Assume not. Then there is a $\eta \in \mathcal{R}$, $\eta > 0$, such that for every $k \in \mathbb{N}$, there exists an $l > k$ such that $\sum_{n=1}^{\infty} a(G_n^l) - \sum_{n=1}^{\infty} a(G_n^k) > \eta$. Fix $k_0 \in \mathbb{N}$ so that $d^{k_0} < \eta$, then for every $k \geqslant k_0$, $\sum_{n=1}^{\infty} a(H_n^k) - \sum_{n=1}^{\infty} a(G_n^k) < d^k \leqslant d^{k_0} < \eta$. Thus, for every $k \geqslant k_0$ in $\mathbb{N}$,

$$\sum_{n=1}^{\infty} a(H_n^k) < \sum_{n=1}^{\infty} a(G_n^k) + \eta. \tag{3.1}$$

However, from above we have that there exists an $l_0 > k_0$ such that $\sum_{n=1}^{\infty} a(G_n^{l_0}) > \sum_{n=1}^{\infty} a(G_n^{k_0}) + \eta$ which implies by equation 3.1 that $\sum_{n=1}^{\infty} a(G_n^{l_0}) > \sum_{n=1}^{\infty} a(H_n^{k_0})$ which is a contradiction because by definition $\bigcup_{n=1}^{\infty} G_n^{l_0} \subset A \subset \bigcup_{n=1}^{\infty} H_n^{k_0}$. Thus, $\left(\sum_{n=1}^{\infty} a(G_n^k)\right)_{k=1}^{\infty}$ is a Cauchy sequence. Using a similar argument, we show that $\left(\sum_{n=1}^{\infty} a(H_n^k)\right)_{k=1}^{\infty}$ is also a Cauchy sequence. Since $\mathcal{R}$ is Cauchy complete it follows that $\lim_{k\to\infty} \sum_{n=1}^{\infty} a(G_n^k)$ and $\lim_{k\to\infty} \sum_{n=1}^{\infty} a(H_n^k)$ both exist; and hence $\lim_{k\to\infty} \left(\sum_{n=1}^{\infty} a\left(H_n^k\right) - \sum_{n=1}^{\infty} a\left(G_n^k\right)\right)$ exists.

For every $k \in \mathbb{N}$, we have that $\bigcup_{n=1}^{\infty} G_n^k \subset \bigcup_{n=1}^{\infty} H_n^k$; it follows that $\sum_{n=1}^{\infty} a(G_n^k) \leqslant \sum_{n=1}^{\infty} a(H_n^k)$; combining this with the fact that for every $k \in \mathbb{N}$, $\sum_{n=1}^{\infty} a(H_n^k) - \sum_{n=1}^{\infty} a(G_n^k) < d^k$, we infer that

$$0 \leqslant \lim_{k\to\infty} \left(\sum_{n=1}^{\infty} a\left(H_n^k\right) - \sum_{n=1}^{\infty} a\left(G_n^k\right)\right) \leqslant \lim_{k\to\infty} d^k = 0$$

and hence

$$\lim_{k\to\infty} \left(\sum_{n=1}^{\infty} a(H_n^k) - \sum_{n=1}^{\infty} a(G_n^k)\right) = 0.$$

It follows that $\lim_{k\to\infty} \sum_{n=1}^{\infty} a(G_n^k) = \lim_{k\to\infty} \sum_{n=1}^{\infty} a(H_n^k)$.

Definition 3.5 (The Measure of a Measurable Set). We define the common limit in Remark 3.4 to be the measure of A and we denote it by $m(A)$. Thus,

$$m(A) = \lim_{k\to\infty} \sum_{n=1}^{\infty} a(G_n^k) = \lim_{k\to\infty} \sum_{n=1}^{\infty} a(H_n^k).$$

PROPOSITION 3.6. *Let $A \subset \mathcal{R}^2$ be a measurable set. Then*

$$m(A) = \inf \left\{ \sum_{n=1}^{\infty} a(H_n) : H_n\text{'s are mutually disjoint simple regions}, A \subset \bigcup_{n=1}^{\infty} H_n, \text{ and } \sum_{n=1}^{\infty} a(H_n) \text{ converges} \right\}$$
$$= \sup \left\{ \sum_{n=1}^{\infty} a(G_n) : G_n\text{'s are mutually disjoint simple regions}, \bigcup_{n=1}^{\infty} G_n \subset A, \text{ and } \sum_{n=1}^{\infty} a(G_n) \text{ converges} \right\}.$$

PROOF. First we show that the infimum exists and is equal to $m(A)$. Since A is a measurable set we know that for every $k \in \mathbb{N}$, there exist two sequences of mutually disjoint simple regions $(G_n^k)_{n=1}^{\infty}$ and $(H_n^k)_{n=1}^{\infty}$ such that

$$\bigcup_{n=1}^{\infty} G_n^k \subset \bigcup_{n=1}^{\infty} G_n^{k+1} \subset A \subset \bigcup_{n=1}^{\infty} H_n^{k+1} \subset \bigcup_{n=1}^{\infty} H_n^k,$$

$\sum_{n=1}^{\infty} a(G_n^k)$ and $\sum_{n=1}^{\infty} a(H_n^k)$ both converge, and $\sum_{n=1}^{\infty} a(H_n^k) - \sum_{n=1}^{\infty} a(G_n^k) < d^k$. By definition,

$$m(A) = \lim_{k \to \infty} \sum_{n=1}^{\infty} a\left(G_n^k\right) = \lim_{k \to \infty} \sum_{n=1}^{\infty} a(H_n^k).$$

Moreover, for every $k \in \mathbb{N}$, we have that

$$\sum_{n=1}^{\infty} a(G_n^k) \leqslant m(A) \leqslant \sum_{n=1}^{\infty} a(H_n^k).$$

It remains to be shown that if $(H_n)_{n=1}^{\infty}$ is a sequence of mutually disjoint simple regions such that $A \subset \bigcup_{n=1}^{\infty} H_n$ and $\sum_{n=1}^{\infty} a(H_n)$ converges, then $\sum_{n=1}^{\infty} a(H_n) \geqslant \lim_{k \to \infty} \sum_{n=1}^{\infty} a(H_n^k) = m(A)$. Suppose not. Then there is a sequence of mutually disjoint simple regions $(H_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} a(H_n)$ converges, $A \subset \bigcup_{n=1}^{\infty} H_n$, and $m(A) > \sum_{n=1}^{\infty} a(H_n)$. Let $k_0 \in \mathbb{N}$ be such that

$$d^{k_0} < \frac{m(A) - \sum_{n=1}^{\infty} a(H_n)}{2}.$$

We have from above that $\bigcup_{n=1}^{\infty} G_n^{k_0} \subset A \subset \bigcup_{n=1}^{\infty} H_n$, and since $(G_n^{k_0})_{n=1}^{\infty}$ and $(H_n)_{n=1}^{\infty}$

are both sequences of mutually disjoint simple regions it follows that $\sum_{n=1}^{\infty} a(G_n^{k_0}) \leqslant \sum_{n=1}^{\infty} a(H_n)$. But $\sum_{n=1}^{\infty} a(H_n^{k_0}) \geqslant m(A)$, and hence $m(A) - \sum_{n=1}^{\infty} a(G_n^{k_0}) \leqslant \sum_{n=1}^{\infty} a(H_n^{k_0}) - \sum_{n=1}^{\infty} a(G_n^{k_0}) < d^{k_0}$. Thus,

$$\begin{aligned}
\sum_{n=1}^{\infty} a(H_n) - \sum_{n=1}^{\infty} a(G_n^{k_0}) &= \left(\sum_{n=1}^{\infty} a(H_n) - m(A)\right) + \left(m(A) - \sum_{n=1}^{\infty} a(G_n^{k_0})\right) \\
&\leqslant \left(\sum_{n=1}^{\infty} a(H_n) - m(A)\right) + d^{k_0} \\
&< \left(\sum_{n=1}^{\infty} a(H_n) - m(A)\right) + \frac{m(A) - \sum_{n=1}^{\infty} a(H_n)}{2} \\
&= \frac{1}{2}\left(\sum_{n=1}^{\infty} a(H_n) - m(A)\right) < 0
\end{aligned}$$

But this contradicts the fact that $\sum_{n=1}^{\infty} a(G_n^{k_0}) \leqslant \sum_{n=1}^{\infty} a(H_n)$, so a sequence such as $(H_n)_{n=1}^{\infty}$ cannot exist.

A similar argument shows that

$$\sup\left\{\sum_{n=1}^{\infty} a(G_n) : G_n\text{'s are mutually disjoint simple regions, } \bigcup_{n=1}^{\infty} G_n \subset A, \sum_{n=1}^{\infty} a(G_n) \text{ converges}\right\}$$

exists and is equal to $m(A)$. □

Proposition 3.7. *Let $A, B \subset \mathcal{R}^2$ be measurable sets with $A \subset B$. Then $m(A) \leq m(B)$.*

Proof. Suppose not. Then $m(A) > m(B)$; let $\eta = m(A) - m(B)$, then $\eta > 0$. Since A is measurable there is a sequence of mutually disjoint simple regions $(G_n)_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} G_n \subset A$, $\sum_{n=1}^{\infty} a(G_n)$ converges, and $m(A) - \sum_{n=1}^{\infty} a(G_n) < \frac{\eta}{4}$. Since B is measurable there is a sequence of mutually disjoint simple regions $(H_n)_{n=1}^{\infty}$ such that $B \subset \bigcup_{n=1}^{\infty} H_n$, $\sum_{n=1}^{\infty} a(H_n)$ converges, and $\sum_{n=1}^{\infty} a(H_n) - m(B) < \frac{\eta}{4}$. It follows that

$$\sum_{n=1}^{\infty} a(H_n) < m(B) + \frac{\eta}{4} < m(A) - \frac{\eta}{4} < \sum_{n=1}^{\infty} a(G_n).$$

However,

$$\bigcup_{n=1}^{\infty} G_n \subset A \subset B \subset \bigcup_{n=1}^{\infty} H_n,$$

so $\sum_{n=1}^{\infty} a(G_n) \leq \sum_{n=1}^{\infty} a(H_n)$ and thus we have reached a contradiction. □

PROPOSITION 3.8. *Let $A \subset \mathcal{R}^2$ be a measurable set with $m(A) = 0$ and let $B \subset A$. Then B is measurable and $m(B) = 0$.*

PROOF. Let $\epsilon > 0$ in $\mathcal{R}$ be given. Since A is measurable and since $m(A) = 0$, then for every $k \in \mathbb{N}$ there exists a sequence of mutually disjoint simple regions $(H_n^k)_{n=1}^\infty$ such that $A \subset \bigcup_{n=1}^\infty H_n^k$, $\sum_{n=1}^\infty a(H_n^k)$ converges, and $\sum_{n=1}^\infty a(H_n^k) - m(A) = \sum_{n=1}^\infty a(H_n^k) < d^k$. For every $n \in \mathbb{N}$, let $G_n = \emptyset$ which is a simple region.

Let $k_0 \in \mathbb{N}$ be such that $d^{k_0} < \epsilon$. Then $\bigcup_{n=1}^\infty G_n \subset B \subset \bigcup_{n=1}^\infty H_n^{k_0}$ and $\sum_{n=1}^\infty a(H_n^{k_0}) - \sum_{n=1}^\infty a(G_n) = \sum_{n=1}^\infty a(H_n^{k_0}) < d^{k_0} < \epsilon$. Hence B is measurable. Since for every $k \in \mathbb{N}$, $B \subset \bigcup_{n=1}^\infty H_n^k$ it follows that $0 \leq m(B) \leqslant \sum_{n=1}^\infty a(H_n^k) < d^k$. Letting $k \to \infty$ we obtain that $m(B) = 0$. □

PROPOSITION 3.9. *Let $A \subset \mathcal{R}^2$ be countable. Then A is measurable and $m(A) = 0$.*

PROOF. Since A is a countable set there is a sequence of points $((x_n, y_n))_{n=1}^\infty$ such that $A = \bigcup_{n=1}^\infty \{(x_n, y_n)\}$. Let $\epsilon > 0$ in $\mathcal{R}$ be given. For every $n \in \mathbb{N}$ let

$$H_n = \left\{(x, y) \in \mathcal{R}^2 : x \in \left]x_n - (d^n\epsilon)^{\frac{1}{2}}, x_n + (d^n\epsilon)^{\frac{1}{2}}\right[, y \in \left]y_n - (d^n\epsilon)^{\frac{1}{2}}, y_n + (d^n\epsilon)^{\frac{1}{2}}\right[\right\}.$$

Then for every $n \in \mathbb{N}$, H_n is a simple region with $a(H_n) = 4d^n\epsilon$. Thus, $\lim_{n\to\infty} a(H_n) = \lim_{n\to\infty} 4d^n\epsilon = 0$, and hence $\sum_{n=1}^\infty a(H_n)$ converges. For every $j \in \mathbb{N}$, let $G_j = \emptyset$. Then $\bigcup_{j=1}^\infty G_j \subset A \subset \bigcup_{i=1}^\infty H_i$, $\sum_{j=1}^\infty a(G_j)$ and $\sum_{i=1}^\infty a(H_i)$ converge, and

$$\sum_{i=1}^\infty a(H_i) - \sum_{j=1}^\infty a(G_j) = \sum_{i=1}^\infty a(H_i) = \sum_{i=1}^\infty 4d^i\epsilon = \frac{4d\epsilon}{1-d} < \epsilon,$$

which proves that A is measurable. Furthermore, since $A \subset \bigcup_{i=1}^\infty H_i$, $m(A) \leqslant \sum_{i=1}^\infty a(H_i) < \epsilon$. Taking the limit as $\epsilon \to 0$ shows that $m(A) = 0$. □

PROPOSITION 3.10. *Let $(H_k)_{k=1}^\infty$ and $(G_n)_{n=1}^\infty$ be sequences of mutually disjoint simple regions such that $\sum_{k=1}^\infty a(H_k)$ and $\sum_{n=1}^\infty a(G_n)$ both converge. Then there exists a sequence of mutually disjoint simple regions $(T_m)_{m=1}^\infty$ such that*

$$\left(\bigcup_{k=1}^\infty H_k\right) \bigcap \left(\bigcup_{n=1}^\infty G_n\right) = \bigcup_{m=1}^\infty T_m$$

and $\sum_{m=1}^\infty a(T_m)$ converges.

PROOF. First note that for every $k, n \in \mathbb{N}$, there is a finite collection of mutually disjoint simple regions $(T_m^{k,n})_{m=1}^{l_{k,n}}$ such that $H_k \cap G_n = \bigcup_{m=1}^{l_{k,n}} T_m^{k,n}$. We show that the collection $(((T_m^{k,n})_{m=1}^{l_{k,n}})_{k=1}^{\infty})_{n=1}^{\infty}$ is mutually disjoint; so consider $k_1, n_1, m_1 \in \mathbb{N}$ and $k_2, n_2, m_2 \in \mathbb{N}$ such that either $k_1 \neq k_2$, $n_1 \neq n_2$ or $m_1 \neq m_2$. Of course if $k_1 = k_2 = k$ and $n_1 = n_2 = n$ then $T_{m_1}^{k_1,n_1}$ and $T_{m_2}^{k_2,n_2}$ are both elements of the set $\left\{T_m^{k,n} : 1 \leq m \leq l_{k,n}\right\}$ and hence they are disjoint. If $n_1 \neq n_2$ then $T_{m_1}^{k_1,n_1} \subset H_{k_1} \cap G_{n_1} \subset G_{n_1}$ and $T_{m_2}^{k_2,n_2} \subset H_{k_2} \cap G_{n_2} \subset G_{n_2}$; and hence $T_{m_1}^{k_1,n_1}$ and $T_{m_2}^{k_2,n_2}$ are disjoint since G_{n_1} and G_{n_2} are. The same argument holds if $k_1 \neq k_2$. Since $(((T_m^{k,n})_{m=1}^{l_{k,n}})_{k=1}^{\infty})_{n=1}^{\infty}$ is a countable collection it may be rewritten as $(T_m)_{m=1}^{\infty}$, and rearrange terms if necessary so that $\sum_{m=1}^{\infty} a(T_m)$ converges. Thus, $(T_m)_{m=1}^{\infty}$ is a collection of mutually disjoint simple regions such that

$$\left(\bigcup_{k=1}^{\infty} H_k\right) \bigcap \left(\bigcup_{n=1}^{\infty} G_n\right) = \bigcup_{m=1}^{\infty} T_m$$

and $\sum_{m=1}^{\infty} a(T_m)$ converges. □

PROPOSITION 3.11. *For every $k \in \mathbb{N}$, let $(G_n^k)_{n=1}^{\infty}$ be a countable sequence of mutually disjoint simple regions such that $\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a(G_n^k)$ converges. Then there exists a collection of mutually disjoint simple regions $(H_m)_{m=1}^{\infty}$ such that*

$$\bigcup_{m=1}^{\infty} H_m = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} G_n^k \text{ and } \sum_{m=1}^{\infty} a(H_m) \text{ converges.}$$

PROOF. First we note that $((G_n^k)_{n=1}^{\infty})_{k=1}^{\infty}$ is a countable collection of simple regions and so may be rewritten as $(H_n^0)_{n=1}^{\infty}$. To obtain the desired sequence $(H_m)_{m=1}^{\infty}$ we begin by defining $H_1 = H_1^0$. Next we observe that, for every $n, j \in \mathbb{N}$, $H_n^0 \backslash H_j^0$ is given by a finite number of mutually disjoint simple regions $(F_i^{n,j})_{i=1}^{t_{n,j}}$. Thus, for every $n \in \mathbb{N}$, we have that

$$H_n^0 \backslash \left(\bigcup_{j=1}^{n-1} H_j^0\right) = \bigcap_{j=1}^{n-1} \left(H_n^0 \backslash H_j^0\right) = \bigcap_{j=1}^{n-1} \bigcup_{i=1}^{t_{n,j}} F_i^{n,j}.$$

However, using the same argument as in the proof of Proposition 3.10, we infer that for every $n \in \mathbb{N}$, $\bigcap_{j=1}^{n-1} \bigcup_{i=1}^{t_{n,j}} F_i^{n,j}$ can be expressed as the union of a finite number of mutually disjoint simple regions $(F_i^n)_{i=1}^{l_n}$.

We define

$$\begin{gathered}
H_2 = F_1^2, \ldots, H_{l_2+1} = F_{l_2}^2 \\
H_{l_2+2} = F_1^3, \ldots, H_{l_2+l_3+1} = F_{l_3}^3 \\
H_{l_2+l_3+2} = F_1^4, \ldots, H_{l_2+l_3+l_4+1} = F_{l_4}^4 \\
\vdots
\end{gathered}$$

Thus, by construction, the H_m's are mutually disjoint and $\lim_{m\to\infty} a(H_m) = 0$, so $\sum_{m=0}^{\infty} a(H_m)$ converges. Moreover,

$$\bigcup_{m=1}^{\infty} H_m = \bigcup_{n=1}^{\infty} H_n^0 = \bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} G_n^k.$$

□

PROPOSITION 3.12. *For each $k \in \mathbb{N}$, let $A_k \subset \mathcal{R}^2$ be measurable, with $\lim_{k\to\infty} m(A_k) = 0$. Then $\bigcup_{k=1}^{\infty} A_k$ is measurable and $m\left(\bigcup_{k=1}^{\infty} A_k\right) \leqslant \sum_{k=1}^{\infty} m(A_k)$. Moreover, if the A_k's are mutually disjoint then $m\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} m(A_k)$.*

PROOF. First note that, since $\lim_{k\to\infty} m(A_k) = 0$, we have that $\sum_{k=1}^{\infty} m(A_k)$ converges. Now let $\epsilon > 0$ in $\mathcal{R}$ be given. Since each A_k is measurable, it follows that, for every $k \in \mathbb{N}$, there are two sequences of mutually disjoint simple regions $(G_n^k)_{n=1}^{\infty}$ and $(H_n^k)_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} G_n^k \subset A_k \subset \bigcup_{n=1}^{\infty} H_n^k$, $\sum_{n=1}^{\infty} a(G_n^k)$ and $\sum_{n=1}^{\infty} a(H_n^k)$ both converge, and $\sum_{n=1}^{\infty} a(H_n^k) - \sum_{n=1}^{\infty} a(G_n^k) < d^k\epsilon$.

Since $\lim_{k\to\infty} m(A_k) = 0$ and since $0 \leq \sum_{n=1}^{\infty} a(G_n^k) \leqslant \sum_{n=1}^{\infty} a(H_n^k) < m(A_k) + d^k\epsilon$, we infer that $\lim_{k\to\infty} \sum_{n=1}^{\infty} a(G_n^k) = \lim_{k\to\infty} \sum_{n=1}^{\infty} a(H_n^k) = 0$. Thus, $\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a(G_n^k)$ and $\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a(H_n^k)$ both converge.

Using the proof of Proposition 3.11, there exist two sequences of mutually disjoint simple regions $(G_n)_{n=1}^{\infty}$ and $(H_n)_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} G_n = \bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} G_n^k$ and $\bigcup_{n=1}^{\infty} H_n = \bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} H_n^k$. Therefore,

$$\left(\bigcup_{n=1}^{\infty} H_n\right) \setminus \left(\bigcup_{n=1}^{\infty} G_n\right) = \left(\bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} H_n^k\right) \setminus \left(\bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} G_n^k\right).$$

For every $k \in \mathbb{N}$ we have that $\bigcup_{n=1}^{\infty} G_n^k \subset \bigcup_{n=1}^{\infty} H_n^k$, and hence

$$\left(\bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} H_n^k\right) \setminus \left(\bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} G_n^k\right) = \bigcup_{k=1}^{\infty}\left(\bigcup_{n=1}^{\infty} H_n^k \setminus \bigcup_{n=1}^{\infty} G_n^k\right).$$

Moreover, since for every $k \in \mathbb{N}$, the sequences $\left(G_n^k\right)_{n=1}^{\infty}$ and $\left(H_n^k\right)_{n=1}^{\infty}$ are both mutually disjoint, we can arrange them in such a way that for every $n \in \mathbb{N}$, $G_n^k \subset H_n^k$. Thus, for every $k \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} H_n^k \setminus \bigcup_{n=1}^{\infty} G_n^k = \bigcup_{n=1}^{\infty} (H_n^k \setminus G_n^k)$. It follows that

$$\left(\bigcup_{n=1}^{\infty} H_n\right) \setminus \left(\bigcup_{n=1}^{\infty} G_n\right) = \bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} (H_n^k \setminus G_n^k).$$

Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty} a(H_n) - \sum_{n=1}^{\infty} a(G_n) = m\left[\left(\bigcup_{n=1}^{\infty} H_n\right) \setminus \left(\bigcup_{n=1}^{\infty} G_n\right)\right] &= m\left(\bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} (H_n^k \setminus G_n^k)\right) \\ &\leqslant \sum_{k=1}^{\infty} m\left(\bigcup_{n=1}^{\infty} (H_n^k \setminus G_n^k)\right) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} m(H_n^k \setminus G_n^k) \\ &= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} (a(H_n^k) - a(G_n^k)) = \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} a(H_n^k) - \sum_{n=1}^{\infty} a(G_n^k)\right) \\ &\leqslant \sum_{k=1}^{\infty} d^k \epsilon = \frac{d}{1-d}\epsilon < \epsilon\end{aligned}$$

which proves that $\bigcup_{k=1}^{\infty} A_k$ is measurable.

Since $\bigcup_{k=1}^{\infty} A_k \subset \bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} H_n^k$ we have that

$$\begin{aligned}m\left(\bigcup_{k=1}^{\infty} A_k\right) \leqslant m\left(\bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} H_n^k\right) &\leqslant \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a(H_n^k) \\ &\leqslant \sum_{k=1}^{\infty}\left(m(A_k) + d^k\epsilon\right) < \sum_{k=1}^{\infty} m(A_k) + \epsilon.\end{aligned}$$

The above holds for any $\epsilon > 0$ in $\mathcal{R}$; and hence $m\left(\bigcup_{k=1}^{\infty} A_k\right) \leqslant \sum_{k=1}^{\infty} m(A_k)$.

Now, assume that the A_k's are mutually disjoint, and let $\epsilon > 0$ in $\mathcal{R}$ be given. There exists a $K \in \mathbb{N}$ such that $\sum_{k>K} m(A_k) < \frac{\epsilon}{2}$. Since $\bigcup_{k=1}^{\infty} A_k$ is measurable there exists a sequence of mutually disjoint simple regions $(H_n)_{n=1}^{\infty}$ such that $\bigcup_{k=1}^{\infty} A_k \subset \bigcup_{n=1}^{\infty} H_n$, $\sum_{n=1}^{\infty} a(H_n)$ converges, and $\sum_{n=1}^{\infty} a(H_n) - m\left(\bigcup_{k=1}^{\infty} A_k\right) < \frac{\epsilon}{2}$.

Because the A_k's and the H_n's are mutually disjoint, and because $\bigcup_{k=1}^{\infty} A_k \subset \bigcup_{n=1}^{\infty} H_n$, we can find for every $k \in \{1, \ldots, K\}$ a sequence of mutually disjoint simple regions $(H_n^k)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} a(H_n^k)$ converges, $A_k \subset \bigcup_{n=1}^{\infty} H_n^k \subset \bigcup_{n=1}^{\infty} H_n$, and $\bigcup_{n=1}^{\infty} H_n^1, \bigcup_{n=1}^{\infty} H_n^2, \ldots, \bigcup_{n=1}^{\infty} H_n^K$ are mutually disjoint. Thus,

$$\begin{aligned}\sum_{k=1}^{K} m(A_k) \leqslant \sum_{k=1}^{K} m\left(\bigcup_{n=1}^{\infty} H_n^k\right) &= \sum_{k=1}^{K}\sum_{n=1}^{\infty} a(H_n^k) \\ &\leqslant \sum_{n=1}^{\infty} a(H_n) < m\left(\bigcup_{k=1}^{\infty} A_k\right) + \frac{\epsilon}{2}.\end{aligned}$$

Therefore,

$$\sum_{k=1}^{\infty} m(A_k) = \sum_{k=1}^{K} m(A_k) + \sum_{k>K} m(A_k) < m\left(\bigcup_{k=1}^{\infty} A_k\right) + \frac{\epsilon}{2} + \frac{\epsilon}{2} = m\left(\bigcup_{k=1}^{\infty} A_k\right) + \epsilon.$$

Taking the limit as $\epsilon \to 0$ yields that $\sum_{k=1}^{\infty} m(A_k) \leqslant m(\bigcup_{k=1}^{\infty} A_k)$.

□

PROPOSITION 3.13. *Let $K \in \mathbb{N}$ be given and for each $k \in \{1, \dots, K\}$, let $A_k \subset \mathcal{R}^2$ be measurable. Then $\bigcap_{k=1}^{K} A_k$ is measurable and*

$$m\left(\bigcap_{k=1}^{K} A_k\right) \leqslant \min\left\{m(A_k) : k \in \{1, \dots, K\}\right\}.$$

PROOF. Using induction on K, it suffices to show that if A and B are measurable sets in $\mathcal{R}^2$ then so is $A \cap B$, and $m(A \cap B) \leq \min\{m(A), m(B)\}$. So let $A, B \subset \mathcal{R}^2$ be measurable and let $\epsilon > 0$ in $\mathcal{R}$ be given. Since A and B are measurable, there exist four sequences of mutually disjoint simple regions $(G_n^A)_{n=1}^{\infty}$, $(G_n^B)_{n=1}^{\infty}$, $(H_n^A)_{n=1}^{\infty}$, and $(H_n^B)_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} G_n^A \subset A \subset \bigcup_{n=1}^{\infty} H_n^A$, $\bigcup_{n=1}^{\infty} G_n^B \subset B \subset \bigcup_{n=1}^{\infty} H_n^B$; $\sum_{n=1}^{\infty} a\left(G_n^A\right)$, $\sum_{n=1}^{\infty} a\left(G_n^B\right)$, $\sum_{n=1}^{\infty} a\left(H_n^A\right)$, and $\sum_{n=1}^{\infty} a\left(H_n^B\right)$ all converge; and

$$\sum_{n=1}^{\infty} a\left(H_n^A\right) - \sum_{n=1}^{\infty} a\left(G_n^A\right) \leqslant \frac{\epsilon}{2} \text{ and } \sum_{n=1}^{\infty} a\left(H_n^B\right) - \sum_{n=1}^{\infty} a\left(G_n^B\right) \leqslant \frac{\epsilon}{2}.$$

By Proposition 3.10, there exist two sequences of mutually disjoint simple regions $(H_n)_{n=1}^{\infty}$ and $(G_n)_{n=1}^{\infty}$ such that

$$\bigcup_{n=1}^{\infty} H_n = \left(\bigcup_{n=1}^{\infty} H_n^A\right) \bigcap \left(\bigcup_{n=1}^{\infty} H_n^B\right) \text{ and } \bigcup_{n=1}^{\infty} G_n = \left(\bigcup_{n=1}^{\infty} G_n^A\right) \bigcap \left(\bigcup_{n=1}^{\infty} G_n^B\right);$$

and $\sum_{n=1}^{\infty} a(H_n)$ and $\sum_{n=1}^{\infty} a(G_n)$ both converge. Obviously $\bigcup_{n=1}^{\infty} G_n \subset A \cap B \subset \bigcup_{n=1}^{\infty} H_n$. Since

$$\begin{aligned}
\bigcup_{n=1}^{\infty} H_n \setminus \bigcup_{n=1}^{\infty} G_n &= \bigcup_{n=1}^{\infty} H_n \setminus \left[\left(\bigcup_{n=1}^{\infty} G_n^A\right) \bigcap \left(\bigcup_{n=1}^{\infty} G_n^B\right)\right] \\
&\subset \left[\left(\bigcup_{n=1}^{\infty} H_n\right) \setminus \left(\bigcup_{n=1}^{\infty} G_n^A\right)\right] \bigcup \left[\left(\bigcup_{n=1}^{\infty} H_n\right) \setminus \left(\bigcup_{n=1}^{\infty} G_n^B\right)\right] \\
&\subset \left[\left(\bigcup_{n=1}^{\infty} H_n^A\right) \setminus \left(\bigcup_{n=1}^{\infty} G_n^A\right)\right] \bigcup \left[\left(\bigcup_{n=1}^{\infty} H_n^B\right) \setminus \left(\bigcup_{n=1}^{\infty} G_n^B\right)\right],
\end{aligned}$$

we have that

$$\begin{aligned}\sum_{n=1}^{\infty} a(H_n) - \sum_{n=1}^{\infty} a(G_n) &\leqslant \left(\sum_{n=1}^{\infty} a\left(H_n^A\right) - \sum_{n=1}^{\infty} a\left(G_n^A\right)\right) \\ &\quad + \left(\sum_{n=1}^{\infty} a\left(H_n^B\right) - \sum_{n=1}^{\infty} a\left(G_n^B\right)\right) \\ &\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

which proves that $A \cap B$ is measurable. Using the result of Proposition 3.7, we have that $m(A \cap B) \leqslant m(A)$ and $m(A \cap B) \leqslant m(B)$. Thus, $m(A \cap B) \leqslant \min\{m(A), m(B)\}$. □

PROPOSITION 3.14. *Let $A, B \subset \mathcal{R}^2$ be measurable. Then*

$$m(A \cup B) = m(A) + m(B) - m(A \cap B).$$

PROOF. First we note that, by Proposition 3.10 and Proposition 3.11, $A \cup B$ and $A \cap B$ are measurable. Now let $\epsilon > 0$ in $\mathcal{R}$ be given. Since $A \cup B$ is measurable there exists a sequence of mutually disjoint simple regions $(H_n)_{n=1}^{\infty}$ such that $A \cup B \subset \bigcup_{n=1}^{\infty} H_n$, $\sum_{n=1}^{\infty} a(H_n)$ converges and $\sum_{n=1}^{\infty} a(H_n) - m(A \cup B) < \frac{\epsilon}{2}$. Since $A \backslash (A \cap B)$, $B \backslash (A \cap B)$, and $A \cap B$ are mutually disjoint subsets of $A \cup B$, there exist three subsequences of $(H_n)_{n=1}^{\infty}$ denoted by $(H_n^1)_{n=1}^{\infty}$, $(H_n^2)_{n=1}^{\infty}$, and $(H_n^3)_{n=1}^{\infty}$ such that $A \backslash (A \cap B) \subset \bigcup_{n=1}^{\infty} H_n^1$, $B \backslash (A \cap B) \subset \bigcup_{n=1}^{\infty} H_n^2$, and $(A \cap B) \subset \bigcup_{n=1}^{\infty} H_n^3$. Note that

$$\left(\bigcup_{n=1}^{\infty} H_n^1\right) \bigcup \left(\bigcup_{n=1}^{\infty} H_n^2\right) \bigcup \left(\bigcup_{n=1}^{\infty} H_n^3\right) = \bigcup_{n=1}^{\infty} H_n.$$

Since $A = [A \backslash (A \cap B)] \bigcup (A \cap B)$, we have that

$$A \subset \left(\bigcup_{n=1}^{\infty} H_n^1\right) \bigcup \left(\bigcup_{n=1}^{\infty} H_n^3\right), \text{ and hence } m(A) \leqslant \sum_{n=1}^{\infty} a(H_n^1) + \sum_{n=1}^{\infty} a(H_n^3).$$

Similarly,

$$B \subset \left(\bigcup_{n=1}^{\infty} H_n^2\right) \bigcup \left(\bigcup_{n=1}^{\infty} H_n^3\right), \text{ and hence } m(B) \leqslant \sum_{n=1}^{\infty} a(H_n^2) + \sum_{n=1}^{\infty} a(H_n^3).$$

Thus,

$$\begin{aligned}m(A) + m(B) &\leqslant \sum_{n=1}^{\infty} a(H_n^1) + \sum_{n=1}^{\infty} a(H_n^3) + \sum_{n=1}^{\infty} a(H_n^2) + \sum_{n=1}^{\infty} a(H_n^3) \\ &= \sum_{n=1}^{\infty} a(H_n) + \sum_{n=1}^{\infty} a(H_n^3)\end{aligned}$$

Since $\sum_{n=1}^{\infty} a(H_n) \leqslant m(A \cup B) + \frac{\epsilon}{2}$, it follows that

$$\sum_{n=1}^{\infty} a(H_n^3) - m(A \cap B) \leq \sum_{n=1}^{\infty} a(H_n) - m(A \cup B) \leq \frac{\epsilon}{2}.$$

Therefore,

$$m(A)+m(B)\leqslant m(A\cup B)+m(A\cap B)+\epsilon.$$

Since this holds for any $\epsilon>0$ in $\mathcal{R}$, we infer that

$$m(A)+m(B)\leqslant m(A\cup B)+m(A\cap B).$$

Next we prove the other inequality. Since A and B are measurable, there exist two sequences of mutually disjoint simple regions $(H_n^A)_{n=1}^\infty$ and $(H_n^B)_{n=1}^\infty$ such that $A\subset\bigcup_{n=1}^\infty H_n^A$, $B\subset\bigcup_{n=1}^\infty H_n^B$; $\sum_{n=1}^\infty a(H_n^A)$ and $\sum_{n=1}^\infty a(H_n^B)$ both converge; and $\sum_{n=1}^\infty a(H_n^A)<m(A)+\frac{\epsilon}{2}$ and $\sum_{n=1}^\infty a(H_n^B)<m(B)+\frac{\epsilon}{2}$.

B can be written as the union of the two mutually disjoint subsets $B\backslash(A\cap B)$ and $A\cap B$. It follows that $(H_n^B)_{n=1}^\infty$ can be split into two subsequences $(H_n^{B,1})_{n=1}^\infty$ and $(H_n^{B,2})_{n=1}^\infty$ where $B\backslash(A\cap B)\subset\bigcup_{n=1}^\infty H_n^{B,1}$ and $A\cap B\subset\bigcup_{n=1}^\infty H_n^{B,2}$. Thus,

$$A\cup B=A\cup[B\backslash(A\cap B)]\subset\left(\bigcup_{n=1}^\infty H_n^A\right)\bigcup\left(\bigcup_{n=1}^\infty H_n^{B,1}\right),$$

and hence

$$m(A\cup B)\leqslant\sum_{n=1}^\infty a(H_n^A)+\sum_{n=1}^\infty a(H_n^{B,1})\leqslant m(A)+\frac{\epsilon}{2}+\sum_{n=1}^\infty a(H_n^{B,1}).$$

Since $A\cap B\subset\bigcup_{n=1}^\infty H_n^{B,2}$, we have that $m(A\cap B)\leqslant\sum_{n=1}^\infty a(H_n^{B,2})$. Thus,

$$\begin{aligned}m(A\cup B)+m(A\cap B)&\leqslant m(A)+\frac{\epsilon}{2}+\sum_{n=1}^\infty a(H_n^{B,1})+\sum_{n=1}^\infty a(H_n^{B,2})\\&\leq m(A)+\frac{\epsilon}{2}+\sum_{n=1}^\infty a(H_n^B)\\&\leq m(A)+m(B)+\epsilon.\end{aligned}$$

Since this holds for any $\epsilon>0$ in $\mathcal{R}$, we infer that

$$m(A\cup B)+m(A\cap B)\leqslant m(A)+m(B).$$

□

3.2. Analytic Functions.

DEFINITION 3.15 (Finite Simple Region and Order of Magnitude). Let $A\subset\mathcal{R}^2$ be a simple region. First assume that A is of the form

$$A=\{(x,y)\in\mathcal{R}^2:y\in I(h_1(x),h_2(x)),x\in I(a,b)\}$$

where $a\leqslant b$, $h_1,h_2:I(a,b)\to\mathcal{R}$ are analytic functions, and $h_1<h_2$. In this case, we define

$$\lambda_x(A)=\lambda(b-a)\text{ and }\lambda_y(A)=i(h_2(x)-h_1(x))$$

where $i(h_2(x)-h_1(x))$ is the index of the analytic function $h_2(x)-h_1(x)$ on $I(a,b)$, given by

$$i(h_2(x)-h_1(x))=\min\{\lambda(h_2(x)-h_1(x)):x\in I(a,b)\},$$

which exists as shown in [**17**]. On the other hand, if A is of the form

$$A=\{(x,y)\in\mathcal{R}^2:x\in I(h_1(y),h_2(y)),y\in I(a,b)\},$$

we define

$$\lambda_y(A) = \lambda(b-a) \text{ and } \lambda_x(A) = i(h_2(y) - h_1(y)).$$

We call $\lambda_x(A)$ and $\lambda_y(A)$ the orders of magnitude of A in x and y, respectively. If $\lambda_x(A) = \lambda_y(A) = 0$ then we say that A is a finite simple region.

DEFINITION 3.16 (Analytic Functions on $\mathcal{R}^2$). Let $A \subset \mathcal{R}^2$ be a simple region. Then we say that $f : A \to \mathcal{R}^2$ is an analytic function on A if, for every $(x_0, y_0) \in A$, there exist a simple region A_0 containing (x_0, y_0) that satisfies $\lambda_x(A_0) = \lambda_x(A)$ and $\lambda_y(A_0) = \lambda_y(A)$, and a regular sequence $(a_{ij})_{i,j=0}^{\infty}$ such that for every $s, t \in \mathcal{R}$, if $(x_0 + s, y_0 + t) \in A \cap A_0$ then

$$f(x_0 + s, y_0 + t) = \sum_{i,j=0}^{\infty} a_{ij} s^i t^j = f(x_0, y_0) + \sum_{\substack{i,j=0 \\ i+j\neq 0}}^{\infty} a_{ij} s^i t^j,$$

where the power series converges in the weak topology [**9**, **15**].

The following proposition follows directly from Definition 3.16.

PROPOSITION 3.17. *Let $A \subset \mathcal{R}^2$ be a simple region and let $f : A \to \mathcal{R}^2$ be an analytic function on A. Then for a fixed x, the function $g(y) := f(x, y)$ is analytic on $I_x = \{y \in \mathcal{R} : (x, y) \in A\}$; and for a fixed y, $h(x) := f(x, y)$ is analytic on $I_y = \{x \in \mathcal{R} : (x, y) \in A\}$.*

PROPOSITION 3.18. *Let $A \subset \mathcal{R}^2$ be a simple region and let $f : A \to \mathcal{R}$ be analytic on A. Then f is bounded on A.*

PROOF. Without loss of generality, we may assume that A is the form

$$A = \{(x, y) \in \mathcal{R}^2 : y \in [h_1(x), h_2(x)], x \in [a, b]\},$$

with $h_1, h_2 : [a, b] \to \mathcal{R}$ analytic on $[a, b]$. Let $F(x, y) : [0, 1]^2 \to \mathcal{R}$ be given by

$$F(x, y) = f((b-a)x + a, (h_2(x) - h_1(x))y + h_1(x)).$$

Then F is analytic on $[0, 1]^2$ and f is bounded on A if and only if F is bounded on $[0, 1]^2$.

For every $(v, w) \in \mathcal{R}^2$ let $N((v, w), \eta) = \{(x, y) \in \mathcal{R}^2 : \sqrt{(x-v)^2 + (y-w)^2} < \eta\}$. Since F is analytic on $[0, 1]^2$, it follows that, for every $(x_0, y_0) \in [0, 1]^2 \cap \mathbb{R}^2$, there exists a real $\eta(x_0, y_0) > 0$ and a regular sequence $(a_{ij}(x_0, y_0))_{i,j=0}^{\infty}$ such that for every $(x, y) \in N((x_0, y_0), \eta(x_0, y_0)) \cap [0, 1]^2$, we have that

$$F(x, y) = \sum_{i,j=0}^{\infty} a_{ij}(x_0, y_0)(x - x_0)^i (y - y_0)^j.$$

The set $\left\{N\left((x, y), \frac{\eta(x,y)}{2}\right) \cap \mathbb{R}^2 : (x, y) \in [0, 1]^2 \cap \mathbb{R}^2\right\}$ is an open cover of $[0, 1]^2 \cap \mathbb{R}^2$ which is a compact subset of the Euclidean space $\mathbb{R}^2$; hence we can select a finite subcover. Thus, there exists a finite set of points $\{(x_k, y_k)\}_{k=1}^{m}$ contained in $[0, 1]^2 \cap \mathbb{R}^2$ such that $[0, 1]^2 \cap \mathbb{R}^2 \subset \bigcup_{k=1}^{m} N\left((x_k, y_k), \frac{\eta(x_k,y_k)}{2}\right) \cap \mathbb{R}^2$. It follows that $[0, 1]^2 \subset \bigcup_{k=1}^{m} N((x_k, y_k), \eta(x_k, y_k))$. Let

$$l = \min_{1\leqslant k\leqslant m} \left\{ \min \left\{ \bigcup_{i,j=0}^{\infty} \operatorname{supp}(a_{ij}(x_k, y_k)) \right\} \right\},$$

which exists by the regularity of the sequence $(a_{ij}(x_k, y_k))$ for each k. It follows from the above that $|F(x, y)| < d^{l-1}$ for every $(x, y) \in [0, 1]^2$. Thus F is bounded on $[0, 1]^2$ and hence f is bounded on A. $\square$

REMARK 3.19. Let A, f, F, $\{(x_k, y_k)\}_{k=1}^m$, and l be as in Proposition 3.18 and the proof thereof. Then

$$l = \min \{\lambda(f(x, y)) : (x, y) \in A\} = \min \left\{\lambda(F(x, y)) : (x, y) \in [0, 1]^2\right\}.$$

Thus, l is independent of our choice of the finite set $\{(x_k, y_k)\}_{k=1}^m$ in $[0, 1]^2 \cap \mathcal{R}^2$ in the proof of Proposition 3.18 above.

DEFINITION 3.20. Let A, f and l be as in Proposition 3.18 and Remark 3.19. Then we call l the index of f on A and we denote it by $i(f)$. Thus,

$$i(f) := l = \min \{\lambda(f(x, y)) : (x, y) \in A\}.$$

PROPOSITION 3.21. *Let $A \subset \mathcal{R}^2$ be a finite simple region, let $f, g : A \to \mathcal{R}^2$ be analytic functions on A, and let $\alpha \in \mathcal{R}$ be given. Then $f + \alpha g$ and $f \cdot g$ are analytic functions on A.*

PROOF. Let $(x_0, y_0) \in A$ be given. Since f and g are analytic on A, there exist finite $\eta_1, \eta_2 > 0$ such that for every $s, t \in \mathcal{R}$, if $s^2 + t^2 < \eta_1^2$ and $(x_0 + s, y_0 + t) \in A$ then $f(x_0 + s, y_0 + t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} s^i t^j$, and if $s^2 + t^2 < \eta_2^2$ and $(x_0 + s, y_0 + t) \in A$ then $g(x_0 + s, y_0 + t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} b_{kl} s^k t^l$. Let $\eta = \min\{\eta_1, \eta_2\}$. Then for every $s, t \in \mathcal{R}$ satisfying $s^2 + t^2 < \eta^2$ and $(x_0 + s, y_0 + t) \in A$, we have that

$$\begin{aligned}
(f + \alpha g)(x_0 + s, y_0 + t) &= f(x_0 + s, y_0 + t) + \alpha g(x_0 + s, y_0 + t) \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} s^i t^j + \alpha \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} b_{kl} s^k t^l \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (a_{ij} + \alpha b_{ij}) s^i t^j \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} s^i t^j,
\end{aligned}$$

where $c_{ij} = a_{ij} + \alpha b_{ij}$. Hence $f + \alpha g$ is analytic on A. Moreover,

$$\begin{aligned}
(f \cdot g)(x_0 + s, y_0 + t) &= f(x_0 + s, y_0 + t) \cdot g(x_0 + s, y_0 + t) \\
&= \left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} s^i t^j \right) \left(\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} b_{kl} s^k t^l \right) \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{i+k=n} \sum_{j+l=m} a_{ij} b_{kl} s^n t^m
\end{aligned}$$

Note that the Cauchy product of two power series converging weakly in $\mathcal{R}$ also converges [**15**]; we infer that the same is true for the Cauchy product of two (weakly) converging power series in two variables. Defining

$$e_{nm} = \sum_{i+k=n} \sum_{j+l=m} a_{ij} b_{kl}$$

yields

$$(f \cdot g)(x_0 + s, y_0 + t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} e_{nm} s^n t^m.$$

Hence $f \cdot g$ is analytic on A. □

COROLLARY 3.22. *Let $A \subset \mathcal{R}^2$ be a simple region, let $f, g : A \to \mathcal{R}^2$ be analytic functions on A, and let $\alpha \in \mathcal{R}$ be given. Then $f + \alpha g$ and $f \cdot g$ are analytic functions on A.*

PROOF. Without loss of generality, we may assume that A is of the form $A = \{(x, y) \in \mathcal{R}^2 : y \in I(h_1(x), h_2(x)), x \in I(a, b)\}$ with $h_1, h_2 : I(a, b) \to \mathcal{R}$ analytic on $I(a, b)$. Let $A_0 = \{(x, y) \in \mathcal{R}^2 : y \in I(d^{-\lambda_y(A)} h_1(d^{\lambda_x(A)} x), d^{-\lambda_y(A)} h_2(d^{\lambda_x(A)} x)), x \in I(d^{-\lambda_x(A)} a, d^{-\lambda_x(A)} b)\}$. Then A_0 is a finite simple region. Moreover, the functions $F, G : A_0 \to \mathcal{R}$, given by $F(x, y) = f(d^{\lambda_x(A)} x, d^{\lambda_y(A)} y)$ and $G(x, y) = g(d^{\lambda_x(A)} x, d^{\lambda_y(A)} y)$ are both analytic on A_0 so by Proposition 3.21 $F + \alpha G$ and $F \cdot G$ are analytic on A_0. It follows that $f + \alpha g$ and $f \cdot g$ are analytic functions on A. □

PROPOSITION 3.23. *Let $A \subset \mathcal{R}^2$ be a finite simple region and let $f : A \to \mathcal{R}$ be an analytic function on A. Let $a, b \in \mathcal{R}$ be such that $a < b$ and $b - a$ is finite, let $g : I(a, b) \to \mathcal{R}$ be an $\mathcal{R}$-analytic function on $I(a, b)$ such that for every $x \in I(a, b)$, $(x, g(x)) \in A$. Then the function $F : I(a, b) \to \mathcal{R}$, given by $F(x) = f(x, g(x))$ is an $\mathcal{R}$-analytic function on $I(a, b)$.*

PROOF. Without loss of generality, we may assume that $i(f) = 0$ on A. Now let $x_0 \in A$ be given. By the definition of analytic functions there exist finite $\eta_1, \eta_2 > 0$ such that if $|h| < \eta_1$ and $x_0 + h \in I(a, b)$ then $g(x_0 + h) = g(x_0) + \sum_{n=1}^{\infty} a_n h^n$ and if $s^2 + t^2 < \eta_2^2$ and $(x_0 + s, g(x_0) + t) \in A$ then $f(x_0 + s, g(x_0) + t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij} s^i t^j$.

Define $F : \mathbb{R} \to \mathbb{R}$ by $F(X) = (X^2 + (\sum_{n=1}^{\infty} a_n X^n)^2)[0]$. Then F is continuous on $\mathbb{R}$ and hence we can choose a real number $\eta \in (0, \frac{\eta_1}{2}]$ such that if $|h| < \eta$ and $x_0 + h \in I(a, b)$ then $F(h[0]) < \frac{\eta_2^2}{2}$, and hence, since $h^2 + (\sum_{n=1}^{\infty} a_n h^n)^2$ is different from $F(h[0])$ by at most an infinitely small amount, it follows that if $|h| < \eta$ and $x_0 + h \in I(a, b)$ then $h^2 + (\sum_{n=1}^{\infty} a_n h^n)^2 < \eta_2^2$. Thus, for any $|h| < \eta$ such that $x_0 + h \in I(a, b)$, we have that $h^2 + (\sum_{n=1}^{\infty} a_n h^n)^2 < \eta_2^2$ and $(x_0 + h, g(x_0) + \sum_{n=1}^{\infty} a_n h^n) \in A$, and hence $f(x_0 + h, g(x_0 + h)) = f(x_0, g(x_0)) + \sum_{\substack{i,j=0 \\ i+j \neq 0}}^{\infty} b_{ij} h^i (\sum_{n=1}^{\infty} a_n h^n)^j$.

Now, for every $i, j \in \mathbb{N}$, let $V_{ij}(h) = b_{ij} h^i (\sum_{n=1}^{\infty} a_n h^n)^j$. Then for every $i, j \in \mathbb{N}$, $V_{ij}(h)$ can be rewritten as $V_{ij}(h) = \sum_{m=1}^{\infty} c_{ijm} h^m$. Since for every $i, j \in \mathbb{N}$, $\sum_{m=1}^{\infty} c_{ijm} h^m$ converges, it follows that for every $i, j \in \mathbb{N}$ and for every $q \in \mathbb{Q}$,

$\sum_{m=1}^{\infty}(c_{ijm}h^m)[q]$ converges absolutely in $\mathbb{R}$ [**15**]. This, in addition to the fact that for every $q \in \mathbb{Q}$, $\sum_{\substack{i,j=0\\i+j\neq 0}}^{\infty}(b_{ij}h^i(\sum_{n=1}^{\infty}a_nh^n)^j)[q] = \sum_{\substack{i,j=0\\i+j\neq 0}}^{\infty}\sum_{m=1}^{\infty}(c_{ijm}h^m)[q]$ converges in $\mathbb{R}$ allows us to change the order of summation in the triple sum. Thus, for every $q \in \mathbb{Q}$, we have that

$$\begin{aligned} f\left(x_0+h, g(x_0)+\sum_{n=1}^{\infty}a_nh^n\right)[q] &= f(x_0,g(x_0))[q] + \sum_{\substack{i,j=0\\i+j\neq 0}}^{\infty}{}^{1}\sum_{m=1}^{\infty}(c_{ijm}h^m)[q] \\ &= f(x_0,g(x_0))[q] + \sum_{m=1}^{\infty}\sum_{\substack{i,j=0\\i+j\neq 0}}^{\infty}(c_{ijm}h^m)[q]; \end{aligned}$$

and hence

$$\begin{aligned} f(x_0+h, g(x_0+h)) &= f(x_0,g(x_0)) + \sum_{m=1}^{\infty}\sum_{\substack{i,j=0\\i+j\neq 0}}^{\infty}c_{ijm}h^m \\ &= f(x_0,g(x_0)) + \sum_{m=1}^{\infty}\left(\sum_{\substack{i,j=0\\i+j\neq 0}}^{\infty}c_{ijm}\right)h^m \end{aligned}$$

Letting $e_m = \sum_{\substack{i,j=0\\i+j\neq 0}}^{\infty}c_{ijm}$, we obtain that $f(x_0+h, g(x_0+h)) = f(x_0,g(x_0)) + \sum_{m=1}^{\infty}e_mh^m$. Thus, for every $x \in I(a,b)$ there exists a finite $\eta > 0$ such that for every $|h| < \eta$, $f(x_0+h, g(x_0+h))$ is given by a power series about x_0, which means that $f(x,g(x))$ is an $\mathcal{R}$-analytic function on $I(a,b)$. □

Corollary 3.24 follows directly from Proposition 3.23 using a scaling argument similar to that for Corollary 3.22.

Corollary 3.24. *Let $A \subset \mathcal{R}^2$ be a simple region and let $f : A \to \mathcal{R}$ be an analytic function on A. Let $a,b \in \mathcal{R}$ be such that $a < b$, let $g : I(a,b) \to \mathcal{R}$ be an $\mathcal{R}$-analytic function on $I(a,b)$ such that for every $x \in I(a,b)$, $(x,g(x)) \in A$. Then the function $F : I(a,b) \to \mathcal{R}$, given by $F(x) = f(x,g(x))$ is an $\mathcal{R}$-analytic function on $I(a,b)$.*

The proof of Proposition 3.25 below is similar to that of Proposition 3.23 and Corollary 3.24 above. We skip the details here and refer the interested reader to [**3**].

Proposition 3.25. *Let $S \subset \mathcal{R}^2$ be a simple region and let $f : S \to \mathcal{R}$ be an analytic function on S. Let $H \subset \mathcal{R}^2$ be a simple region and let $g : H \to \mathcal{R}$ be an analytic function on H such that for every $(x,y) \in H$, $(x,g(x,y)) \in S$. Then the function $F : H \to \mathcal{R}$, given by $F(x,y) = f(x,g(x,y))$ is analytic on H.*

3.3. Measurable Functions on $\mathcal{R}^2$.

DEFINITION 3.26 (Measurable Function). Let $A \subset \mathcal{R}^2$ be a measurable set and let $f : A \to \mathcal{R}$ be bounded on A. Then we say that f is measurable on A if for every $\epsilon > 0$ in $\mathcal{R}$ there exists a sequence of mutually disjoint simple regions $(G_n)_{n=1}^{\infty}$ such that $\bigcup_{n=0}^{\infty} G_n \subset A$, $\sum_{n=1}^{\infty} a(G_n)$ converges, $m(A) - \sum_{n=1}^{\infty} a(G_n) < \epsilon$, and for all $n \in \mathbb{N}$ f is analytic on G_n.

PROPOSITION 3.27. *Let $A \subset \mathcal{R}^2$ be a measurable set and let $f : A \to \mathcal{R}$ be a measurable function on A. Then f is given locally by a power series almost everywhere on A.*

PROOF. Let $A_0 = \{(x, y) \in A : f$ is not given locally by a power series about $(x, y)\}$ We show that A_0 is measurable and $m(A_0) = 0$. Let $\epsilon > 0$ be given in $\mathcal{R}$. Since f is measurable on A, there exists a sequence of mutually disjoint open simple regions $(G_n)_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} G_n \subset A$, $\sum_{n=1}^{\infty} a(G_n)$ converges, $m(A) - \sum_{n=1}^{\infty} a(G_n) \leqslant \frac{\epsilon}{2}$, and f is analytic on G_n for all n.

Also, since A is measurable, there exists a sequence of mutually disjoint simple regions $(H_n)_{n=1}^{\infty}$ such that $A \subset \bigcup_{n=1}^{\infty} H_n$, $\sum_{n=1}^{\infty} a(H_n)$ converges, $\sum_{n=1}^{\infty} a(H_n) - m(A) \leqslant \frac{\epsilon}{2}$.

Since f is given by a power series around every point in $\bigcup_{n=1}^{\infty} G_n$ and since $A_0 \subset A$, we have that

$$A_0 \subset A \backslash \left(\bigcup_{n=1}^{\infty} G_n \right) \subset \left(\bigcup_{n=1}^{\infty} H_n \right) \backslash \left(\bigcup_{n=1}^{\infty} G_n \right).$$

Since both $(G_n)_{n=1}^{\infty}$ and $(H_n)_{n=1}^{\infty}$ consist of mutually disjoint sets, we can rearrange the members of the sequences so that for every $n \in \mathbb{N}$, $G_n \subset H_n$. Thus,

$$\left(\bigcup_{n=1}^{\infty} H_n \right) \backslash \left(\bigcup_{n=1}^{\infty} G_n \right) = \bigcup_{n=1}^{\infty} (H_n \backslash G_n).$$

The $H_n \backslash G_n$'s are mutually disjoint, and hence for every $n \in \mathbb{N}$, $H_n \backslash G_n$ may be expressed as the union of a finite number of mutually disjoint simple regions. It follows that $\bigcup_{n=1}^{\infty} H_n \backslash \bigcup_{n=1}^{\infty} G_n$ may be rewritten as the union of countably many mutually disjoint simple regions $(H_n^0)_{n=1}^{\infty}$.

For every $n \in \mathbb{N}$, let $G_n^0 = \emptyset$. Then we have $\bigcup\limits_{n=1}^{\infty} G_n^0 \subset A_0 \subset \bigcup\limits_{n=1}^{\infty} H_n^0$. Moreover,

$$\begin{aligned}
\sum_{n=1}^{\infty} a(H_n^0) - \sum_{n=1}^{\infty} a(G_n^0) &= \sum_{n=1}^{\infty} a(H_n^0) \\
&= \sum_{n=1}^{\infty} a(H_n) - \sum_{n=1}^{\infty} a(G_n) \\
&= \left(\sum_{n=1}^{\infty} a(H_n) - m(A)\right) + \left(m(A) - \sum_{n=1}^{\infty} a(G_n)\right) \\
&\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

So A_0 is measurable. Since $A_0 \subset \bigcup\limits_{n=1}^{\infty} H_n^0$ we have that $m(A_0) \leqslant \sum\limits_{n=1}^{\infty} a(H_n^0) \leqslant \epsilon$. This is true for all $\epsilon > 0$ in $\mathcal{R}$; hence $m(A_0) = 0$. $\square$

Proposition 3.28. *Let $A \subset \mathcal{R}^2$ be a simple region and let $f : A \to \mathcal{R}$ be a measurable function such that $\frac{\partial}{\partial x} f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ both exist and $\frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial y} f(x, y) = 0$ everywhere in A (see* [**12**] *for the definition of partial derivatives). Then $f(x, y)$ is constant on A*

Proof. Fix two arbitrary points $(x_0, y_0), (x_1, y_1) \in A$. By Proposition 3.27, we have that $f(x, y)$ is analytic almost everywhere on A, and hence by Corollary 3.17, we have that for a fixed y, the function $g_y(x) := f(x, y)$ is $\mathcal{R}$-analytic almost everywhere on $\{x \in \mathcal{R} : (x, y) \in A\}$ and for a fixed x, the function $h_x(y) := f(x, y)$ is $\mathcal{R}$-analytic almost everywhere on $\{y \in \mathcal{R} : (x, y) \in A\}$. Therefore, $g_{y_0}(x) = f(x, y_0)$ is $\mathcal{R}$-analytic almost everywhere on $\{x \in \mathcal{R} : (x, y_0) \in A\}$ and since $\frac{\partial}{\partial x} f(x, y_0) = g'_{y_0}(x) = 0$ everywhere on $\{x \in \mathcal{R} : (x, y_0) \in A\}$ it follows that $g_{y_0}(x)$ is constant [**8**]. Thus $f(x_0, y_0) = f(x_1, y_0)$. Using a similar argument, making use of the fact that $\frac{\partial}{\partial y} f(x, y) = 0$ everywhere in A, we show that $f(x_1, y_0) = f(x_1, y_1)$, and hence $f(x_0, y_0) = f(x_1, y_1)$. $\square$

Proposition 3.29. *Let $A, B \subset \mathcal{R}^2$ be measurable sets and let $f : A, B \to \mathcal{R}$ be a measurable function on both A and B. Then f is a measurable function on $A \cap B$ and $A \cup B$.*

Proof. Let $\epsilon > 0$ in $\mathcal{R}$ be given. Since f is measurable on A there exists a sequence of mutually disjoint simple regions $(G_n^A)_{n=1}^{\infty}$ such that for every $n \in \mathbb{N}$, f is analytic on G_n^A, $\bigcup\limits_{n=1}^{\infty} G_n^A \subset A$, $\sum\limits_{n=1}^{\infty} a(G_n^A)$ converges, and $m(A) - \sum\limits_{n=1}^{\infty} a(G_n^A) < \frac{\epsilon}{2}$. Also, since f is measurable on B there exists a sequence of mutually disjoint simple regions $(G_n^B)_{n=1}^{\infty}$ such that for every $n \in \mathbb{N}$, f is analytic on G_n^B, $\bigcup\limits_{n=1}^{\infty} G_n^B \subset B$, $\sum\limits_{n=1}^{\infty} a(G_n^B)$ converges, and $m(B) - \sum\limits_{n=1}^{\infty} a(G_n^B) < \frac{\epsilon}{2}$.

$\left(\bigcup\limits_{n=1}^{\infty} G_n^A\right) \cup \left(\bigcup\limits_{n=1}^{\infty} G_n^B\right)$ can be rewritten as the union of mutually disjoint simple regions $\bigcup\limits_{n=1}^{\infty} G_n^0 \subset A \cup B$ such that $\sum\limits_{n=1}^{\infty} a(G_n^0)$ converges and for every $n \in \mathbb{N}$, f

is analytic on G_n^0. Since $\bigcup_{n=1}^{\infty} G_n^0 = \left(\bigcup_{n=1}^{\infty} G_n^A\right) \bigcup \left(\bigcup_{n=1}^{\infty} G_n^B\right) \subset A \cup B$ and since the G_n^0's are mutually disjoint, it follows that

$$m(A \cup B) - \sum_{n=1}^{\infty} a(G_n^0) \leqslant m(A) - \sum_{n=1}^{\infty} a(G_n^A) + m(B) - \sum_{n=1}^{\infty} a(G_n^B) \leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} \leqslant \epsilon.$$

Thus, f is measurable on $A \cup B$.

Also, $\left(\bigcup_{n=1}^{\infty} G_n^A\right) \bigcap \left(\bigcup_{n=1}^{\infty} G_n^B\right)$ can be rewritten as a countable union of mutually disjoint simple regions $\bigcup_{n=1}^{\infty} G_n^1 \subset A \cap B$, such that $\sum_{n=1}^{\infty} a(G_n^1)$ converges and for every $n \in \mathbb{N}$, f is analytic on G_n^1.

Using Proposition 3.14 we have that $m(A \cap B) = m(A) + m(B) - m(A \cup B)$, and

$$m\left(\bigcup_{n=1}^{\infty} G_n^1\right) = m\left(\bigcup_{n=1}^{\infty} G_n^A\right) + m\left(\bigcup_{n=1}^{\infty} G_n^B\right) - m\left(\bigcup_{n=1}^{\infty} G_n^0\right).$$

It follows that

$$\begin{aligned}
& m(A \cap B) - \sum_{n=1}^{\infty} a(G_n^1) = m(A \cap B) - m\left(\bigcup_{n=1}^{\infty} G_n^1\right) \\
& = m(A) + m(B) - m(A \cup B) + m\left(\bigcup_{n=1}^{\infty} G_n^0\right) - m\left(\bigcup_{n=1}^{\infty} G_n^A\right) - m\left(\bigcup_{n=1}^{\infty} G_n^B\right) \\
& = m(A) - \sum_{n=1}^{\infty} a(G_n^A) + m(B) - \sum_{n=1}^{\infty} a(G_n^B) - m(A \cup B) + \sum_{n=1}^{\infty} a(G_n^0) \\
& \leqslant m(A) - \sum_{n=1}^{\infty} a(G_n^A) + m(B) - \sum_{n=1}^{\infty} a(G_n^B) \\
& \leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus, f is measurable on $A \cap B$. $\square$

PROPOSITION 3.30. *Let $A \subset \mathcal{R}^2$ be measurable, let $f, g : A \to \mathcal{R}$ be both measurable on A, and let $\alpha \in \mathcal{R}$ be given. Then $f + \alpha g$ and $f \cdot g$ are measurable on A.*

PROOF. Fix $\epsilon > 0$ in $\mathcal{R}$. Since f and g are both measurable on A, there exist sequences of mutually disjoint simple regions $(G_n^f)_{n=1}^{\infty}$ and $(G_n^g)_{n=1}^{\infty}$ such that, for every $n \in \mathbb{N}$, f is analytic on G_n^f and g is analytic on G_n^g, $\bigcup_{n=1}^{\infty} G_n^f \subset A$, $\bigcup_{n=1}^{\infty} G_n^g \subset A$, $\sum_{n=1}^{\infty} a(G_n^f)$ and $\sum_{n=1}^{\infty} a(G_n^g)$ both converge, $m(A) - \sum_{n=1}^{\infty} a(G_n^f) < \epsilon/2$, and $m(A) - \sum_{n=1}^{\infty} a(G_n^g) < \epsilon/2$.

$\left(\bigcup_{n=1}^{\infty} G_n^f\right) \bigcap \left(\bigcup_{n=1}^{\infty} G_n^g\right)$ can be rewritten as the union of mutually disjoint simple regions $\bigcup_{i=1}^{\infty} T_i \subset A$ such that $\sum_{i=1}^{\infty} a(T_i)$ converges, and for every $i \in \mathbb{N}$, f and g are analytic on T_i.

Since $\bigcup_{i=1}^{\infty} T_i = \left(\bigcup_{n=1}^{\infty} G_n^f\right) \bigcap \left(\bigcup_{n=1}^{\infty} G_n^g\right)$ and since the T_i's are mutually disjoint, the G_n^f's are mutually disjoint and the G_n^g's are mutually disjoint, we obtain that

$$\begin{aligned} m(A) - \sum_{i=1}^{\infty} a(T_i) &= m(A) - m\left(\bigcup_{i=1}^{\infty} T_i\right) \\ &\leqslant \left[m(A) - m\left(\bigcup_{n=1}^{\infty} G_n^f\right)\right] + \left[m(A) - m\left(\bigcup_{n=1}^{\infty} G_n^g\right)\right] \\ &= \left(m(A) - \sum_{n=1}^{\infty} a(G_n^f)\right) + \left(m(A) - \sum_{n=1}^{\infty} a(G_n^g)\right) \\ &< \epsilon. \end{aligned}$$

We know from Corollary 3.22 that for each $i \in \mathbb{N}$, $f + \alpha g$ and $f \cdot g$ are analytic on T_i. Thus, there exists a sequence of mutually disjoint simple regions $(T_i)_{i=1}^{\infty}$ such that $\bigcup_{i=1}^{\infty} T_i \subset A$, $\sum_{i=1}^{\infty} a(T_i)$ converges, $m(A) - \sum_{i=1}^{\infty} a(T_i) < \epsilon$, and for every $i \in \mathbb{N}$, $f + \alpha g$ and $f \cdot g$ are $\mathcal{R}$-analytic on T_i. Therefore, $f + \alpha g$ and $f \cdot g$ are measurable on A. $\square$

3.4. Integration on $\mathcal{R}^2$.

DEFINITION 3.31 (Integration of Analytic Functions on Simple Regions). Let $H \subset \mathcal{R}^2$ be a simple region and let $f : H \to \mathcal{R}$ be an analytic function. First assume that $H = \{(x, y) \in \mathcal{R}^2 : y \in I(h_1(x), h_2(x)), x \in I(a, b)\}$, where $a, b \in \mathcal{R}$, $a \leqslant b$, and $h_1, h_2 : I(a, b) \to \mathcal{R}$ are analytic on $I(a, b)$ with $h_1 < h_2$. We define the integral of f over H as the iterated integral

$$\iint_{(x,y)\in H} f(x, y) = \int_{x\in I(a,b)} \left[\int_{y\in I(h_1(x),h_2(x))} f(x, y)\right].$$

We note that, for each $x \in I(a, b)$, $f(x, y)$ is $\mathcal{R}$-analytic on $I(h_1(x), h_2(x))$; hence $\int_{y\in I(h_1(x),h_2(x))} f(x, y)$ is well-defined and it yields an $\mathcal{R}$-analytic function $F(x)$ on $I(a, b)$. Then

$$\iint_{(x,y)\in H} f(x, y) = \int_{x\in I(a,b)} F(x)$$

is well-defined.

On the other hand, if H is given by $H = \{(x, y) \in \mathcal{R}^2 : x \in I(h_1(y), h_2(y)), y \in I(a, b)\}$ then we define the integral of f over H as the iterated integral

$$\iint_{(x,y)\in H} f(x, y) = \int_{y\in I(a,b)} \left[\int_{x\in I(h_1(y),h_2(y))} f(x, y)\right].$$

LEMMA 3.32. *Let $G \subset \mathcal{R}^2$ be a simple region and let $\alpha \in \mathcal{R}$ be given. Then*

$$\iint_{(x,y)\in G} \alpha = \alpha a(G).$$

Proof. Without loss of generality, we may assume that $G = \{(x,y) \in \mathcal{R}^2 : y \in I(g_1(x), g_2(x)), x \in I(a,b)\}$, where $a, b \in \mathcal{R}$, $a \leqslant b$, and $g_1, g_2 : I(a,b) \to \mathcal{R}$ are $\mathcal{R}$-analytic on I(a,b) with $g_1 < g_2$. Thus, by the linearity of the single integral [**11**, **16**], we have that

$$\iint_{(x,y)\in G} \alpha = \int_{x\in I(a,b)} \left[\int_{y\in I(g_1(x),g_2(x))} \alpha \right] = \int_{x\in I(a,b)} \alpha \left[\int_{y\in I(g_1(x),g_2(x))} 1 \right]$$
$$= \alpha \int_{x\in I(a,b)} \left[\int_{y\in I(g_1(x),g_2(x))} 1 \right] = \alpha \int_{x\in I(a,b)} [g_2(x) - g_1(x)] = \alpha a(G).$$

□

Using Definition 3.31 and the linearity of the single integral, we readily obtain the following result which will be used later to prove the linearity property for the integral of a measurable function over a measurable subset of $\mathcal{R}^2$.

Lemma 3.33. *Let $H \subset \mathcal{R}^2$ be a simple region, let $f, g : H \to \mathcal{R}$ be analytic functions on H, and let $\alpha \in \mathcal{R}$ be given. Then*

$$\iint_{(x,y)\in H} (f + \alpha g)(x,y) = \iint_{(x,y)\in H} f(x,y) + \alpha \iint_{(x,y)\in H} g(x,y).$$

Lemma 3.34. *Let $G \subset \mathcal{R}^2$ be a simple region and let $f : G \to \mathcal{R}$ be analytic such that $f \leq 0$ on G. Then $\iint_{(x,y)\in G} f(x,y) \leqslant 0$.*

Proof. Without loss of generality we may assume that $G = \{(x,y) \in \mathcal{R}^2 : y \in I(g_1(x), g_2(x)), x \in I(a,b)\}$ where $a, b \in \mathcal{R}$, $a \leqslant b$, and $g_1, g_2 : I(a,b) \to \mathcal{R}$ are analytic on $I(a,b)$ with $g_1 < g_2$ on $I(a,b)$. Since $f \leq 0$ on G, then for every $x \in I(a,b)$, we have that $f(x,y)$ is a non-positive $\mathcal{R}$-analytic function on $I(g_1(x), g_2(x))$. It follows from Proposition 4.4 in [**16**] that for every $x \in I(a,b)$, $\int_{y\in I(g_1(x),g_2(x))} f(x,y) \leqslant 0$. Thus, using Proposition 4.4 of [**16**] again, we obtain that

$$\iint_{(x,y)\in G} f(x,y) = \int_{x\in I(a,b)} \left[\int_{y\in I(g_1(x),g_2(x))} f(x,y) \right] \leqslant 0.$$

□

Corollary 3.35. *Let $G \subset \mathcal{R}^2$ be a simple region and let $f, h : G \to \mathcal{R}$ be analytic functions such that $f \leqslant g$ on G. Then $\iint_{(x,y)\in G} f(x,y) \leqslant \iint_{(x,y)\in G} h(x,y)$.*

Corollary 3.36. *Let G be a simple region, let $f : G \to \mathcal{R}$ be an analytic function, and let M be a bound for $|f|$ on G. Then*

$$\left| \iint_{(x,y)\in G} f(x,y) \right| \leq \iint_{(x,y)\in G} |f(x,y)| \leqslant M a(G).$$

Definition 3.37. [The Integral of a Measurable Function over a Measurable Set] Let $A \subset \mathcal{R}^2$ be a measurable set, let $f : A \to \mathcal{R}$ be a measurable function, and let M be a bound for $|f|$ on A. Since f is measurable then for every $k \in \mathbb{N}$ there exists a sequence of mutually disjoint simple regions $(G_n^k)_{n=1}^{\infty}$ such that for every $n \in \mathbb{N}$, f is analytic on G_n^k, $\bigcup\limits_{n=1}^{\infty} G_n^k \subset A$, $\sum\limits_{n=1}^{\infty} a(G_n^k)$ converges, and $m(A) - \sum\limits_{n=1}^{\infty} a(G_n^k) \leqslant d^k$. Without loss of generality we may assume that for every k and n in $\mathbb{N}$, $G_n^k \subset G_n^{k+1}$.

Since $\sum\limits_{n=1}^{\infty} a(G_n^k)$ converges we have that $\lim\limits_{n\to\infty} a(G_n^k) = 0$. From Corollary 3.36 we have that $|\iint\limits_{(x,y)\in G_n^k} f(x,y)| \leqslant Ma(G_n^k)$, and hence $\lim\limits_{n\to\infty} \iint\limits_{(x,y)\in G_n^k} f(x,y) = 0$. Therefore, for every $k \in \mathbb{N}$, $\sum\limits_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^k} f(x,y)$ converges. Now, fix $\epsilon > 0$ in $\mathcal{R}$ and select a $k_0 \in \mathbb{N}$ such that $Md^{k_0} \leqslant \epsilon$. Let $i > j \geqslant k_0$ be given in $\mathbb{N}$. We can write $\bigcup\limits_{n=1}^{\infty} G_n^i \setminus \bigcup\limits_{n=1}^{\infty} G_n^j$ as the union of mutually disjoint simple regions $(G_n^{i,j})_{n=1}^{\infty}$ such that $\sum\limits_{n=1}^{\infty} a(G_n^{i,j})$ converges and

$$\begin{aligned}\sum_{n=1}^{\infty} a(G_n^{i,j}) &= \sum_{n=1}^{\infty} a(G_n^{i}) - \sum_{n=1}^{\infty} a(G_n^{j})\\ &\leqslant m(A) - \sum_{n=1}^{\infty} a(G_n^{j}) \leqslant d^j \leqslant d^{k_0}.\end{aligned}$$

It follows that

$$\begin{aligned}&\left|\sum_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^i} f(x,y) - \sum_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^j} f(x,y)\right| = \left|\sum_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^{i,j}} f(x,y)\right|\\ &\leqslant \sum_{n=1}^{\infty}\left|\iint\limits_{(x,y)\in G_n^{i,j}} f(x,y)\right| \leqslant \sum_{n=1}^{\infty} Ma(G_n^{i,j}) = M\sum_{n=1}^{\infty} a(G_n^{i,j}) \leqslant Md^{k_0} \leqslant \epsilon,\end{aligned}$$

Thus, $\left(\sum\limits_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^k} f(x,y)\right)_{k=1}^{\infty}$ is a Cauchy sequence and hence it converges. We call the limit of this sequence the integral of f over A and we denote it by $\iint\limits_{(x,y)\in A} f(x,y)$. Thus,

$$\iint\limits_{(x,y)\in A} f(x,y) = \lim_{\substack{\sum_{n=1}^{\infty} a(G_n)\to m(A),\\ \bigcup_{n=1}^{\infty} G_n\subset A,\, G_n\text{'s mutually disjoint,}\\ f \text{ is analytic on } G_n \text{ for every n}}} \sum_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n} f(x,y)$$

PROPOSITION 3.38. *Let $A \subset \mathcal{R}^2$ be measurable and let $\alpha \in \mathcal{R}$ be given. Then the function $f : A \to \{\alpha\}$ is measurable on A and*

$$\iint\limits_{(x,y)\in A} f = \iint\limits_{(x,y)\in A} \alpha = \alpha m(A).$$

PROOF. Since A is measurable, for every $k \in \mathbb{N}$ there exists a sequence of mutually disjoint simple regions $(G_n^k)_{n=1}^{\infty}$ such that $\bigcup\limits_{n=1}^{\infty} G_n^k \subset A$, $\sum\limits_{n=1}^{\infty} a(G_n^k)$ converges, and $m(A) - \sum\limits_{n=1}^{\infty} a(G_n^k) < d^k$. f is analytic on G_n^k for each n and for each k; moreover,

$$\iint_{(x,y)\in G_n^k} f = \iint_{(x,y)\in G_n^k} \alpha = \alpha a(G_n^k).$$

From Definition 3.37, it follows that

$$\iint\limits_{(x,y)\in A} f = \lim_{k\to\infty} \sum_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^k} f = \lim_{k\to\infty} \sum_{n=1}^{\infty} \alpha a(G_n^k) = \alpha \lim_{k\to\infty} \sum_{n=1}^{\infty} a(G_n^k) = \alpha m(A).$$

□

PROPOSITION 3.39. *Let $A \subset \mathcal{R}^2$ be measurable and let $f : A \to \mathcal{R}$ be a measurable function such that $f \le 0$ on A. Then*

$$\iint\limits_{(x,y)\in A} f(x,y) \leqslant 0.$$

PROOF. Let $\epsilon > 0$ in $\mathcal{R}$ be given. There exists a sequence of mutually disjoint simple regions $(G_n)_{n=1}^{\infty}$ such that f is analytic on G_n for each n, $\bigcup\limits_{n=1}^{\infty} G_n \subset A$, $\sum\limits_{n=1}^{\infty} a(G_n)$ converges, and $m(A) - \sum\limits_{n=1}^{\infty} a(G_n) \leqslant \epsilon$. We have from Lemma 3.34 that for every $n \in \mathbb{N}$,

$$\iint_{(x,y)\in G_n} f(x,y) \leqslant 0.$$

Thus,

$$\sum_{n=1}^{\infty} \iint_{(x,y)\in G_n} f(x,y) \leqslant 0.$$

It follows that

$$\iint\limits_{(x,y)\in A} f(x,y) = \lim_{\epsilon\to 0} \sum_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n} f(x,y) \leqslant 0.$$

□

PROPOSITION 3.40. *Let $A \subset \mathcal{R}^2$ be measurable, let $f, g : A \to \mathcal{R}$ be measurable functions on A, and let $\alpha \in \mathcal{R}$ be given. Then*

$$\iint\limits_{(x,y)\in A} (f + \alpha g)(x,y) = \iint\limits_{(x,y)\in A} f(x,y) + \alpha \iint\limits_{(x,y)\in A} g(x,y).$$

Proof. Let $\epsilon > 0$ in $\mathcal{R}$ be given. Since f and g are measurable on A there exist two sequences of mutually disjoint simple regions $(G_n^f)_{n=1}^\infty$ and $(G_n^g)_{n=1}^\infty$ such that $\bigcup_{n=1}^{\infty} G_n^f$ and $\bigcup_{n=1}^{\infty} G_n^g$ are both subsets of A, for every $n \in \mathbb{N}$ f is analytic on G_n^f and g is analytic on G_n^g, $\sum_{n=1}^{\infty} a(G_n^f)$ and $\sum_{n=1}^{\infty} a(G_n^g)$ converge, and $m(A) - \sum_{n=1}^{\infty} a(G_n^f) \leqslant \frac{\epsilon}{2}$ and $m(A) - \sum_{n=1}^{\infty} a(G_n^g) \leqslant \frac{\epsilon}{2}$.

By Proposition 3.10 we can write $\left(\bigcup_{n=1}^{\infty} G_n^f\right) \bigcap \left(\bigcup_{n=1}^{\infty} G_n^g\right)$ as the union of mutually disjoint simple regions $(T_n)_{n=1}^\infty$ such that $\sum_{n=1}^{\infty} a(T_n)$ converges, $m(A) - \sum_{n=1}^{\infty} a(T_n) \leqslant \epsilon$, and for every $n \in \mathbb{N}$ f and g are both analytic on T_n . It follows from Lemma 3.33 that, for every $n \in \mathbb{N}$,

$$\iint_{(x,y)\in T_n} (f+\alpha g)(x,y) = \iint_{(x,y)\in T_n} f(x,y) + \alpha \iint_{(x,y)\in T_n} g(x,y).$$

Therefore,

$$\sum_{n=1}^{\infty} \iint_{(x,y)\in T_n} (f+\alpha g)(x,y) = \sum_{n=1}^{\infty} \iint_{T_n} f(x,y) + \alpha \sum_{n=1}^{\infty} \iint_{(x,y)\in T_n} g(x,y).$$

Thus,

$$\begin{aligned}
\iint_{(x,y)\in A} (f+\alpha g)(x,y) &= \lim_{\epsilon\to 0} \sum_{n=1}^{\infty} \iint_{(x,t)\in T_n} (f+\alpha g)(x,y) \\
&= \lim_{\epsilon\to 0} \sum_{n=1}^{\infty} \iint_{(x,y)\in T_n} f(x,y) + \alpha \lim_{\epsilon\to 0} \sum_{n=1}^{\infty} \iint_{(x,y)\in T_n} g(x,y) \\
&= \iint_{(x,y)\in A} f(x,y) + \alpha \iint_{(x,y)\in A} g(x,y).
\end{aligned}$$

□

Corollary 3.41. *Let $A \subset \mathcal{R}^2$ be measurable and let $f, g : A \to \mathcal{R}$ be measurable functions such that $f \leqslant g$ on A. Then*

$$\iint_{(x,y)\in A} f(x,y) \leqslant \iint_{(x,y)\in A} g(x,y).$$

Corollary 3.42. *Let $A \subset \mathcal{R}^2$ be measurable, let $f : A \to \mathcal{R}$ be a measurable function on A, and let M be a bound for $|f|$ on A. Then*

$$\left| \iint_{(x,y)\in A} f(x,y) \right| \leqslant \iint_{(x,y)\in A} |f(x,y)| \leqslant M m(A).$$

Proposition 3.43. *Let $A, B \subset \mathcal{R}^2$ be measurable sets and let f be a measurable function on A and B. Then*

$$\iint\limits_{(x,y)\in A\cup B} f(x,y) = \iint\limits_{(x,y)\in A} f(x,y) + \iint\limits_{(x,y)\in B} f(x,y) - \iint\limits_{(x,y)\in A\cap B} f(x,y).$$

Proof. Let $\epsilon > 0$ in $\mathcal{R}$ be given. Since f is a measurable function on $A \cup B$ there exists a sequence of mutually disjoint simple regions $(G_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} a(G_n)$ converges, $\bigcup_{n=1}^{\infty} G_n \subset A \cup B$, $m(A\cup B) - \sum_{n=1}^{\infty} a(G_n) \leqslant \frac{\epsilon}{2}$, and for every $n \in \mathbb{N}$, f is analytic on G_n.

We can arrange $(G_n)_{n=1}^{\infty}$ into three sequences of mutually disjoint simple regions $(G_n^1)_{n=1}^{\infty}$, $(G_n^2)_{n=1}^{\infty}$, and $(G_n^3)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} a(G_n^1)$, $\sum_{n=1}^{\infty} a(G_n^2)$, and $\sum_{n=1}^{\infty} a(G_n^3)$ converge; $\bigcup_{n=1}^{\infty} G_n^1 \subset A\backslash(A\cap B)$, $\bigcup_{n=1}^{\infty} G_n^2 \subset B\backslash(A\cap B)$, and $\bigcup_{n=1}^{\infty} G_n^3 \subset (A\cap B)$; $m(A\cup B) - \sum_{n=1}^{\infty} a(G_n^1) - \sum_{n=1}^{\infty} a(G_n^2) - \sum_{n=1}^{\infty} a(G_n^3) \leqslant \epsilon$. Thus,

$$\begin{aligned}
\iint\limits_{(x,y)\in A\cup B} f(x,y) &= \lim_{\epsilon\to 0}\sum_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n} f(x,y) \\
&= \lim_{\epsilon\to 0}\sum_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^1} f(x,y) + \lim_{\epsilon\to 0}\sum_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^2} f(x,y) \\
&+ \lim_{\epsilon\to 0}\sum_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^3} f(x,y) \\
&= \lim_{\epsilon\to 0}\sum_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^1} f(x,y) + \lim_{\epsilon\to 0}\sum_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^3} f(x,y) \\
&+ \lim_{\epsilon\to 0}\sum_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^2} f(x,y) + \lim_{\epsilon\to 0}\sum_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^3} f(x,y) \\
&- \lim_{\epsilon\to 0}\sum_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^3} f(x,y) \\
&= \lim_{\epsilon\to 0}\sum_{n=1}^{\infty} \left(\iint\limits_{(x,y)\in G_n^1} f(x,y) + \iint\limits_{(x,y)\in G_n^3} f(x,y) \right) \\
&+ \lim_{\epsilon\to 0}\sum_{n=1}^{\infty} \left(\iint\limits_{(x,y)\in G_n^2} f(x,y) + \iint\limits_{(x,y)\in G_n^3} f(x,y) \right) \\
&- \lim_{\epsilon\to 0}\sum_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^3} f(x,y).
\end{aligned}$$

By our choice of $(G_n^1)_{n=1}^\infty$, $(G_n^2)_{n=1}^\infty$, and $(G_n^3)_{n=1}^\infty$,

$$\lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\left(\iint_{(x,y)\in G_n^1} f(x,y)+\iint_{(x,y)\in G_n^3} f(x,y)\right)=\iint_{(x,y)\in A} f(x,y),$$

$$\lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\left(\iint_{(x,y)\in G_n^2} f(x,y)+\iint_{(x,y)\in G_n^3} f(x,y)\right)=\iint_{(x,y)\in B} f(x,y),$$

and

$$\lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\iint_{(x,y)\in G_n^3} f(x,y)=\iint_{(x,y)\in A\cap B} f(x,y).$$

Thus, we finally get that

$$\iint_{(x,y)\in A\cup B} f(x,y)=\iint_{(x,y)\in A} f(x,y)+\iint_{(x,y)\in B} f(x,y)-\iint_{(x,y)\in A\cap B} f(x,y).$$

$\square$

PROPOSITION 3.44. *Let $A\subset\mathcal{R}^2$ be measurable, let $f:A\to\mathcal{R}$, and for each $k\in\mathbb{N}$ let $f_k:A\to\mathcal{R}$ be a measurable function on A such that the sequence $(f_k)_{k=1}^\infty$ converges uniformly to f on A. Then $\lim_{k\to\infty}\iint_{(x,y)\in A} f_k(x,y)$ exists; moreover, if f is measurable on A then*

$$\lim_{k\to\infty}\iint_{(x,y)\in A} f_k(x,y)=\iint_{(x,y)\in A} f(x,y).$$

PROOF. Let $\epsilon>0$ in $\mathcal{R}$ be given and let

$$\epsilon_0=\begin{cases}\frac{\epsilon}{m(A)} & \text{if } m(A)\neq 0\\ \epsilon & \text{if } m(A)=0\end{cases}.$$

Then there exists a $k_0\in\mathbb{N}$ such that for every $i,j\geqslant k_0$, $|f_i(x,y)-f_j(x,y)|\leqslant\epsilon_0$ for every $(x,y)\in A$. Thus,

$$\begin{aligned}\left|\iint_{(x,y)\in A} f_i(x,y)-\iint_{(x,y)\in A} f_j(x,y)\right| &= \left|\iint_{(x,y)\in A}(f_i(x,y)-f_j(x,y))\right|\\ &\leqslant \iint_{(x,y)\in A}|f_i(x,y)-f_j(x,y)|\leqslant\epsilon_0 m(A)\leqslant\epsilon.\end{aligned}$$

Thus, $\left(\iint_{(x,y)\in A} f_k(x,y)\right)_{k=1}^{\infty}$ is a Cauchy sequence; since $\mathcal{R}$ is Cauchy complete, it follows that $\lim_{k\to\infty}\iint_{(x,y)\in A} f_k(x,y)$ exists.

Now assume that f is measurable on A. Let $\epsilon>0$ be given in $\mathcal{R}$ and let ϵ_0 be defined as above. Since $(f_k)_{k=1}^\infty$ converges uniformly to f, there exists a $k\in\mathbb{N}$ such that, for every $i \geqslant k$, we have that $|f_i(x,y) - f(x,y)| \leqslant \epsilon_0$ for every $(x,y) \in A$. It follows that

$$\left| \iint_{(x,y)\in A} f_i(x,y) - \iint_{(x,y)\in A} f(x,y) \right| \leqslant \iint_{(x,y)\in A} |f_i(x,y) - f(x,y)| \leqslant \epsilon_0 m(A) \leqslant \epsilon,$$

and hence $\lim_{k\to\infty} \iint_{(x,y)\in A} f_k(x,y) = \iint_{(x,y)\in A} f(x,y)$. □

Just as we used the measure theory and integration in one dimension to extend the results to two dimensions, we can use the measure theory and integration in two dimensions developed above to extend the results to three dimensions (and, by induction, to higher dimensions). In the following section, we only present the key steps needed for going from two to three dimensions; the details of the theory in three dimensions are similar to those in two dimensions and hence we will leave out those details but we refer the interested reader to [**3**].

4. Measure Theory and Integration on $\mathcal{R}^3$

Definition 4.1 (Simple Region). Let $S \subset \mathcal{R}^3$. Then we say S is a simple region in $\mathcal{R}^3$ if there exists a simple region $A \subset \mathcal{R}^2$ and two analytic functions $h_1, h_2 : A \to \mathcal{R}$ such that $h_1 < h_2$ everywhere on A and

$$S = \{(x,y,z) \in \mathcal{R}^3 : z \in I(h_1(x,y), h_2(x,y)), (x,y) \in A\}$$

or

$$S = \{(x,y,z) \in \mathcal{R}^3 : y \in I(h_1(x,z), h_2(x,z)), (x,z) \in A\}$$

or

$$S = \{(x,y,z) \in \mathcal{R}^3 : x \in I(h_1(y,z), h_2(y,z)), (y,z) \in A\}.$$

Definition 4.2 (Volume of a Simple Region). Let $S \subset \mathcal{R}^3$ be a simple region with A, h_1 and h_2 as in Definition 4.1. If S is of the form

$$S = \{(x,y,z) \in \mathcal{R}^3 : z \in I(h_1(x,y), h_2(x,y)), (x,y) \in A\}$$

then we define the volume of S, denoted by $v(S)$, as follows:

$$v(S) = \iint_{(x,y)\in A} [h_2(x,y) - h_1(x,y)].$$

We define $v(S)$ similarly if $S = \{(x,y,z) \in \mathcal{R}^3 : y \in I(h_1(x,z), h_2(x,z)), (x,z) \in A\}$ or $S = \{(x,y,z) \in \mathcal{R}^3 : x \in I(h_1(y,z), h_2(y,z)), (y,z) \in A\}$.

Definition 4.3 (Finite Simple Region and Order of Magnitude). Let S be a simple region given by

$$S = \{(x,y,z) \in \mathcal{R}^3 : z \in I(h_1(x,y), h_2(x,y)), (x,y) \in A\}.$$

We define $\lambda_x(S) = \lambda_x(A)$, $\lambda_y(S) = \lambda_y(A)$, and $\lambda_z(S) = i(h_2(x,y) - h_1(x,y))$, the index of the analytic function $h_2 - h_1$ on A; we call these the orders of magnitude of S in x, y and z, respectively. We say that S is a finite region in $\mathcal{R}^3$ if $\lambda_x(S) = \lambda_y(S) = \lambda_z(S) = 0$.

Definition 4.4 (Analytic Function in $\mathcal{R}^3$). Let $S \subset \mathcal{R}^3$ be a simple region and let $f : S \to \mathcal{R}$. Then we say that f is an analytic function on S if for every $(x_0, y_0, z_0) \in S$, there exists a simple region $A \subset \mathcal{R}^3$ containing (x_0, y_0, z_0) and a regular sequence $(a_{ijk})_{i,j,k=0}^{\infty}$ in $\mathcal{R}$ such that $\lambda_x(A) = \lambda_x(S)$, $\lambda_y(A) = \lambda_y(S)$, $\lambda_z(A) = \lambda_z(S)$, and if $(x_0 + r, y_0 + s, z_0 + t) \in S \cap A$ then

$$f(x_0 + r, y_0 + s, z_0 + t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{ijk} r^i s^j t^k,$$

where the power series converges in the weak topology.

Exactly as in the two-dimensional case, we can show that if $A \subset \mathcal{R}^3$ is a simple region, $f, g : A \to \mathcal{R}$ are two analytic functions on A, and $\alpha \in \mathcal{R}$ then $f + \alpha g$ and $f \cdot g$ are analytic functions on A. Moreover, we have the following result whose proof is similar to that of the corresponding result in two dimensions above and which is is useful for defining the iterated integral of an analytic function on a simple region in $\mathcal{R}^3$.

Proposition 4.5. *Let $A \subset \mathcal{R}^3$ be a simple region, let $f : A \to \mathcal{R}$ ba an analytic function on A, let $B \subset \mathcal{R}^2$ be a simple region, and let $g : B \to \mathcal{R}$ be an analytic function on B such that for every $(x, y) \in B$, $(x, y, g(x, y)) \in A$. Then $F : B \to \mathcal{R}$, given by $F(x, y) = f(x, y, g(x, y))$, is an analytic function on B.*

Definition 4.6 (Measurable Set). Let $S \subset \mathcal{R}^3$. Then we say that S is a measurable set if for every $\epsilon > 0$ there exists two sequences of mutually disjoint simple regions, $(G_n)_{n=1}^{\infty}$ and $(H_n)_{n=1}^{\infty}$, such that $\bigcup_{n=1}^{\infty} G_n \subset S \subset \bigcup_{n=1}^{\infty} H_n$, $\sum_{n=1}^{\infty} v(G_n)$ and $\sum_{n=1}^{\infty} v(H_n)$ converge, and $\sum_{n=1}^{\infty} v(H_n) - \sum_{n=1}^{\infty} v(G_n) < \epsilon$.

Definition 4.7 (Measure of a Measurable Set). Let $S \subset \mathcal{R}^3$ be a measurable set. Then for every $k \in \mathbb{N}$, there exist two sequences of mutually disjoint simple regions, $(G_n^k)_{n=1}^{\infty}$ and $(H_n^k)_{n=1}^{\infty}$, such that $\bigcup_{n=1}^{\infty} G_n^k \subset S \subset \bigcup_{n=1}^{\infty} H_n^k$, $\sum_{n=1}^{\infty} v(G_n^k)$ and $\sum_{n=1}^{\infty} v(H_n^k)$ converge, and $\sum_{n=1}^{\infty} v(H_n^k) - \sum_{n=1}^{\infty} v(G_n^k) < d^k$. We note that since for every $k \in \mathbb{N}$, $(G_n^k)_{n=1}^{\infty}$ and $(H_n^k)_{n=1}^{\infty}$ are mutually disjoint we can arrange them so that $\bigcup_{n=1}^{\infty} G_n^k \subset \bigcup_{n=1}^{\infty} G_n^{k+1} \subset S \subset \bigcup_{n=1}^{\infty} H_n^{k+1} \subset \bigcup_{n=1}^{\infty} H_n^k$.

We show that $\left(\sum_{n=1}^{\infty} v(G_n^k)\right)_{k=1}^{\infty}$ is a Cauchy sequence. Let $\epsilon > 0$ in $\mathcal{R}$ be given. Let $k_0 \in \mathbb{N}$ be large enough so that $d^{k_0} < \epsilon$. Then for $l > k > k_0$, we have that $\bigcup_{n=1}^{\infty} G_n^l \subset S \subset \bigcup_{n=1}^{\infty} H_n^k$ and hence $\sum_{n=1}^{\infty} v(G_n^l) \leqslant \sum_{n=1}^{\infty} v(H_n^k)$. Thus,

$$0 \leqslant \sum_{n=1}^{\infty} v(G_n^l) - \sum_{n=1}^{\infty} v(G_n^k) \leqslant \sum_{n=1}^{\infty} v(H_n^k) - \sum_{n=1}^{\infty} v(G_n^k) < d^k < d^{k_0} < \epsilon.$$

A similar argument shows that the sequence $(\sum_{n=1}^{\infty} v(H_n^k))_{k=1}^{\infty}$ is Cauchy.

Since $\mathcal{R}$ is Cauchy complete $\lim_{k\to\infty}\sum_{n=1}^{\infty} v(G_n^k)$ and $\lim_{k\to\infty}\sum_{n=1}^{\infty} v(H_n^k)$ both exist, and hence $\lim_{k\to\infty}\left(\sum_{n=1}^{\infty} v(H_n^k)-\sum_{n=1}^{\infty} v(G_n^k)\right)$ exists; moreover,

$$\lim_{k\to\infty}\left(\sum_{n=1}^{\infty} v(H_n^k)-\sum_{n=1}^{\infty} v(G_n^k)\right)=\lim_{k\to\infty}\sum_{n=1}^{\infty} v(H_n^k)-\lim_{k\to\infty}\sum_{n=1}^{\infty} v(G_n^k).$$

Furthermore, for every $k\in\mathbb{N}$, we have that

$$0\leqslant\sum_{n=1}^{\infty} v(H_n^k)-\sum_{n=1}^{\infty} v(G_n^k)<d^k,$$

and hence

$$0\leqslant\lim_{k\to\infty}\left(\sum_{n=1}^{\infty} v(H_n^k)-\sum_{n=1}^{\infty} v(G_n^k)\right)\leqslant 0.$$

It follows that

$$\lim_{k\to\infty}\left(\sum_{n=1}^{\infty} v(H_n^k)-\sum_{n=1}^{\infty} v(G_n^k)\right)=0;\ \text{and hence}\ \lim_{k\to\infty}\sum_{n=1}^{\infty} v(G_n^k)=\lim_{k\to\infty}\sum_{n=1}^{\infty} v(H_n^k).$$

We call the common limit the measure of S and we denote it by $m(S)$.

PROPOSITION 4.8. *Let $S\subset\mathcal{R}^3$ be a measurable set. Then*

$$\begin{aligned}m(S)=\inf&\left\{\sum_{n=1}^{\infty} v(H_n): H_n\text{'s are mutually disjoint simple regions,}\right.\\&\left.S\subset\bigcup_{n=1}^{\infty}H_n,\ \text{and}\ \sum_{n=1}^{\infty} v(H_n)\ \text{converges}\right\}\\&=\sup\left\{\sum_{n=1}^{\infty} v(G_n): G_n\text{'s are mutually disjoint simple regions,}\right.\\&\left.\bigcup_{n=1}^{\infty}G_n\subset S,\ \text{and}\ \sum_{n=1}^{\infty} v(G_n)\ \text{converges}\right\}.\end{aligned}$$

DEFINITION 4.9 (Measurable Function). Let $S\subset\mathcal{R}^3$ be a measurable set and let $f:S\to\mathcal{R}$ be bounded on S. Then we say that f is measurable on S if for every $\epsilon>0$, there exists a sequence of mutually disjoint simple regions $(G_n)_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty}G_n\subset S$, $\sum_{n=1}^{\infty} v(G_n)$ converges, $m(S)-\sum_{n=1}^{\infty} v(G_n)<\epsilon$, and f is analytic on G_n for every $n\in\mathbb{N}$.

DEFINITION 4.10 (Integral of an analytic function over a simple region in $\mathcal{R}^3$). Let $S\subset\mathcal{R}^3$ be a simple region, and let $f:S\to\mathcal{R}$ be an analytic function on S. First assume that S is of the form

$$S=\{(x,y,z)\in\mathcal{R}^3: z\in I(h_1(x,y),h_2(x,y)),(x,y)\in A\}$$

where A is a simple region in $\mathcal{R}^2$ and where $h_1, h_2 : A \to \mathcal{R}$ are analytic on A. Then we define

$$\iiint_{(x,y,z)\in S} f(x,y,z) = \iint_{(x,y)\in A} \left[\int_{z\in I(h_1(x,y),h_2(x,y))} f(x,y,z) \right]$$

and we call this the integral of f over S. Note that for fixed x and y, $f(x,y,z)$ is an $\mathcal{R}$-analytic function on the interval $I(h_1(x,y), h_2(x,y))$, and hence

$$\int_{z\in I(h_1(x,y),h_2(x,y))} f(x,y,z)$$

is well defined. Moreover, $F(x,y) := \int_{z\in I(h_1(x,y),h_2(x,y))} f(x,y,z)$ is an analytic function on A; thus, the integral is well-defined.

If S is of the form

$$S = \{(x,y,z) \in \mathcal{R}^3 : y \in I(h_1(x,z), h_2(x,z)), (x,z) \in A\}$$

then we define the integral of f over S by

$$\iiint_{(x,y,z)\in S} f(x,y,z) = \iint_{(x,z)\in A} \left[\int_{y\in I(h_1(x,z),h_2(x,z))} f(x,y,z) \right].$$

Finally, if S is of the form

$$S = \{(x,y,z) \in \mathcal{R}^3 : x \in I(h_1(y,z), h_2(y,z)), (y,z) \in A\}$$

then we define the integral of f over S by

$$\iiint_{(x,y,z)\in S} f(x,y,z) = \iint_{(y,z)\in A} \left[\int_{x\in I(h_1(y,z),h_2(y,z))} f(x,y,z) \right].$$

It follows readily from Definition 4.10 that if $S \subset \mathcal{R}^3$ is a simple region and $M \in \mathcal{R}$ a constant then $\iiint_{(x,y,z)\in S} M = Mv(S)$.

Definition 4.11 (Integral of a Measurable Function over a Measurable Set). Let $S \subset \mathcal{R}^3$ be measurable and let $f : S \to \mathcal{R}$ be measurable on S. Then the integral of f over S, denoted by $\iiint_{(x,y,z)\in S} f(x,y,z)$ or $\iiint_S f$, is given by

$$\iiint_S f = \lim_{\substack{\sum_{n=1}^{\infty} v(G_n) \to m(S) \\ \cup_{n=1}^{\infty} G_n \subset S \\ (G_n) \text{ are mutually disjoint} \\ f \text{ is analytic on } G_n\ \forall\ n}} \sum_{n=1}^{\infty} \iiint_{G_n} f.$$

That the limit exists follows the same arguments as in the one-dimensional and two-dimensional cases.

With the definition above, the triple integral has similar properties to those of the single and double integrals discussed in Section 2 and Section 3 above. In particular,

- If $S \subset \mathcal{R}^3$ is measurable, $f, g : S \to \mathcal{R}$ are measurable on S, and $\alpha \in \mathcal{R}$ then $f + \alpha g$ is measurable on S, with

$$\iiint_{(x,y,z)\in S} [f(x,y,z) + \alpha g(x,y,z)] = \iiint_{(x,y,z)\in S} f(x,y,z) + \alpha \iiint_{(x,y,z)\in S} g(x,y,z).$$

- If $S \subset \mathcal{R}^3$ is measurable and $M \in \mathcal{R}$ a given constant then

$$\iiint_{(x,y,z)\in S} M = Mm(S).$$

- If $S \subset \mathcal{R}^3$ is measurable and $f, g : S \to \mathcal{R}$ are measurable with $f \leqslant g$ everywhere on S then

$$\iiint_{(x,y,z)\in S} f(x,y,z) \leqslant \iiint_{(x,y,z)\in S} g(x,y,z).$$

- If $S \subset \mathcal{R}^3$ is measurable and $f : S \to \mathcal{R}$ is a measurable function satisfying $|f| \leq M$ on S then

$$\left| \iiint_{(x,y,z)\in S} f(x,y,z) \right| \leqslant \iiint_{(x,y,z)\in S} |f(x,y,z)| \leqslant Mm(S).$$

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