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Characterization of compact and self-adjoint operators on free Banach spaces of countable type over the complex Levi-Civita field

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Let $\mathcal{C}$ be the complex Levi-Civita field and let E be a free Banach space over $\mathcal{C}$ of countable type. Then E is isometrically isomorphic to $c_0(\mathbb{N}, \mathcal{C}, s) := \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathcal{C}; \lim_{n\to\infty} |x_n| s(n) = 0\}$, where $s : \mathbb{N} \to (0, \infty)$. If the range of s is contained in $|\mathcal{C} \setminus \{0\}|$, it is enough to study $c_0(\mathbb{N}, \mathcal{C})$, which will be denoted by $c_0(\mathcal{C})$ or, simply, c_0. In this paper, we define a natural inner product on c_0, which induces the sup-norm of c_0. Of course, c_0 is not orthomodular, so we characterize those closed subspaces of c_0 with an orthonormal complement with respect to this inner product; that is, those closed subspaces M of c_0 such that $c_0 = M \oplus M^{\perp}$. Such a subspace, together with its orthonormal complement, defines a special kind of projection, the so-called normal projection. We present a characterization of such normal projections as well as a characterization of another kind of operators, the compact operators on c_0. © *2013 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4789541]

I. INTRODUCTION

Two of the most useful and beautiful mathematical theories in real or complex functional analysis have been Hilbert spaces and continuous linear operators. These theories have exactly matched the needs of many branches of physics, biology, and other fields of science.

The importance of Hilbert spaces over real or complex fields has led many researchers to try and extend the concept to non-Archimedean fields. One of the first attempts to define an appropriate non-Archimedean inner product was made by G. K. Kalisch.[4] Two of the most recent papers about non-Archimedean Hilbert spaces are those of L. Narici and E. Beckenstein[5] and J. Aguayo and M. Nova.[1] They define a non-Archimedean inner product on a vector space X over a non-Archimedean valued field $\mathbb{K}$ as a non-degenerate $\mathbb{K}$-function in $X \times X$, which is linear in the first variable and satisfies what they call the Cauchy-Schwarz type inequality. The main problem that these researchers have faced is the orthomodular property, that is, for any subspace M of X; $M = M^{\perp\perp} \Leftrightarrow X = M \oplus M^{\perp}$. It is well known that real and complex Hilbert spaces are orthomodular. The existence of infinite-dimensional non-classical orthomodular spaces was an open question until the following interesting theorem was proved by M. P. Solér:[9] "Let X be an orthomodular space and suppose it contains an orthonormal sequence $e_1, e_2, \ldots$ (in the sense of the inner product). Then the base field is $\mathbb{R}$ or $\mathbb{C}$." Based on the result of Solér, if $\mathbb{K}$ is a non-Archimedean, complete valued field and $c_0(\mathbb{N}, \mathbb{K})$, or simply c_0, is the space of all null sequences $x = (x_n)_{n\in\mathbb{N}}$, $x_n \in \mathbb{K}$, equipped with the inner product defined by $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$, then c_0 is not an orthomodular space. It was proved in

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Ref. 5 that the inner product defined above induces $\|\cdot\|_\infty$ if and only if the residue class field of $\mathbb{K}$ is formally real. Unlike classical Hilbert spaces, however, c_0 is not orthomodular.

Let $\mathcal{C}$ be the complex Levi-Civita field and let E be a free Banach space of countable type over $\mathcal{C}$. It is known that any such free Banach of countable type space is isometrically isomorphic to

$$c_0(\mathbb{N}, \mathcal{C}, s) := \left\{(x_n)_{n\in\mathbb{N}} : x_n \in \mathcal{C};\ \lim_{n\to\infty} |x_n| s(n) = 0\right\},$$

where $s : \mathbb{N} \to (0, \infty)$. Of course, it could be that, for some $i \in \mathbb{N}$, $s(i) \notin |\mathcal{C} \setminus \{0\}|$. But, if the range of s is contained in $|\mathcal{C} \setminus \{0\}|$, it is enough to study $c_0(\mathbb{N}, \mathcal{C})$ [taking s to be the constant function 1], which will be denoted by $c_0(\mathcal{C})$ or, simply, c_0.

In this paper, we consider the complex Levi-Civita field $\mathcal{C}$ as $\mathbb{K}$, whose residue class field is not formally real. In $\mathcal{C}$, we take the natural involution $z \to \overline{z}$, and we analyze the space $c_0(\mathbb{N}, \mathcal{C}) = c_0$, which, we already know, is not orthomodular. Among other results, we characterize those closed subspaces of c_0 with an orthonormal complement with respect to this inner product; that is, those closed subspaces M of c_0 such that $c_0 = M \oplus M^\perp$. Such a subspace, together with its orthonormal complement, defines a special kind of projection, the so-called normal projection. We present a characterization of such normal projections as well as a characterization of another kind of operators, the compact operators on c_0.

II. A INNER PRODUCT IN c_0

Throughout this paper, $\mathcal{R}$ (resp. $\mathcal{C}$) will denote the real (resp. complex) Levi-Civita field; for a detailed study of $\mathcal{R}$ (and $\mathcal{C}$), we refer the reader to Refs. 7 and 8 and the references therein. Any $z \in \mathcal{C}$ (resp. $\mathcal{R}$) is a function from $\mathbb{Q}$ into $\mathbb{C}$ (resp. $\mathbb{R}$) with left-finite support. For $w \in \mathcal{R}$ (resp. $\mathcal{C}$), we will denote by $\lambda(w) = \min(\operatorname{supp}(w))$, $w \neq 0$ and $\lambda(0) = +\infty$. On the other hand, since each $z \in \mathcal{C}$ can be written as $z = x + iy$, where $x, y \in \mathcal{R}$, we have that $\lambda(z) = \min\{\lambda(x), \lambda(y)\}$. If we define

$$|z| = \begin{cases} e^{-\lambda(z)} & if \quad z \neq 0 \\ 0 & if \quad z = 0 \end{cases},$$

then $|\cdot|$ is a non-Archimedean absolute value in $\mathcal{C}$. It is not hard to prove that $(\mathcal{C}, \Delta)$, where Δ is the induced metric by $|\cdot|$, is a complete metric space. Note that, for any $z \in \mathcal{C}$,

$$|z| = e^{-\lambda(z)} = e^{-\min\{\lambda(x),\lambda(y)\}} = \max\left\{e^{-\lambda(x)}, e^{-\lambda(y)}\right\} = \max\{|x|, |y|\}.$$

In other words, $\mathcal{C}$ is topologically isomorphic to $\mathcal{R}^2$ provided with the product topology induced by $|\cdot|$ in $\mathcal{R}$.

We denote by $c_0(\mathcal{C})$, or simply c_0, the space

$$c_0 = \left\{z = (z_n)_{n\in\mathbb{N}} : z_n \in \mathcal{C};\ \lim_{n\to\infty} z_n = 0\right\}.$$

A natural non-Archimedean norm on c_0 is $\|z\|_\infty = \sup\{|z_n| : n \in \mathbb{N}\}$ and $(c_0, \|\cdot\|_\infty)$ is a Banach space. Writing $z_n = x_n + iy_n$ and $x = (x_n)_{n\in\mathbb{N}}$, $y = (y_n)_{n\in\mathbb{N}}$, we also have the equality

$$\|z\|_\infty = \max\left\{\|x\|_\infty, \|y\|_\infty\right\}.$$

Recall that a topological space is called separable if it has a countable dense subset. In the class of real or complex Hilbert spaces, we can distinguish two types: those spaces which are separable and those which are not separable. If E is a separable normed space over a non-Archimedean valued field $\mathbb{K}$, then each one-dimensional subspace is homeomorphic to $\mathbb{K}$, so $\mathbb{K}$ must be separable too. Nevertheless, we know that there exist non-Archimedean fields, which are not separable. Thus, for non-Archimedean normed spaces, the concept of separability cannot be used if $\mathbb{K}$ is not separable. However, by linearizing the notion of separability, we obtain a generalization, useful for every scalar field $\mathbb{K}$. A normed space E over a non-Archimedean valued field is said to be *of countable type* if it contains a countable subset whose linear hull is dense in E. An example of a normed space of countable type is $(c_0, \|\cdot\|_\infty)$, in particular, when $\mathbb{K}$ is the complex Levi-Civita field $\mathcal{C}$.

Let us consider the following form:

$$\langle \cdot, \cdot \rangle : c_0 \times c_0 \to \mathcal{C};\ \langle z, w \rangle = \sum_{n=1}^{\infty} z_n \overline{w_n}.$$

This form is well-defined since $lim_{n\to\infty} z_n \overline{w_n} = 0$ and, at the same time, it satisfies:

I.1 $z \neq 0 \Rightarrow \langle z, z \rangle \neq 0$;
I.2 $\langle a z^1 + b z^2, w \rangle = a\langle z^1, w \rangle + b\langle z^2, w \rangle$ for $a, b \in \mathcal{C}$ and $z^1, z^2, w \in c_0$;
I.3 $\langle z, w \rangle = \overline{\langle w, z \rangle}$ for $z, w \in c_0$;
I.4 $|\langle z, w \rangle|^2 \leq |\langle z, z \rangle|\,|\langle w, w \rangle|$ (the Cauchy-Schwarz inequality).

It follows from the definition that

$$\langle x, y \rangle = 0,\ \forall y \in c_0 \Rightarrow x = \theta,$$

which is referred to as the non-degeneracy condition.

Let

$$\|z\| := \sqrt{|\langle z, z \rangle|}.$$

Then, since $|2| = 1$, $\| \cdot \|$ is a non-Archimedean norm on c_0 (see Ref. 2).

The next theorem was proved in Ref. 5 and tells us when the non-Archimedean norm in a Banach space is induced by an inner product.

Theorem 1: *Let $(X, \| \cdot \|)$ be a $\mathbb{K}$-Banach space. Then, if $\|X\| \subset |\mathbb{K}|^{1/2}$ and every one-dimensional subspace of X is orthocomplemted, then X admits an inner product that induces the norm $\| \cdot \|$.*

If $X = c_0$ and $\mathbb{K} = \mathcal{C}$ then the conditions of the theorem above are satisfied. In fact, if $z \in c_0, z \neq \theta$, then $\lim_{n\to\infty} z_n = 0$, which implies that

$$\|z\| = \max\left\{\left|z_j\right| : j \in \mathbb{N}\right\} = \left|z_{j_0}\right| \in |\mathcal{C}|.$$

Now, since $|\mathcal{C}| \subset |\mathcal{C}|^{1/2}$, $\|c_0\| \subset |\mathcal{C}|^{1/2}$. The other condition is guaranteed by Lemma 2.3.19, p. 34 in Ref. 6. Let us show that $\langle \cdot, \cdot \rangle$ is one of the inner products that induce the $\| \cdot \|_\infty$ norm on c_0. First of all, note that if $z \in c_0$ and $e_n = \left(\delta_{i,n}\right)_{i\in\mathbb{N}}$, where $\delta_{i,n}$ is the Kronecker delta, then

$$z = \sum_{n=1}^{\infty} \langle z, e_n \rangle e_n,$$

whose convergence is with respect to the $\| \cdot \|_\infty$.

It is well-known that the residue class field of the Levi-Civita $\mathcal{R}$ is a formally real field; that is, for any finite collection $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of $\mathcal{R}$,

$$\left|\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2\right| = \max\left\{\left|\lambda_1^2\right|, \left|\lambda_2^2\right|, \ldots, \left|\lambda_n^2\right|\right\}. \tag{2.1}$$

This condition allows us to prove that $\|x\| = \sqrt{|\langle x, x \rangle|}$, for any $x \in c_0(\mathcal{R})$. Of course, the equality in Eq. (2.1) is not necessarily true in $\mathcal{C}$ since $|i^2 + 1| = 0$. However, we have the following lemma:

Lemma 1: *If $\{z_1, z_2, \ldots, z_n\} \subset \mathcal{C}$, then*

$$|z_1\overline{z_1} + z_2\overline{z_2} + \cdots + z_n\overline{z_n}| = \max\{|z_1\overline{z_1}|, |z_2\overline{z_2}|, \ldots, |z_n\overline{z_n}|\}.$$

Proof: If we denote by $|\cdot|_o$ the ordinary modulus in $\mathcal{C}$, then $|z_i|_o^2 = z_i\overline{z_i} \in \mathcal{R}$. Applying Eq. (2.1) to $\{|z_1|_o, |z_2|_o, \ldots, |z_n|_o\}$, we get the result. □

It follows from the above that $|\langle z, z \rangle| = \|z\|_\infty^2$ for any $z \in \mathcal{C}$; in fact,

$$|\langle z, z \rangle| = \left|\sum_{n=1}^{\infty} z_n\overline{z_n}\right| = \lim_{n\to\infty}\left|\sum_{i=1}^{n} z_i\overline{z_i}\right| = \lim_{n\to\infty}\max_{1\leq i\leq n}|z_i\overline{z_i}| = \max_{n\in\mathbb{N}}|z_n|^2 = \|z\|_\infty^2.$$

Definition 1: *A subset D of* c_0 *such that for all* $x, y \in D$, $x \neq y \Rightarrow \langle x, y\rangle = 0$, *is called a normal family. A countable normal family* $\{x_n : n \in \mathbb{N}\}$ *of unit vectors is called an orthonormal sequence.*

If $A \subset c_0$, then $[A]$ and $cl[A]$ will denote the linear and the closed linear span of A, respectively. If M is a subspace of c_0, then M^p will denote the subspace of all $y \in c_0$ such that $\langle y, x\rangle = 0$, for all $x \in M$. Since the definition of the inner product given in Ref. 6, p. 38, is included in the definition of inner product given here, the Gram-Schmidt procedure can be used.

Theorem 2: *If* (z_n) *is a sequence of linearly independent vectors in* c_0, *then there exists an orthonormal sequence* (y_n) *such that* $[\{z_1, \ldots, z_n\}] = [\{y_1, \ldots, y_n\}]$ *for every* $n \in \mathbb{N}$.

Definition 2: *A sequence* $(z^n)_{n\in\mathbb{N}}$ *of non-null vectors of* c_0 *has the Riemann-Lebesgue property (RLP) if for all* $z \in c_0$,

$$\lim_{n\to\infty} \langle z^n, z\rangle = 0.$$

Obviously, any orthonormal basis of c_0 has this property. The following theorem has been proved in Ref. 5.

Theorem 3: *If* $S \subset c_0$ *is a finite orthonormal subset, say* $\{z_1, \ldots, z_n\}$, *or is an orthonormal sequence* $(z_n)_{n\in\mathbb{N}}$, *which satisfies the RLP, then S can be extended to an orthonormal basis for* c_0; *that is, there exists a countable orthonormal sequence* $(w_n)_{n\in\mathbb{N}}$ *(possibly finite) such that* $S \cup \{w_n : n \in \mathbb{N}\}$ *is an orthonormal basis for* c_0.

The next lemma shows that a normal sequence in c_0 is also an orthogonal sequence in the van Rooij's sense (see Ref. 10 p. 57).

Lemma 2: *Let* $(x_n)_{n\in\mathbb{N}}$ *be a normal sequence in* c_0. *Then, for all* $k \in \mathbb{N}$, *we have that*

$$\left\| \sum_{j=1}^{k} \alpha_j x_{n_j} \right\|^2 = \max\left\{ \left\| \alpha_j x_{n_j} \right\|^2 : j = 1, ..., k \right\}.$$

Proof: Let $\{x_{n_j} : j = 1, \ldots, k\}$ be a finite subset of $\{x_n : n \in \mathbb{N}\}$. Then

$$\begin{aligned}
\left\| \sum_{j=1}^{k} \alpha_j x_{n_j} \right\|^2 &= \left| \left\langle \sum_{i=1}^{k} \alpha_i x_{n_i}, \sum_{j=1}^{k} \alpha_j x_{n_j} \right\rangle \right| \\
&= \left| \sum_{j=1}^{k} \langle \alpha_j x_{n_j}, \alpha_j x_{n_j}\rangle \right| = \left| \sum_{j=1}^{k} \alpha_j \overline{\alpha_j} \langle x_{n_j}, x_{n_j}\rangle \right| \\
&= \lim_{i\to\infty} \left| \sum_{j=1}^{k} \alpha_j \overline{\alpha_j} \sum_{l=1}^{i} x_{l,n_j} \overline{x_{l,n_j}} \right| \\
&= \lim_{i\to\infty} \left| \sum_{j=1}^{k} \sum_{l=1}^{i} \alpha_j x_{l,n_j} \overline{\alpha_j x_{l,n_j}} \right| \\
&= \lim_{i\to\infty} \max_{1\le l\le i; 1\le j\le k} \left| \alpha_j x_{l,n_j} \overline{\alpha_j x_{l,n_j}} \right| \\
&= \lim_{i\to\infty} \max_{1\le l\le i; 1\le j\le k} \left| \alpha_j x_{l,n_j} \right|^2
\end{aligned}$$

$$= \max_{1\le j\le k}\left(\lim_{i\to\infty}\max_{1\le l\le i}\left|\alpha_j x_{l,n_j}\right|^2\right)$$

$$= \max_{1\le j\le k}\max_{l\in\mathbb{N}}\left|\alpha_j x_{l,n_j}\right|^2$$

$$= \max_{1\le j\le k}\left\|\alpha_j x_{n_j}\right\|^2 .$$

□

Using the Gram-Schmidt process, we have the following theorem proved in Ref. 1:

Theorem 4: *Every closed subspace D of c_0 admits a countable orthonormal base, that is, an orthonormal sequence (y_n) such that $D = cl\,[\{y_n : n \in \mathbb{N}\}]$.*

A nonempty subset C of a normed space E over $\mathbb{K}$ is absolutely convex if $x, y \in C$, $\alpha, \beta \in \mathbb{K}$, $|\alpha| \le 1, |\beta| \le 1$ implies $\alpha x + \beta y \in C$. The absolutely (and closed) convex hull of A is denoted by $co(A)$ $(\overline{co}\,(A))$.

If E and F are normed spaces over $\mathbb{K}$, then $\mathcal{L}(E, F)$ will be the normed space consisting of all continuous linear maps from E into F. $\mathcal{L}(E, \mathbb{K})$ will be denoted by E' and $\mathcal{L}(E, E)$ will be denoted by $\mathcal{L}(E)$.

A nonempty subset C of a normed space E is convex if $x, y, z \in C$, $\alpha, \beta, \gamma \in \mathbb{K}$, $\alpha + \beta + \gamma = 1$ implies $\alpha x + \beta y + \gamma z \in C$.

A convex set in $\mathbb{K} = \mathcal{C}$ consisting of at least two points contains line segments, which are homeomorphic to the unit ball $B_{\mathcal{C}} = \{\alpha \in \mathcal{C} : |\alpha| \le 1\}$. Since $\mathcal{C}$ is not locally compact, convex compact sets of E are trivial. To overcome this difficulty we "convexify" the (pre)compactness concept as follows: A set C of E is called *compactoid* if for every $\epsilon > 0$ there exists a finite subset $S \subset E$ such that $C \subset B_E(\theta, \epsilon) + \overline{co}\,(S)$, where $B_E(\theta, \epsilon) = \{x \in E\colon \|x\| \le \epsilon\}$.

Let E and F be two normed spaces over K; a linear operator T from E into F is said to be compact if $T(B_E)$ is compactoid, where $B_E = \{x \in E\colon \|x\| \le 1\}$ is the unit ball of E. It was proved in Ref. 10 that if E and F are Banach spaces, then T is compact if and only if, for each $\epsilon > 0$, there exists a continuous linear operator of finite-dimensional range S such that $\|T - S\| \le \epsilon$.

It is well-known that the dual of c_0 is $c_0' \cong l^\infty$.

Definition 3: *A functional $f \in c_0'$ is called a Riesz functional if there exists $z \in c_0$ such that $f = \langle\cdot, z\rangle$. The space of all Riesz functionals of c_0' will be denoted by $(c_0)_{RF}$, i.e.,*

$$(c_0)_{RF} = \left\{f \in c_0' : f = \langle\cdot, z\rangle \text{ for some } z \in c_0\right\}.$$

Proposition 1: *Let $f \in c_0'$. Then, $f \in (c_0)_{RF}$ if and only if $\lim_{n\to\infty} f(e_n) = 0$. In this case, $f = \langle\cdot, z\rangle$ where $z = \left(\overline{f\,(e_n)}\right)_{n\in\mathbb{N}}$.*

Proof: If $f \in (c_0)_{RF}$ and $z \in c_0$ is such that $f = \langle\cdot, z\rangle$, $\lim_{n\to\infty} f(e_n) = \lim_{n\to\infty}\langle e_n, z\rangle = 0$. Conversely, if we take $z = \left(\overline{f\,(e_n)}\right)_{n\in\mathbb{N}} \in c_0$, then for any $w = (w_n)_{n\in\mathbb{N}} \in c_0$,

$$f\,(w) = \sum_{n=1}^{\infty} w_n f\,(e_n) = \sum_{n=1}^{\infty} w_n \overline{\overline{f\,(e_n)}} = \sum_{n=1}^{\infty} w_n \overline{z_n} = \langle w, z\rangle .$$

□

Remark 1: *Note that if we define $\Phi\colon (c_0)_{RF} \to c_0$ by $\Phi\,(f) = \left(\overline{f\,(e_n)}\right)_{n\in\mathbb{N}} = z$, then*

$$\|\Phi\,(f)\| = \|z\| = \frac{|\langle z, z\rangle|}{\|z\|} \le \|f\|$$

and on the other hand, for any $x \neq 0$ in c_0, we have that

$$\frac{|f(x)|}{\|x\|} = \frac{|\langle x, z\rangle|}{\|x\|} \leq \frac{\|x\|\,\|z\|}{\|x\|} = \|z\| = \|\Phi(f)\|,$$

from which we infer that $\|f\| \leq \|\Phi(f)\|$. Thus, summarizing, Φ is a linear isometry and bijective. Since c_0 is a closed subspace of l^∞, we have that $(c_0)_{RF}$ is also closed in l^∞.

Now we define the linear functional $e_i' : c_0 \to \mathcal{C};\ x \to e_i'(x) = x_i$, where $x = \sum_{j=1}^{\infty} x_j e_j$. It is clear that $e_i' \in c_0'$ and that $\left\|e_i'\right\| = 1$. Moreover, e_i' is a Riesz functional, since $\langle x, e_i\rangle = x_i = e_i'(x)$.

Proposition 2: $(c_0)_{RF} = cl\left[\left\{e_i' : i \geq 1\right\}\right] \cong c_0$.

Proof: By previous remark, $(c_0)_{RF}$ is closed in l^∞ and then $cl\left[\left\{e_i' : i \geq 1\right\}\right] \subseteq (c_0)_{RF}$. On the other hand, if $f \in (c_0)_{RF}$, there exists $z \in c_0$ such that $f = \langle \cdot, z\rangle$; since $z = \sum_{i=1}^{\infty} z_i e_i$, we have

$$\left| f(\cdot) - \sum_{i=1}^{n} \overline{z_i}\,\langle \cdot, e_i\rangle \right| = \left| \left\langle \cdot, z - \sum_{i=1}^{n} z_i e_i \right\rangle \right|$$

$$\leq \|\cdot\| \left\| z - \sum_{i=1}^{n} z_i e_i \right\|.$$

It follows that $f \in cl\left[\left\{e_i' : i \geq 1\right\}\right]$. Hence, $(c_0)_{RF} = cl\left[\left\{e_i' : i \geq 1\right\}\right]$. □

Using arguments given in Ref. 5, we can prove the next proposition.

Proposition 3: *If $f \in (c_0)_{RF}$, then any orthonormal basis of $N(f)$ has the Riemann-Lebesgue property. As a consequence, $N(f)$ has a normal complement: $N(f)^p = \{y \in c_0\colon \langle y, x\rangle = 0$ for all $x \in N(f)\}$.*

Remark 2: *The converse of Proposition 3 is also true and it was proved in Ref. 5.*

III. NORMAL PROJECTIONS AND ORTHONORMAL COMPLEMENTED SUBSPACES

Throughout the rest of the paper, $N(T)$ and $R(T)$ will denote the kernel and range of a linear operator $T\colon c_0 \to c_0$.

Definition 4: *A linear operator $P\colon c_0 \to c_0$ is said to be a normal projection if*

a. *P is continuous;*
b. $P^2 = P$;
c. *$\langle z, w\rangle = 0$, for all $z \in N(P)$ and for all $w \in R(P)$.*

Remark 3: *Assume that $P \neq \theta$ is a normal projection on c_0.*

1. *It follows from Definition 4 that $N(P)$ and $R(P)$ are both closed and $c_0 = N(P) \oplus R(P)$.*
2. *If I denotes the identity operator, then $I - P$ is also a normal projection. Moreover,*

$$N(I-P) = R(P) = N(P)^p \quad \text{and} \quad R(I-P) = N(P) = R(P)^p.$$

3. *Note that for any $z \in c_0$, $z - Pz \in N(P)$. Thus,*

$$\langle z - Pz, Pz\rangle = 0; \quad \text{and hence} \quad |\langle z, Pz\rangle| = |\langle Pz, Pz\rangle|.$$

By the Cauchy-Schwarz inequality, it follows that

$$\|Pz\|^2 = |\langle z, Pz\rangle| \leq \|z\|\,\|Pz\|$$

and hence

$$\|Px\| \leq \|x\| \quad \text{for all} \quad x \in c_0,$$

which implies that P is an orthoprojection in the van Rooij sense (see Ref. 10 p. 63). On the other hand, since $\|Py\| = \|y\|$ for any $y \in R(P)$, we have $\|P\| = 1$.

Theorem 5: *Let P be a normal projection. If $\{z_n : n \in \mathbb{N}\}$ is an orthonormal base of $N(P)$, then it has the Riemann-Lebesgue property.*

Proof: Take an arbitrary $z \in c_0$. We have to prove that $\lim_{n\to\infty}\langle z, z_n\rangle = 0$. If $z \in N(P)$, then $\lim_{n\to\infty}\langle z, z_n\rangle = 0$, since $\{z_n : n \in \mathbb{N}\}$ is a base on $N(P)$. Suppose that $z \notin N(P)$; since P is a normal projection, we have $z - Pz \in N(P)$ and $\langle Pz, z_n\rangle = 0$. From this fact, we have

$$\begin{aligned}\langle z, z_n\rangle &= \langle z - Pz, z_n\rangle + \langle Pz, z_n\rangle \\ &= \langle z - Pz, z_n\rangle,\end{aligned}$$

and then

$$\lim_{n\to\infty}\langle z, z_n\rangle = \lim_{n\to\infty}\langle z - Pz, z_n\rangle = 0.$$

□

Proposition 4: *Let M be a closed subspace of c_0. If M has an orthonormal base with the Riemann-Lebesgue property, then there exists a normal projection P such that $M = N(P)$.*

Proof: By Corollary 8.2 of Ref. 5, we have $c_0 = M \oplus M^p$. If $z \in c_0$, then there exists a unique pair $(u, v) \in M \times M^p$ such that $z = u + v$. If we define $P(z) = v$, then P is the normal projection that satisfies the statement. □

Corollary 1: *Let M be an infinite dimensional closed subspace of c_0. Then, the following statements are equivalent:*

1. *M has a normal complement.*
2. *M has an orthonormal base with the Riemann-Lebesgue property.*
3. *There exists a normal projection P such that $N(P) = M$.*

Corollary 2: *Let M be a closed subspace of c_0. Then,*

1. *If M is finite dimensional or has an orthonormal base with the Riemann-Lebesgue property, then M^p has an orthonormal base with the Riemann-Lebesgue property or is finite dimensional.*
2. *If M has an orthonormal base with the Riemann-Lebesgue property, then any other orthonormal base has the same property.*

Proof: These statements are direct consequences of Proposition 4, Theorem 5, Remark 3 (part (2)), and Corollary 1. □

For $i, j \in \mathbb{N}$, we define $e'_j \otimes e_i : c_0 \to c_0$ by $e'_j \otimes e_i(z) = \langle z, e_j\rangle e_i$. Note that $e'_j \otimes e_i$ is a linear operator and

$$\left\|e'_j \otimes e_i(z)\right\| = \left\|\langle z, e_j\rangle e_i\right\| \leq \|z\|,$$

that is, $e'_j \otimes e_i \in \mathcal{L}(c_0)$. Moreover, $\|e'_j \otimes e_i\| = 1$.

Diarra in Ref. 3 has proved the following two lemmas:

Lemma 3: *Suppose that $(\alpha_{i,j})_{i,j\geq 1}$ is a bounded sequence of elements of $\mathcal{C}$ such that $\lim_{i\to\infty}\alpha_{ij} = 0$, for each $j \in \mathbb{N}$. Then, the operator u: $c_0 \to c_0$ defined by $u = \sum_{i,j\geq 1}\alpha_{ij}e'_j \otimes e_i$ is a continuous linear operator. Conversely, if $u \in \mathcal{L}(c_0)$, then $u = \sum_{i,j\geq 1}\alpha_{ij}e'_j \otimes e_i$ for some bounded sequence $(\alpha_{i,j})_{i,j\geq 1}$ of elements of $\mathcal{C}$ such that $\lim_{i\to\infty}\alpha_{ij} = 0$, for each $j \in \mathbb{N}$.*

It follows from lemma 3 that any continuous linear operator $u \in \mathcal{L}(c_0)$ can be identified with the following matrix whose columns converge to 0:

$$[u] = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1j} & \cdots \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots & \alpha_{2j} & \cdots \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \cdots & \alpha_{3j} & \cdots \\ \vdots & & & \ddots & & \\ \alpha_{i1} & \alpha_{i2} & \alpha_{i3} & \cdots & \alpha_{ij} & \cdots \\ \vdots & & & & & \ddots \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \cdots \\ 0 & 0 & 0 & & 0 & \end{pmatrix}.$$

Definition 5: *A linear operator $v : c_0 \to c_0$ is said to be an adjoint of a given operator $u \in \mathcal{L}(c_0)$ if $\langle u(x), y\rangle = \langle x, v(y)\rangle$, for all $x, y \in c_0$. In that case, we will say that u admits an adjoint v. We will also say that u is self-adjoint if $v = u$.*

Proposition 5: *If a continuous linear operator u has an adjoint, then it is unique and continuous.*

Proof: If v and $\widetilde{v}$ are adjoints of u and $y \in c_0$,then $\langle x, v(y) - \widetilde{v}(y)\rangle = 0$, for all x, which implies that $v = \widetilde{v}$. On the other hand, for $y \in c_0, y \neq \theta$, we have that

$$\begin{aligned} \|v(y)\|^2 &= |\langle v(y), v(y)\rangle| = |\langle u(v(y)), y\rangle| \\ &\leq \|u(v(y))\| \|y\| \leq \|u\| \|v(y)\| \|y\|, \end{aligned}$$

which implies that

$$\frac{\|v(y)\|}{\|y\|} \leq \|u\|.$$

Thus, $v \in \mathcal{L}(c_0)$. □

Lemma 4: *Let $u \in \mathcal{L}(c_0)$ with associated matrix $(\alpha_{i,j})_{i,j\geq 1}$. Then, u admits an adjoint operator v if and only if $\lim_{j\to\infty} \alpha_{ij} = 0$, for each $i \in \mathbb{N}$. In matrix terms, this means that*

$$[u] = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1j} & \cdots & \to 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots & \alpha_{2j} & \cdots & \to 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \cdots & \alpha_{3j} & \cdots & \to 0 \\ \vdots & & & \ddots & & & \\ \alpha_{i1} & \alpha_{i2} & \alpha_{i3} & \cdots & \alpha_{ij} & \cdots & \to 0 \\ \vdots & & & & & \ddots & \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \cdots & \\ 0 & 0 & 0 & & 0 & & \end{pmatrix}.$$

In the classical Hilbert space theory, any continuous linear operator admits an adjoint. This is not true in the non-Archimedean case. For example, the operator $u \in \mathcal{L}(c_0)$ given by the matrix:

$$\begin{pmatrix} b & b^2 & b^3 & \cdots & b^j & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & & & & & \ddots \end{pmatrix},$$

with $1 < |b|$, does not admit an adjoint, by Lemma 4.

The following result will be useful for the purposes of this section.

Theorem 6: *Let $P \in \mathcal{L}(c_0)$. If P is a normal projection, then P is self-adjoint. Conversely, if P is self-adjoint and $P^2 = P$ then it is a normal projection.*

Proof: ($\Rightarrow$): Assume P is a normal projection. Then, for any $x, y \in c_0$, we have that $x - Px, y - Py \in N(P)$; hence,

$$\langle x, Py\rangle = \langle Px, Py\rangle \text{ and } \langle y, Px\rangle = \langle Py, Px\rangle .$$

It follows that

$$\langle Px, y\rangle = \overline{\langle y, Px\rangle} = \overline{\langle Py, Px\rangle} = \langle Px, Py\rangle = \langle x, Py\rangle ;$$

and hence P is self-adjoint.

($\Leftarrow$): Suppose that $P^2 = P$ and $\langle x, Py\rangle = \langle Px, y\rangle$, for any $x, y \in c_0$. Let $x \in N(P)$ and $y \in R(P)$. Then $y = Pz$ for some $z \in c_0$; hence,

$$\langle x, y\rangle = \langle x, Pz\rangle = \langle Px, z\rangle = \langle 0, z\rangle = 0.$$

Therefore, P is a normal projection. □

Let f be a Riesz functional; that is, $f = \langle \cdot , y\rangle$, for some $y \in c_0$, $y = (y_n)_{n\in\mathbb{N}} \neq 0$. The operator u: $c_0 \to c_0$, defined by the following matrix:

$$[u] = \begin{pmatrix} \overline{y_1} & \overline{y_2} & \overline{y_3} & \overline{y_4} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

admits an adjoint operator.

The operator P, described by the matrix

$$[P] = \frac{1}{\langle y, y\rangle}\left[u^* u\right] = \frac{1}{\langle y, y\rangle}\begin{pmatrix} \overline{y}_1 y_1 & \overline{y}_2 y_1 & \overline{y}_3 y_1 & \overline{y}_4 y_1 & \cdots \\ \overline{y}_1 y_2 & \overline{y}_2 y_2 & \overline{y}_3 y_2 & \overline{y}_4 y_2 & \cdots \\ \overline{y}_1 y_3 & \overline{y}_2 y_3 & \overline{y}_3 y_3 & \overline{y}_4 y_3 & \cdots \\ \overline{y}_1 y_4 & \overline{y}_2 y_4 & \overline{y}_3 y_4 & \overline{y}_4 y_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is a normal projection with $N(P) = N(\langle \cdot , y\rangle)$ and $Pz = \frac{\langle z, y\rangle}{\langle y, y\rangle} y$. Our purpose next is to generalize this result.

First, let $\{y^1, y^2, \ldots, y^n\}$ be a finite subset of non-null elements of c_0 such that $\langle y^i, y^j\rangle = 0$, $i \neq j$. The operator defined by

$$P(\cdot) = \sum_{i=1}^{n} \frac{\langle \cdot, y^i\rangle}{\langle y^i, y^i\rangle} y^i$$

is a normal projection with $N(P) = \cap_{i=1}^{n} N\left(\langle \cdot, y^i\rangle\right)$.

Now, suppose that $(y^n)_{n\in\mathbb{N}}$ is an orthonormal sequence in c_0 with the RLP. Then, as before, for each $i \in \mathbb{N}$, we define the normal projection

$$P_i(\cdot) = \frac{\langle \cdot, y^i\rangle}{\langle y^i, y^i\rangle} y^i.$$

Since $\lim_{i\to\infty} P_i(x) = \lim_{i\to\infty} \frac{\langle x, y^i\rangle}{\langle y^i, y^i\rangle} y^i = 0$, we have that $\sum_{i=1}^{\infty} \frac{\langle \cdot, y^i\rangle}{\langle y^i, y^i\rangle} y^i$ converges pointwise in c_0. Now, if we define $P(\cdot) = \sum_{i=1}^{\infty} \frac{\langle \cdot, y^i\rangle}{\langle y^i, y^i\rangle} y^i$, then P is linear, continuous and $R(P) = \overline{\left[\left\{y^i : i \in \mathbb{N}\right\}\right]}$. Since $P(y^j) = y^j$, we also have

$$P^2(x) = P(Px) = P\left(\sum_{i=1}^{\infty} \frac{\langle x, y^i\rangle}{\langle y^i, y^i\rangle} y^i\right) = \sum_{i=1}^{\infty} \frac{\langle x, y^i\rangle}{\langle y^i, y^i\rangle} P\left(y^i\right) = \sum_{i=1}^{\infty} \frac{\langle x, y^i\rangle}{\langle y^i, y^i\rangle} y^i = Px,$$

and

$$\langle x, Py\rangle = \left\langle x, \sum_{i=1}^{\infty} \frac{\langle y, y^i\rangle}{\langle y^i, y^i\rangle} y^i\right\rangle = \sum_{i=1}^{\infty} \frac{\overline{\langle y, y^i\rangle}}{\langle y^i, y^i\rangle} \langle x, y^i\rangle = \sum_{i=1}^{\infty} \frac{\langle y^i, y\rangle}{\langle y^i, y^i\rangle} \langle x, y^i\rangle$$

$$= \sum_{i=1}^{\infty} \frac{\langle x, y^i\rangle}{\langle y^i, y^i\rangle} \langle y^i, y\rangle = \left\langle \sum_{i=1}^{\infty} \frac{\langle x, y^i\rangle}{\langle y^i, y^i\rangle} y^i, y\right\rangle = \langle Px, y\rangle.$$

That is, P is a projection and self-adjoint. By Theorem 6, P is a normal projection.

The next result states a characterization for normal projections:

Theorem 7: *If $P: c_0 \to c_0$ is a normal projection with $R(P) = \overline{[\{y_1, y_2, \ldots\}]}$, where $\{y_1, y_2, \ldots\}$ is an orthonormal finite subset of c_0 or an orthonormal sequence with the Riemann-Lebesgue property, then $Px = \sum_{i=1}^{\infty} \frac{\langle x, y_i\rangle}{\langle y_i, y_i\rangle} y_i$.*

Proof: Let $x \in c_0$; hence $Px = \sum_{i=1}^{\infty} \alpha_i(x) y_i$, where $\alpha_i \in c_0'$, for each $i \in \mathbb{N}$. Now, for this x, we have

$$\langle x, y_j\rangle = \langle x, Py_j\rangle = \langle Px, y_j\rangle = \left\langle \sum_{i=1}^{\infty} \alpha_i(x) y_i, y_j\right\rangle$$

$$= \sum_{i=1}^{\infty} \alpha_i(x) \langle y_i, y_j\rangle = \alpha_j(x) \langle y_j, y_j\rangle.$$

Thus, $\alpha_j(x) = \frac{\langle x, y_j\rangle}{\langle y_j, y_j\rangle}$.

IV. CHARACTERIZATION OF COMPACT OPERATORS

Recall that a continuous linear operator T is compact if and only if T is the uniform limit of continuous linear operators of finite dimensional range (see Ref. 10). Moreover, each of the previous statements is equivalent to "$R(T)$ contains no infinite-dimensional subspace that is closed in c_0'' (see Theorem 4.40-γ in Ref. 10). It follows that if P is a normal projection and $\dim R(P) = \infty$ then P cannot be compact, since $R(P)$ is closed. Hence, the convergence of $P(\cdot) = \sum_{i=1}^{\infty} \frac{\langle \cdot, y^i\rangle}{\langle y^i, y^i\rangle} y^i$ is only pointwise; otherwise, P would be a uniform limit of continuous linear operators with finite-dimensional range, say $\left(\sum_{i=1}^{n} P_i(\cdot)\right)_{n\in\mathbb{N}}$, and therefore P would be compact.

The following theorem provides a way to construct compact operators starting from an orthonormal sequence.

Theorem 8: *Let $(y_i)_{i\in\mathbb{N}}$ be an orthonormal sequence in c_0. Then, for any $\lambda = (\lambda_i)_{i\in\mathbb{N}}$ in c_0 such that $\lambda_i \in \mathcal{R}$, the map T: $c_0 \to c_0$ defined by*

$$T(\cdot) = \sum_{i=1}^{\infty} \lambda_i P_i(\cdot)$$

is a compact and self-adjoint operator, where $P_i(\cdot) = \frac{\langle \cdot, y_i\rangle}{\langle y_i, y_i\rangle} y_i$.

Proof: Clearly, T is well-defined and linear. Note that if $x \in c_0$, then

$$\begin{aligned}\left\| Tx - \sum_{i=1}^{n} \lambda_i P_i x \right\| &= \left\| \sum_{j=n+1}^{\infty} \lambda_j \frac{\langle x, y_j\rangle}{\langle y_j, y_j\rangle} y_j \right\| \\ &\leq \max\left\{ \left|\lambda_j\right| \left| \frac{\langle x, y_j\rangle}{\langle y_j, y_j\rangle} \right| \left\| y_j \right\| : j = n+1, n+2, \ldots \right\} \\ &\leq \max\left\{ \left|\lambda_j\right| \|x\| : j = n+1, n+2, \ldots \right\} \\ &= \|x\| \max\left\{ \left|\lambda_j\right| : j = n+1, n+2, \ldots \right\}.\end{aligned}$$

Then,

$$\begin{aligned}\left\| T - \sum_{i=1}^{n} \lambda_i P_i \right\| &= \sup_{x\neq\theta} \frac{\left\| Tx - \sum_{i=1}^{n} \lambda_i P_i(x) \right\|}{\|x\|} \\ &\leq \max\{|\lambda_i| : i = n+1, n+2, \ldots\} \underset{n\to\infty}{\to} 0.\end{aligned}$$

Since each P_i has finite-dimensional range and the convergence is uniform, T should be a compact operator. The self-adjoint condition follows easily. □

Let us consider the operator M_a, with $a = (a_i)_{i\in\mathbb{N}} \in c_0$, whose associated matrix is given by:

$$[M_a] = \begin{pmatrix} a_1 & 0 & 0 & 0 & \cdots \\ 0 & a_2 & 0 & 0 & \cdots \\ 0 & 0 & a_3 & 0 & \cdots \\ 0 & 0 & 0 & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.1}$$

Thus, in terms of linear operators, $M_a = \sum_{i=1}^{\infty} a_i P_i(\cdot)$, where $P_i(\,\cdot\,) = \langle\,\cdot\,, e_i\rangle e_i$.

Clearly, M_a is a self-adjoint and compact operator, but it is not, in general, a normal projection, since

$$M_a^2(e_i) = a_i^2 e_i \neq a_i e_i = M_a(e_i).$$

Therefore, there are compact operators, which are not projections.

Now, for a self-adjoint and compact linear operator T, the following question can be formulated: Does there exist an element $(\lambda_i)_{i\in\mathbb{N}} \in c_0(\mathcal{R})$ and a sequence $(P_i)_{i\in\mathbb{N}}$ of normal projections such that $T = \sum_{i=1}^{\infty} \lambda_i P_i$?

The rest of this section will be devoted to answer this question. First, we consider compact linear operators with finite range.

Theorem 9: *Let T be a compact linear operator of finite-dimensional range and self-adjoint. Then, there exists a finite collection $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ in $\mathcal{R}$ such that*

$$T = \sum_{i=1}^{n} \lambda_i P_i,$$

where, for each $i \in \{1, 2, \ldots, n\}$, P_i is a normal projection.

Proof: Since the range of T is finite-dimensional, we have that $R(T)$ is generated by a finite subset $\{y_1, y_2, \ldots, y_n\}$; that is, $R(T) = [\{y_1, y_2, \ldots, y_n\}]$. Without loss of generality, we can assume that $\{y_1, y_2, \ldots, y_n\}$ is linearly independent. By the Gram-Schmidt process, we can also assume that $\langle y_i, y_j \rangle = 0$, for $i \neq j$ and $\|y_i\| = 1$. Now, for any $x \in c_0$, we have that

$$T(x) = \sum_{i=1}^{n} \alpha_i(x)\, y_i.$$

By the continuity of T, $\alpha_i \in c_0'$ for $i = 1, \ldots, n$. Thus, since T is self-adjoint and $\langle y_i, y_j \rangle = 0$ for $i \neq j$, we have that

$$\langle x, T y_j \rangle = \langle T x, y_j \rangle = \left\langle \sum_{i=1}^{n} \alpha_i(x)\, y_i, y_j \right\rangle = \alpha_j(x) \langle y_j, y_j \rangle .$$

Therefore, $\alpha_j(\cdot) = \frac{\langle \cdot, T y_j \rangle}{\langle y_j, y_j \rangle}$ and

$$T(x) = \sum_{i=1}^{n} \frac{\langle x, T y_i \rangle}{\langle y_i, y_i \rangle} y_i. \tag{4.2}$$

Now, for $y_j \in c_0$, let us solve the equation for the unknowns λ_j, $j = 1, \ldots, n$

$$T y_j = \lambda_j y_j.$$

Thus,

$$\sum_{i=1}^{n} \frac{\langle y_j, T y_i \rangle}{\langle y_i, y_i \rangle} y_i = \lambda_j y_j,$$

or

$$\sum_{i \neq j} \frac{\langle y_j, T y_i \rangle}{\langle y_i, y_i \rangle} y_i + \left[\frac{\langle y_j, T y_j \rangle}{\langle y_j, y_j \rangle} - \lambda_j \right] y_j = 0.$$

Since $\{y_1, y_2, \ldots, y_n\}$ is linearly independent, we infer that

$$\langle y_j, T y_i \rangle \underset{i \neq j}{=} 0 \quad \wedge \quad \lambda_j = \frac{\langle y_j, T y_j \rangle}{\langle y_j, y_j \rangle}$$

for each $j = 1, \ldots, n$. Note that

$$\overline{\langle y_j, T y_j \rangle} = \langle T y_j, y_j \rangle = \langle y_j, T y_j \rangle,$$

and hence, for each j, $\lambda_j \in \mathcal{R}$. With $\lambda_j = \frac{\langle y_j, T y_j\rangle}{\langle y_j, y_j\rangle}$ for $j = 1, \ldots, n$, we note that

$$\sum_{i=1}^{n} \lambda_i \frac{\langle x, y_i\rangle}{\langle y_i, y_i\rangle} y_i = \sum_{i=1}^{n} \frac{\langle y_i, T y_i\rangle}{\langle y_i, y_i\rangle} \frac{\langle x, y_i\rangle}{\langle y_i, y_i\rangle} y_i$$

$$= \sum_{i=1}^{n} \frac{\left\langle x, \frac{\langle y_i, T y_i\rangle}{\langle y_i, y_i\rangle} y_i \right\rangle}{\langle y_i, y_i\rangle} y_i$$

$$= \sum_{i=1}^{n} \frac{\langle x, T y_i\rangle}{\langle y_i, y_i\rangle} y_i = T(x).$$

Therefore,

$$T(x) = \sum_{i=1}^{n} \frac{\langle x, T y_i\rangle}{\langle y_i, y_i\rangle} y_i = \sum_{i=1}^{n} \lambda_i \frac{\langle x, y_i\rangle}{\langle y_i, y_i\rangle} y_i.$$

□

Before we go forward to study compact operators with infinite-dimensional range, we recall some important facts that will be useful in the proof.

Given a compact operator $T \in \mathcal{L}(c_0)$, the subset $X = T(B)$ is compactoid, where B is the unit ball of c_0. Now, since X is also absolutely convex, we have that each orthogonal sequence of elements of X is convergent to 0 (Theorem 4.37 in Ref. 10 and Lemma 2 above).

On the other hand, it was proved in Ref. 1 that each linear subspace of c_0 is of countable type, in particular, $\overline{R(T)}$.

Theorem 10: *Let T: $c_0 \to c_0$ be a compact, self-adjoint linear operator of infinite dimensional range. Then there exists an element $\lambda = (\lambda_n)_{n\in\mathbb{N}} \in c_0(\mathcal{R})$ and an orthonormal sequence $(y_n)_{n\in\mathbb{N}}$ in c_0 such that*

$$T = \sum_{n=1}^{\infty} \lambda_n P_n,$$

where

$$P_n = \frac{\langle \cdot, y_n\rangle}{\langle y_n, y_n\rangle} y_n$$

is a normal projection defined by y_n.

Proof: Let us denote by X the set $T(B)$, where B is the unit ball of c_0. Then X is compactoid in c_0. Let $s \in \mathcal{C}$ be such that $0 < |s| < 1$; that is, $\lambda(s) > 0$. Choose $\beta, \alpha_1, \alpha_2, \ldots \in \mathcal{C}$ and $v_1, v_2, \ldots \in (0, \infty)$ such that

$$|\beta|^2 < |s|^{-1};$$

$$1 < v_n < |\alpha_n|; \quad |\alpha_1 \alpha_2 \cdots \alpha_n| < |\beta| \quad \forall n \in \mathbb{N}.$$

First step: We will construct inductively an orthogonal sequence $(a_n)_{n\in\mathbb{N}}$ in c_0, such that

$$\|a_n\| \geq v_n^{-1} \sup_{x\in X} \|H_{n-1} x\|,$$

where H_k are normal projections of the form

$$H_k(x) = x - \sum_{i=1}^{k} \frac{\langle x, a_i\rangle}{\langle a_i, a_i\rangle} a_i$$

with the properties

$$H_n H_k = H_n; \;\; k = 0, \ldots, n,$$

$$H_{k-1} - H_k \text{ are normal projections for } k = 0, \ldots, n$$

and

$$\left\langle a_i, a_j \right\rangle = 0; \; i, j = 1, \ldots, n \text{ and } i \neq j.$$

In fact, set $H_0 = I$ (I the identity operator) and choose $a_1 \in H_0(X) = X$ such that

$$\|a_1\| \geq v_1^{-1} \sup_{x \in X} \|H_0 x\| = v_1^{-1} \sup_{x \in X} \|x\| .$$

Since T is non-null, such an a_1 is not θ. By one of the properties of c_0, the subspace $[\{a_1\}]$ has a normal complement, say $[\{a_1\}]^p$. We define the operator H_1 by

$$H_1 x = x - \frac{\langle x, a_1 \rangle}{\langle a_1, a_1 \rangle} a_1,$$

which is a normal projection whose range is $[\{a_1\}]^p$. Also, H_1 satisfies

$$H_1 H_0 = H_1; \;\; H_1 H_1 = H_1, \text{ and } H_0 - H_1 \text{ is a normal projection.}$$

Suppose that we have obtained $a_n \in H_{n-1}(X)$ such that

$$\|a_n\| \geq v_n^{-1} \sup_{x \in X} \|H_{n-1} x\| ,$$

where H_{n-1} is the normal projection given by

$$H_{n-1}(x) = x - \sum_{i=1}^{n-1} \frac{\langle x, a_i \rangle}{\langle a_i, a_i \rangle} a_i,$$

whose range is $[\{a_1, \ldots, a_{n-1}\}]^p$ and

$$H_{n-1} H_i = H_{n-1}, \; i = 0, 1, \ldots, n-1; \quad H_{n-2} - H_{n-1} \text{ is a normal projection; and}$$

$$\left\langle a_i, a_j \right\rangle = 0, \; i, j = 1, \ldots, n, \; i \neq j$$

Since $a_n \neq \theta$, we construct

$$H_n(x) = H_{n-1}(x) - \frac{\langle x, a_n \rangle}{\langle a_n, a_n \rangle} a_n = x - \sum_{i=1}^{n} \frac{\langle x, a_i \rangle}{\langle a_i, a_i \rangle} a_i,$$

which is also a normal projection and its range is $[\{a_1, \ldots, a_n\}]^p$. Choose $a_{n+1} \in H_n(X) \subset H_n(c_0) = [\{a_1, \ldots, a_n\}]^p$ such that

$$\|a_{n+1}\| \geq v_{n+1}^{-1} \sup_{x \in X} \|H_n x\| .$$

Clearly, $\langle a_{n+1}, a_i \rangle = 0$ for all $i \in \{0, 1, \ldots, n\}$. We claim that $H_n H_i = H_n$. In fact,

$$\begin{aligned} H_n H_i(x) &= \left(H_{n-1} - \frac{\langle \cdot, a_n \rangle}{\langle a_n, a_n \rangle} a_n \right) H_i(x) \\ &= H_{n-1} H_i(x) - \frac{\langle H_i(x), a_n \rangle}{\langle a_n, a_n \rangle} a_n \\ &= H_{n-1}(x) - \frac{\langle x, a_n \rangle}{\langle a_n, a_n \rangle} a_n \text{ (by the induction hypothesis)} \\ &= H_n(x) . \end{aligned}$$

Summarizing,

$$H_n H_i = H_n,\ i = 0, 1, \ldots, n; \quad H_{n-1} - H_n \text{ is a normal projection; and}$$

$$\langle a_i, a_j \rangle = 0,\ i, j = 1, \ldots, n+1,\ i \neq j;$$

and the induction is complete.

Second step: We will prove that $\lim_{n\to\infty} a_n = 0$. Observe that $H_n(c_0) \subset H_{n-1}(c_0)$, since $[\{a_1, \ldots, a_n\}]^p \subset [\{a_1, \ldots, a_{n-1}\}]^p$. On the other hand, since $\langle a_i, a_n\rangle = 0$ for $i \neq n$, it follows that

$$i > n \Rightarrow H_n(a_i) = a_i,$$

$$i \leq n \Rightarrow H_n(a_i) = \theta.$$

Now, if we call $Q_{n-1} = H_{n-1} - H_n = \frac{\langle \cdot, a_n\rangle}{\langle a_n, a_n\rangle} a_n$, then

$$Q_{n-1}(a_i) = \begin{cases} \theta & if \quad i \neq n \\ a_n & if \quad i = n; \end{cases}$$

$$Q_{n-1} H_{n-1} = Q_{n-1};$$

$$\|Q_{n-1}\| \leq 1.$$

For $x \in X$, let $\lambda = \frac{\langle x, a_n\rangle}{\langle a_n, a_n\rangle}$. Then,

$$\begin{aligned} |\lambda|\,\|a_n\| &= \|\lambda a_n\| = \|Q_{n-1}(x)\| = \|Q_{n-1}H_{n-1}(x)\| \\ &\leq \|Q_{n-1}\|\,\|H_{n-1}(x)\| \leq \|H_{n-1}(x)\| \\ &\leq \sup_{\widetilde{x}\in X} \|H_{n-1}(\widetilde{x})\| \leq v_n\,\|a_n\|, \end{aligned}$$

which implies $|\lambda| \leq v_n$. Since $v_n < |\alpha_n|$, we have that $|\lambda| < |\alpha_n|$ and then, $Q_{n-1}(x) \in (\alpha_n B_{\mathcal{C}})\, a_n \subset \alpha_n H_{n-1}(X)$, where $B_{\mathcal{C}}$ is the unit ball of $\mathcal{C}$.

It follows that

$$\begin{aligned} H_n(X) &= (H_{n-1} - Q_{n-1})(X) \\ &\subseteq H_{n-1}(X) - Q_{n-1}(X) \\ &\subseteq H_{n-1}(X) - \alpha_n H_{n-1}(X) \\ &\subseteq \alpha_n H_{n-1}(X), \end{aligned}$$

since X is absolutely convex and $1 < |\alpha_n|$. Going down with these arguments, we get

$$H_n(X) \subseteq \alpha_n \cdot \alpha_{n-1} \cdot \ldots \cdot \alpha_1 H_0(X) = \alpha_n \cdot \alpha_{n-1} \cdot \ldots \cdot \alpha_1 X$$

and, since $|\alpha_n \cdot \alpha_{n-1} \cdot \ldots \cdot \alpha_1| < |\beta|$, we obtain that $H_n(X) \subseteq \beta X$. Now, since n was arbitrary and $a_n \in H_{n-1}(X)$, $\{a_n : n \in \mathbb{N}\} \subset \beta X$. Therefore, since $\{a_n : n \in \mathbb{N}\}$ is orthogonal and βX is compactoid, by Theorem 4.38, p. 141 of Ref. 10, $\lim_{n\to\infty} a_n = 0$.

Third step: We will prove that $X \subset \overline{co}(\{z_n : n \in \mathbb{N}\})$, where $z_n = \beta a_n$. First of all, note that $\lim_{n\to\infty} z_n = 0, z_n \in \beta^2 X \subset s^{-1} X$ and $\langle z_i, z_j\rangle = 0$ for $i \neq j$. Let $x \in X$ be given; then

$$\|H_n(x)\| \leq \sup_{\widetilde{x}\in X} \|H_n \widetilde{x}\| \leq v_n\,\|a_n\| \leq |\alpha_n|\,\|a_n\| < |\beta|\,\|a_n\|$$

implies

$$\lim_{n\to\infty} H_n(x) = 0.$$

On the other hand,

$$x = H_0(x) = \sum_{n=1}^{\infty} [H_{n-1}(x) - H_n(x)] = \sum_{n=1}^{\infty} Q_{n-1}(x)$$

and $Q_{n-1}(x) \in (\alpha_n B_{\mathcal{C}}) a_n \subset \left(\frac{\alpha_n}{\beta} B_{\mathcal{C}}\right) \beta a_n = \left(\frac{\alpha_n}{\beta} B_{\mathcal{C}}\right) z_n \subset B_{\mathcal{C}} z_n$; that is, $Q_{n-1}(x) = \eta_n(x) z_n$, with $\eta_n(x) \in B_{\mathcal{C}}$. Thus,

$$x = \sum_{n=1}^{\infty} \eta_n(x) z_n \in \overline{co}(\{z_n : n \in \mathbb{N}\}).$$

Fourth step: We will prove that

$$T(\cdot) = \sum_{n=1}^{\infty} \frac{\langle \cdot, T(z_n)\rangle}{\langle z_n, z_n\rangle} z_n.$$

For an element $u = (u_n)_{n\in\mathbb{N}} \in c_0 \setminus \{\theta\}$, there exists some $n \in \mathbb{N}$ such that $\|u\| = |u_n|$. Let $n_\circ = \min\{n \in \mathbb{N} : \|u\| = |u_n|\}$. By the linearity and continuity of T,

$$T(u) = u_{n_\circ} T\left(\frac{1}{u_{n_\circ}} u\right) = \sum_{n=1}^{\infty} u_{n_\circ} \eta_n \left(T\left(\frac{1}{u_{n_\circ}} u\right)\right) z_n$$

$$= \sum_{n=1}^{\infty} g_n(u) z_n,$$

where

$$g_n(u) = u_{n_\circ} \eta_n \left(T\left(\frac{1}{u_{n_\circ}} u\right)\right).$$

Now, since $\langle z_n, z_k\rangle = 0, n \neq k$, we have that

$$\langle T(u), z_k\rangle = g_k(u) \langle z_k, z_k\rangle \text{ or } g_k(u) = \frac{\langle T(u), z_k\rangle}{\langle z_k, z_k\rangle} = \frac{\langle u, T(z_k)\rangle}{\langle z_k, z_k\rangle};$$

that is, g_k is a Riesz functional. Also, since

$$|g_n(u)| = \left|u_{n_\circ}\right| \left|\eta_n \left(T\left(\frac{1}{u_{n_\circ}} u\right)\right)\right| \leq \|u\|,$$

we get that $\|g_n\| \leq 1$. Thus,

$$T(\cdot) = \sum_{n=1}^{\infty} \frac{\langle \cdot, T(z_n)\rangle}{\langle z_n, z_n\rangle} z_n.$$

Fifth step: We claim that the equation

$$T z_i = \lambda_i z_i,$$

has a solution in $\mathcal{C}$. In fact,

$$\sum_{n=1}^{\infty} \frac{\langle z_i, T(z_n)\rangle}{\langle z_n, z_n\rangle} z_n = T z_i = \lambda_i z_i$$

$$\Rightarrow \sum_{n\neq i} \frac{\langle z_i, T(z_n)\rangle}{\langle z_n, z_n\rangle} z_n + \left[\frac{\langle z_i, T z_i\rangle}{\langle z_i, z_i\rangle} - \lambda_i\right] z_i = 0.$$

Since $\{z_n : n \in \mathbb{N}\}$ is an orthogonal set, we have that

$$\langle z_i, T(z_n)\rangle = 0 \text{ for } n \neq i; \text{ and } \lambda_i = \frac{\langle z_i, T z_i\rangle}{\langle z_i, z_i\rangle}.$$

Since T is self-adjoint, $\lambda_i \in \mathcal{R}$ and

$$|\lambda_n| = \left|\frac{\langle T z_n, z_n\rangle}{\langle z_n, z_n\rangle}\right| = |g_n(z_n)| \leq \|g_n\| \, \|z_n\| \leq \|z_n\| \underset{n\to\infty}{\to} 0,$$

which implies that $(\lambda_n)_{n\in\mathbb{N}} \in c_0(\mathcal{R})$. On the other hand,

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda_n \frac{\langle u, z_n\rangle}{\langle z_n, z_n\rangle} z_n &= \sum_{n=1}^{\infty} \frac{\langle z_n, T z_n\rangle}{\langle z_n, z_n\rangle} \frac{\langle u, z_n\rangle}{\langle z_n, z_n\rangle} z_n \\ &= \sum_{n=1}^{\infty} \frac{\left\langle u, \frac{\langle z_n, T z_n\rangle}{\langle z_n, z_n\rangle} z_n\right\rangle}{\langle z_n, z_n\rangle} z_n \\ &= \sum_{n=1}^{\infty} \frac{\langle u, T z_n\rangle}{\langle z_n, z_n\rangle} z_n = T(u).\end{aligned}$$

Therefore,

$$T(u) = \sum_{n=1}^{\infty} \lambda_n \frac{\langle u, z_n\rangle}{\langle z_n, z_n\rangle} z_n.$$

Sixth step: Finally, since the operators

$$P_n = \frac{\langle \cdot, z_n\rangle}{\langle z_n, z_n\rangle} z_n$$

are normal projections and since $\|z_n\| = |\mu_n|$ for some $\mu_n \in \mathcal{C} \setminus \{0\}$, we have that $y_n = \frac{z_n}{\mu_n}$ are unitary vectors, $(y_n)_{n\in\mathbb{N}}$ is an orthonormal sequence, and

$$P_n = \frac{\langle \cdot, y_n\rangle}{\langle y_n, y_n\rangle} y_n.$$

This finishes the proof of the theorem. □

Remark 4: *If T is a compact and self-adjoint operator then, by the previous theorem, there exist an element $(\lambda_n)_{n\in\mathbb{N}} \in c_0$ and an orthonormal sequence $(y_n)_{n\in\mathbb{N}}$ in c_0 such that*

$$T = \sum_{n=1}^{\infty} \lambda_n P_n,$$

where

$$P_n = \frac{\langle \cdot, y_n\rangle}{\langle y_n, y_n\rangle} y_n \ \text{ and } \ \lambda_n = \frac{\langle y_n, T y_n\rangle}{\langle y_n, y_n\rangle}.$$

Since $\|P_n\| = 1$, for all $n \in \mathbb{N}$, and $\{y_1, y_2, \ldots\}$ is, in particular, an orthonormal collection, we have

$$\begin{aligned}\|T x\| &= \left\|\sum_{n=1}^{\infty} \lambda_n P_n x\right\| = \max \|\lambda_n P_n x\| \\ &\leq \|x\| \max |\lambda_n| = \|x\| \left\|(\lambda_n)_{n\in\mathbb{N}}\right\|,\end{aligned}$$

that is, $\|T\| \leq \left\|(\lambda_n)_{n\in\mathbb{N}}\right\|$. *On the other hand, since* $|\lambda_n| = \|\lambda_n y_n\| = \|Ty_n\| \leq \|T\|$, *we have* $\left\|(\lambda_n)_{n\in\mathbb{N}}\right\| \leq \|T\|$. *Therefore,* $\|T\| = \left\|(\lambda_n)_{n\in\mathbb{N}}\right\|$.

Proposition 6: *Let* $T = \sum_{n=1}^{\infty} \lambda_n \frac{\langle \cdot, y_n\rangle}{\langle y_n, y_n\rangle} y_n$ *be a compact and self-adjoint operator and let* $\mu \neq 0$ *in* $\mathcal{C}$ *be an eigenvalue of T. Then* $\mu = \lambda_n$ *for some n.*

Proof: Let $v \in c_0(\mathcal{C})$ an eigenvector corresponding to μ. Then

$$Tv = \sum_{n=1}^{\infty} \lambda_n \frac{\langle v, y_n\rangle}{\langle y_n, y_n\rangle} y_n$$

and

$$T(Tv) = T(\mu v) = \mu T v.$$

It follows from the last equation that

$$T\left(\sum_{n=1}^{\infty} \lambda_n \frac{\langle v, y_n\rangle}{\langle y_n, y_n\rangle} y_n\right) = \mu\left(\sum_{n=1}^{\infty} \lambda_n \frac{\langle v, y_n\rangle}{\langle y_n, y_n\rangle} y_n\right).$$

Thus,

$$\sum_{n=1}^{\infty} \lambda_n^2 \frac{\langle v, y_n\rangle}{\langle y_n, y_n\rangle} y_n = \sum_{n=1}^{\infty} \mu\lambda_n \frac{\langle v, y_n\rangle}{\langle y_n, y_n\rangle} y_n;$$

and hence

$$\sum_{\substack{n=1\\ \langle v, y_n\rangle \neq 0}}^{\infty} \lambda_n^2 \frac{\langle v, y_n\rangle}{\langle y_n, y_n\rangle} y_n = \sum_{\substack{n=1\\ \langle v, y_n\rangle \neq 0}}^{\infty} \mu\lambda_n \frac{\langle v, y_n\rangle}{\langle y_n, y_n\rangle} y_n.$$

Since $Tv = \mu v \neq 0$, it follows that $\langle v, y_n\rangle \neq 0$ for some n. Hence,

$$\bigcup_{\substack{n=1\\ \langle v, y_n\rangle \neq 0}}^{\infty} \{\lambda_n\} \neq \emptyset.$$

Thus,

$$\sum_{\substack{n=1\\ \langle v, y_n\rangle \neq 0}}^{\infty} \lambda_n(\lambda_n - \mu) \frac{\langle v, y_n\rangle}{\langle y_n, y_n\rangle} y_n = 0.$$

The normality of the sequence $\{y_n\}$ then entails that

$$\lambda_n(\lambda_n - \mu) \frac{\langle v, y_n\rangle}{\langle y_n, y_n\rangle} = 0 \text{ for all } n \in \bigcup_{\substack{n=1\\ \langle v, y_n\rangle \neq 0}}^{\infty} \{n\}.$$

Since the eigenvectors corresponding to different eigenvalues are normal and since $\mu \neq 0$, it follows that

$$\lambda_n \neq 0 \text{ for all } n \in \bigcup_{\substack{n=1\\ \langle v, y_n\rangle \neq 0}}^{\infty} \{n\}.$$

Hence,

$$\lambda_n - \mu = 0 \text{ for all } n \in \bigcup_{\substack{n=1\\ \langle v, y_n\rangle \neq 0}}^{\infty} \{n\}.$$

It follows from the above that

$$\emptyset \neq \bigcup_{\substack{n=1 \\ \langle v, y_n \rangle \neq 0}}^{\infty} \{\lambda_n\} = \{\mu\}.$$

□

From Theorem 10, Remark 4, and Proposition 6, it follows that $\mathcal{A}_2(c_0)$, the collection of all compact and self-adjoint operators on c_0, is isometrically isomorphic to $c_0(\mathcal{R})$:

$$\mathcal{A}_2(c_0) \cong c_0(\mathcal{R}).$$

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