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RESEARCH ARTICLES

Positive Operators on a Free Banach Space over the Complex Levi-Civita Field*

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Abstract—Let $\mathcal{C}$ be the complex Levi-Civita field and let $c_0(\mathcal{C})$ or, simply, c_0 denote the space of all null sequences of elements of $\mathcal{C}$. A non-Archimedean norm is defined naturally on c_0 with respect to which c_0 is a Banach space. In this paper, we study the properties of positive operators on c_0 which are similar to those of positive operators in classical functional analysis; however the proofs of many of the results are nonclassical. Then we use our study of positive operators to introduce a partial order on the set of compact and self-adjoint operators on c_0 and study the properties of that partial order.

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1. INTRODUCTION

Two of the most useful and interesting mathematical theories in real or complex functional analysis have been Hilbert spaces and continuous linear operators. These theories have exactly matched the needs of many branches of physics, biology, and other fields of science.

The importance of Hilbert spaces over the real or complex fields has led many researchers to try and extend the concept to non-Archimedean fields. One of the first attempts to define an appropriate non-Archimedean inner product was made by G. K. Kalisch [2]. Two of the most recent papers about non-Archimedean Hilbert spaces are those of L. Narici and E. Beckenstein [3] and the authors [1]. They define a non-Archimedean inner product on a vector space E over a complete non-Archimedean and non-trivially valued field $\mathbb{K}$ as a non-degenerated $\mathbb{K}$-function in $E \times E$, which is linear in the first variable and satisfies what they call the Cauchy-Schwarz type inequality. Recall that a vector space E is said to be orthomodular if for every closed subspace M of E, we have that E is the directed sum of M and its normal complement. The existence of infinite-dimensional non-classical orthomodular spaces was an open question until the following interesting theorem was proved by M. P. Solèr [7]: "Let X be an orthomodular space and suppose it contains an orthonormal sequence $e_1, e_2, \cdots$ (in the sense of the inner product). Then the base field is $\mathbb{R}$ or $\mathbb{C}$". Based on the result of Solèr, if $\mathbb{K}$ is a non-Archimedean, complete valued field and $\mathcal{L}(c_0)$ is the space of all continuous linear operators on c_0, then there exist $T \in \mathcal{L}(c_0)$ which does not have an adjoint. For example, $T(x) = \left(\sum_{i=1}^{\infty} x_i\right) e_1$ is such a linear operator; on the other hand, the normal projections (see the definition below) admit adjoints.

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Throughout this paper, we will use the following notations: Given a valued field $(\mathbb{K}, |\cdot|)$ and a subset B of $\mathbb{K}$, we denote by $|B|$ the set $\{|x| : x \in B\}$. Moreover, given a normed $\mathbb{K}$-vector space E and a subspace F of E, we denote by $\|F\|$ the set $\{\|x\| : x \in F\}$.

In this paper, we consider the complex Levi-Civita field $\mathcal{C}$ as $\mathbb{K}$; in $\mathcal{C}$, we take the natural involution $z \to \overline{z}$ (complex conjugation) when defining an inner product on c_0. Recall that a free Banach space E is a non-Archimedean Banach space for which there exists a family $(e_i)_{i\in I}$ in $E \setminus \{\mathbf{0}\}$ such that any element $x \in E$ can be written in the form of a convergent sum $x = \sum_{i\in I} x_i e_i,\ x_i \in \mathbb{K}$, i.e., $\lim_{i\in I} x_i e_i = 0$ (the limit is with respect to the Fréchet filter on I) and $\|x\| = \sup_{i\in I} |x_i| \|e_i\|$. The family $(e_i)_{i\in I}$ is called an orthogonal basis. Now, if E is a free Banach space of countable type over $\mathcal{C}$, then it is known that E is isometrically isomorphic to

$$c_0(\mathbb{N}, \mathcal{C}, s) := \left\{(x_n)_{n\in\mathbb{N}} : x_n \in \mathcal{C};\ \lim_{n\to\infty} |x_n| s(n) = 0\right\},$$

where $s : \mathbb{N} \to (0, \infty)$. Of course, it could be that, for some $i \in \mathbb{N}$, $s(i) \notin |\mathcal{C} \setminus \{0\}|$. But, if the range of s is contained in $|\mathcal{C} \setminus \{0\}|$, it is enough to study $c_0(\mathbb{N}, \mathcal{C})$ [taking s to be the constant function 1], which will be denoted by $c_0(\mathcal{C})$ or, simply, c_0. We already know that c_0 is not orthomodular.

In a previous paper [1], we characterized closed subspaces of c_0 with a normal complement; that is, we characterized those non-trivial closed subspaces M which admit a non-trivial closed subspace N such that

a. $c_0 = M \oplus N$, and

b. for $x \in M$ and $y \in N$, $\langle x, y\rangle = 0$.

N is actually the subspace $M^p = \{y \in c_0 : \langle x, y\rangle = 0 \text{ for all } x \in M\}$ and then $c_0 = M \oplus M^p$. Such a subspace, together with its normal complement, defines a special kind of projection, the so-called normal projection; that is, a linear operator $P : c_0 \to c_0$ such that

i. P is continuous;

ii. $P^2 = P$;

iii. $\langle z, w\rangle = 0$, for all $z \in N(P)$ and for all $w \in R(P)$.

Actually these concepts are not exclusive to c_0; if E is a vector space with an inner product, then "normal complements" and "normal projections" have similar meaning.

Throughout this paper $\mathcal{R}$ (resp. $\mathcal{C}$) will denote the real (resp. complex) Levi-Civita field; for a detailed study of $\mathcal{R}$ (and $\mathcal{C}$), we refer the reader to [5, 6] and the references therein. Any $z \in \mathcal{C}$ (resp. $\mathcal{R}$) is a function from $\mathbb{Q}$ into $\mathbb{C}$ (resp. $\mathbb{R}$) with left-finite support. For $w \in \mathcal{R}$ (resp. $\mathcal{C}$), we will denote by $\lambda(w) = \min(\operatorname{supp}(w))$, for $w \neq 0$, and $\lambda(0) = +\infty$. On the other hand, since each $z \in \mathcal{C}$ can be written as $z = x + iy$, where $x, y \in \mathcal{R}$, we have that $\lambda(z) = \min\{\lambda(x), \lambda(y)\}$. If we define

$$|z| = \begin{cases} e^{-\lambda(z)} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases},$$

then $|\cdot|$ is a non-Archimedean absolute value in $\mathcal{C}$. It is not hard to prove that $(\mathcal{C}, \Delta)$, where Δ is the metric induced by $|\cdot|$, is a complete metric space. Now let $z = x + iy$ in $\mathcal{C}$ be given. If $x \neq 0 \neq y$ then

$$|z| = e^{-\lambda(z)} = e^{-\min\{\lambda(x),\lambda(y)\}} = \max\left\{e^{-\lambda(x)}, e^{-\lambda(y)}\right\} = \max\{|x|, |y|\}.$$

We can easily also check that $|z| = \max\{|x|, |y|\}$ when $x = 0$ or $y = 0$. Thus,

$$|z| = \max\{|x|, |y|\} \text{ for all } z = x + iy \in \mathcal{C}.$$

In other words, $\mathcal{C}$ is topologically isomorphic to $\mathcal{R}^2$ provided with the product topology induced by $|\cdot|$ in $\mathcal{R}$.

We denote by $c_0(\mathcal{C})$, or simply c_0, the space

$$c_0 = \left\{ z = (z_n)_{n\in\mathbb{N}} : z_n \in \mathcal{C};\ \lim_{n\to\infty} z_n = 0 \right\}.$$

A natural non-Archimedean norm on c_0 is $\|z\|_\infty = \sup\{|z_n| : n \in \mathbb{N}\}$. Writing $z_n = x_n + iy_n$ and $x = (x_n)_{n\in\mathbb{N}},\ y = (y_n)_{n\in\mathbb{N}}$, we also have the equality

$$\|z\|_\infty = \max\{\|x\|_\infty, \|y\|_\infty\}.$$

It follows that $(c_0, \|\cdot\|_\infty)$ is a Banach space. For a detailed study of non-Archimedean Banach spaces, in general, we refer the reader to [8].

Recall that a topological space is called separable if it has a countable dense subset. In the class of real or complex Hilbert spaces, we can distinguish two types: those spaces which are separable and those which are not separable. If E is a separable normed space over $\mathbb{K}$, then each one-dimensional subspace is homeomorphic to $\mathbb{K}$, so $\mathbb{K}$ must be separable too. Nevertheless, we know that there exist non-Archimedean fields which are not separable, for example, the Levi-Civita fields $\mathcal{R}$ and $\mathcal{C}$. Thus, for non-Archimedean normed spaces the concept of separability cannot be used if $\mathbb{K}$ is not separable. However, by linearizing the notion of separability, we obtain a generalization, useful for each non-Archimedean valued field $\mathbb{K}$. A normed space E over $\mathbb{K}$ is said to be *of countable type* if it contains a countable subset whose linear hull is dense in E. An example of a normed space of countable type is $(c_0(\mathbb{K}), \|\cdot\|_\infty)$, for any non-Archimedean valued field $\mathbb{K}$, in particular, when $\mathbb{K}$ is the complex Levi-Civita field $\mathcal{C}$.

Let us consider the following form:

$$\langle \cdot,\cdot \rangle : c_0 \times c_0 \to \mathcal{C};\ \langle z, w \rangle = \sum_{n=1}^{\infty} z_n \overline{w_n}.$$

This form is well-defined since $\lim_{n\to\infty} z_n\overline{w_n} = 0$ and, at the same time, $\langle \cdot,\cdot \rangle$ satisfies Definition 2.4.1, p. 38, in [4].

Let

$$\|z\| := \sqrt{|\langle z, z \rangle|}.$$

Then, since $|2| = 1$, $\|\cdot\|$ is a non-Archimedean norm on c_0 (Theorem 2.4.2 (ii) in [4]).

It follows easily that

$$\langle x, y \rangle = 0,\ \forall y \in c_0 \Rightarrow x = \mathbf{0}$$

which is referred to as the non-degeneracy condition.

The next theorem was proved in [3] and tells us when the non-Archimedean norm in a Banach space is induced by an inner product.

Theorem 1.1. *Let $(E, \|\cdot\|)$ be a $\mathbb{K}$-Banach space. Then, if $\|E\| \subset |\mathbb{K}|^{1/2}$ and every one-dimensional subspace of E admits a normal complement, then E has, at least, an inner product that induces the norm $\|\cdot\|$.*

If $E = c_0$ and $\mathbb{K} = \mathcal{C}$, then the conditions of the theorem above are satisfied. In fact, if $z \in c_0,\ z \neq \mathbf{0}$, then $\lim_{n\to\infty} z_n = 0$, which implies that there exists $j_o \in \mathbb{N}$ such that

$$\|z\|_\infty = \max\{|z_j| : j \in \mathbb{N}\} = |z_{j_0}| \in |\mathcal{C}|.$$

Now, since $|\mathcal{C}| \subset |\mathcal{C}|^{1/2}$, $\|c_0\| \subset |\mathcal{C}|^{1/2}$. The other condition is guaranteed by Lemma 2.3.19, p. 34 in [4].

It was proved in [1] that $\langle \cdot,\cdot \rangle$ is one of the inner products that induce the $\|\cdot\|_\infty$ norm on c_0. Such a result was guaranteed thanks to the following lemma which will be useful also in this paper.

Lemma 1.2. *If $\{z_1, z_2, \cdots, z_n\} \subset \mathcal{C}$, then*

$$|z_1\overline{z_1} + z_2\overline{z_2} + \cdots + z_n\overline{z_n}| = \max\{|z_1\overline{z_1}|, |z_2\overline{z_2}|, \cdots, |z_n\overline{z_n}|\}.$$

Definition 1.3. *A subset D of c_0 such that for all $x, y \in D$, $x \neq y \Rightarrow \langle x, y\rangle = 0$, is called a normal family. A countable normal family $\{x_n : n \in \mathbb{N}\}$ of unit vectors is called an orthonormal sequence.*

If $A \subset c_0$, then $[A]$ and $cl\,[A]$ will denote the linear and the closed linear span of A, respectively. If M is a subspace of c_0, then M^p will denote the subspace of all $y \in c_0$ such that $\langle y, x\rangle = 0$, for all $x \in M$. Since the definition of the inner product given in [4], p.38, coincides with the definition of inner product given here, the Gram-Schmidt procedure can be used.

Theorem 1.4. *If $(z_n)_{n\in\mathbb{N}}$ is a sequence of linearly independent vectors in c_0, then there exists an orthonormal sequence $(y_n)_{n\in\mathbb{N}}$ such that $[\{z_1, \cdots, z_n\}]$ =$[\{y_1, \cdots, y_n\}]$ for every $n \in \mathbb{N}$.*

Lemma 1.5. *If $(z_n)_{n\in\mathbb{N}}$ is an orthonormal sequence in c_0, then $(z_n)_{n\in\mathbb{N}}$ is orthogonal in the van Rooij's sense (see [8] p. 57).*

If E and F are normed spaces over $\mathbb{K}$, then $\mathcal{L}(E, F)$ will be the normed space consisting of all continuous linear maps from E into F. $\mathcal{L}(E, \mathbb{K})$ will be denoted by E' and $\mathcal{L}(E, E)$ will be denoted by $\mathcal{L}(E)$. For a $T \in \mathcal{L}(E, F)$, $N(T)$ and $R(T)$ will denote the Kernel and the range of T, respectively. It is well-known that the dual of c_0 is $c_0' \cong l^\infty$, where l^∞ denotes the space of all bounded sequences of elements of $\mathcal{C}$.

Definition 1.6. *A linear map T from E into F is said to be compact if, for each $\epsilon > 0$, there exists a continuous linear map of finite-dimensional range S such that $\|T - S\| \leq \epsilon$.*

Any continuous linear operator $u \in \mathcal{L}(c_0)$ can be identified with a bounded infinite matrix whose columns converge to 0:

$$[u] = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1j} & \cdots \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots & \alpha_{2j} & \cdots \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \cdots & \alpha_{3j} & \cdots \\ \vdots & & & \ddots & & \\ \alpha_{i1} & \alpha_{i2} & \alpha_{i3} & \cdots & \alpha_{ij} & \cdots \\ \vdots & & & & & \ddots \\ \downarrow_0 & \downarrow_0 & \downarrow_0 & \cdots & \downarrow_0 & \cdots \end{pmatrix}.$$

Definition 1.7. *A linear operator $v : c_0 \to c_0$ is said to be an adjoint of a given linear operator $u \in \mathcal{L}(c_0)$ if $\langle u(x), y\rangle = \langle x, v(y)\rangle$, for all $x, y \in c_0$. In that case, we will say that u admits an adjoint v. We will also say that u is self-adjoint if $v = u$.*

In [1] we showed that if a continuous linear operator u has an adjoint, then the adjoint is unique and continuous.

Lemma 1.8. *Let $u \in \mathcal{L}(c_0)$ with associated matrix $(\alpha_{i,j})_{i,j\in\mathbb{N}}$. Then, u admits an adjoint operator v if and only if $\lim_{j\to\infty} \alpha_{ij} = 0$, for each $i \in \mathbb{N}$. In terms of matrices, this means that*

$$[u] = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1j} & \cdots \to 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots & \alpha_{2j} & \cdots \to 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \cdots & \alpha_{3j} & \cdots \to 0 \\ \vdots & & & \ddots & & \\ \alpha_{i1} & \alpha_{i2} & \alpha_{i3} & \cdots & \alpha_{ij} & \cdots \to 0 \\ \vdots & & & & & \ddots \\ \downarrow_0 & \downarrow_0 & \downarrow_0 & \cdots & \downarrow_0 & \cdots \end{pmatrix}.$$

In the classical Hilbert space theory, any continuous linear operator admits an adjoint. This is not true in the non-Archimedean case. For example, the operator $u \in \mathcal{L}(c_0)$ given by the matrix:

$$\begin{pmatrix} b & b^2 & b^3 & \cdots & b^j & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & & & & & \ddots \end{pmatrix},$$

with $1 < |b|$, does not admit an adjoint, by Lemma 1.8.

The following theorem (proved in [1]) provides a way to construct compact and self-adjoint operators starting from an orthonormal sequence.

Theorem 1.9. *Let $(y_i)_{i\in\mathbb{N}}$ be an orthonormal sequence in c_0. Then, for any $\lambda = (\lambda_i)_{i\in\mathbb{N}}$ in c_0 such that $\lambda_i \in \mathcal{R}$, the map $T : c_0 \to c_0$ defined by*

$$T(\cdot) = \sum_{i=1}^{\infty} \lambda_i P_i(\cdot),$$

where $P_i(\cdot) = \frac{\langle \cdot, y_i\rangle}{\langle y_i, y_i\rangle} y_i$, is a compact and self-adjoint operator.

The converse is also true, as the following theorem shows.

Theorem 1.10. *Let $T : c_0 \to c_0$ be a compact, self-adjoint linear operator of infinite dimensional range. Then there exists an element $\lambda = (\lambda_n)_{n\in\mathbb{N}} \in c_0(\mathcal{R})$ and an orthonormal sequence $(y_n)_{n\in\mathbb{N}}$ in c_0 such that*

$$T = \sum_{n=1}^{\infty} \lambda_n P_n,$$

where

$$P_n = \frac{\langle \cdot, y_n\rangle}{\langle y_n, y_n\rangle} y_n$$

is a normal projection defined by y_n.

The uniqueness of the element $(\lambda_n)_{n\in\mathbb{N}}$ of $c_0(\mathcal{R})$ in Theorem 1.10 is shown by the following proposition, also proved in [1].

Proposition 1.11. *Let $T = \sum_{n=1}^{\infty} \lambda_n \frac{\langle \cdot, y_n\rangle}{\langle y_n, y_n\rangle} y_n$ be a compact and self-adjoint operator and let $\mu \neq 0$ in $\mathcal{C}$ be an eigenvalue of T. Then $\mu = \lambda_n$ for some n.*

We use $\mathcal{A}_0$, $\mathcal{A}_1$, and $\mathcal{A}_2$ to denote the following closed subsets of $\mathcal{L}(c_0)$:

$$\begin{aligned} \mathcal{A}_0 &= \{T \in \mathcal{L}(c_0) : T \text{ has an adjoint}\}; \\ \mathcal{A}_1 &= \{T \in \mathcal{A}_0 : T \text{ is compact}\}; \\ \mathcal{A}_2 &= \{T \in \mathcal{A}_1 : T = T^*\} = \{T \in \mathcal{L}(c_0) : T \text{ is compact and self-adjoint}\}. \end{aligned}$$

In this paper, we will study the properties of positive operators on $c_0(\mathcal{C})$, obtaining results that are similar to those from classical functional analysis but many of which have non-classical proofs. Then we will use our study of positive operators to introduce a partial order on $\mathcal{A}_2$ and study the properties of that partial order.

2. POSITIVE OPERATORS

We recall that the Levi-Civita $\mathcal{R}$ is a totally ordered field. The order on $\mathcal{R}$ is defined as follows: $x \geq 0$ if and only if $x = 0$ or $[x \neq 0$ and $x[\lambda(x)] > 0]$.

Definition 2.1. *For $T \in \mathcal{A}_1$, we say that T is positive and write $T \geq 0$ if $\langle Tx, x\rangle \in \mathcal{R}$ and $\langle Tx, x\rangle \geq 0$ for all $x \in c_0(\mathcal{C})$.*

Lemma 2.2. *Let $T \in \mathcal{A}_1$ be positive. Then T is self-adjoint; that is $T \in \mathcal{A}_2$. Moreover, all eigenvalues of T are in $\mathcal{R}$ and non-negative.*

Proof. For all $x, y \in c_0(\mathcal{C})$ we have that

$$\begin{aligned}\langle Tx, y\rangle = &\frac{1}{4}\left[\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle\right] \\ &+ \frac{i}{4}\left[\langle T(x+iy), x+iy\rangle - \langle T(x-iy), x-iy\rangle\right]\end{aligned}$$

and

$$\begin{aligned}\langle Ty, x\rangle = &\frac{1}{4}\left[\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle\right] \\ &- \frac{i}{4}\left[\langle T(x+iy), x+iy\rangle - \langle T(x-iy), x-iy\rangle\right].\end{aligned}$$

Since $T \geq 0$ it follows that $\langle T(x+y), x+y\rangle$, $\langle T(x-y), x-y\rangle$, $\langle T(x+iy), x+iy\rangle$ and $\langle T(x-iy), x-iy\rangle$ are all (non-negative) elements of $\mathcal{R}$. Thus, for all $x, y \in c_0(\mathcal{C})$ we have that $\langle Ty, x\rangle = \overline{\langle Tx, y\rangle} = \langle y, Tx\rangle$; and hence $\langle y, T^*x\rangle = \langle y, Tx\rangle$ for all $x, y \in c_0(\mathcal{C})$. Thus, given $x \in c_0(\mathcal{C})$, we have that

$$\langle y, (T^* - T)x\rangle = 0 \text{ for all } y \in c_0(\mathcal{C}).$$

It follows, in particular, that

$$\langle (T^* - T)x, (T^* - T)x\rangle = 0, \text{ and hence } (T^* - T)x = 0.$$

This is true for all $x \in c_0(\mathcal{C})$. Thus, $T^* - T = 0$, or $T^* = T$.

Now let λ be an eigenvalue of T and let $v \in c_0(\mathcal{C})$ be a corresponding eigenvector. Then $\langle Tv, v\rangle \in \mathcal{R}$ and $0 \leq \langle Tv, v\rangle = \langle \lambda v, v\rangle = \lambda\langle v, v\rangle$. Since $\langle v, v\rangle > 0$, it follows that $\lambda \in \mathcal{R}$ and $\lambda \geq 0$. □

The proofs of the following two lemmas are straightforward; therefore, we only state them without proof here but we note that, for the proof of Lemma 2.4, we need the fact that if $T \in \mathcal{A}_1$ then $T^* \in \mathcal{A}_1$ [1] and hence TT^* and T^*T are both elements of $\mathcal{A}_1$.

Lemma 2.3. *Let $S, T \geq 0$ in $\mathcal{A}_1$ and $\alpha \geq 0$ in $\mathcal{R}$ be given. Then $\alpha S + T \geq 0$.*

Lemma 2.4. *For all $T \in \mathcal{A}_1$, both TT^* and T^*T are positive.*

Proposition 2.5. *Let $T \in \mathcal{A}_1$ be positive. Then*

$$|\langle Tx, y\rangle|^2 \leq |\langle Tx, x\rangle||\langle Ty, y\rangle|$$

for all $x, y \in c_0(\mathcal{C})$, where $|\cdot|$ denotes the ultrametric absolute value; that is, $|z| = e^{-\lambda(z)}$ for $z \in \mathcal{C}$.

Proof. Let $x, y \in c_0(\mathcal{C})$ be given. First assume that $\langle Tx, y\rangle \in \mathcal{R}$. Then for all $\lambda \in \mathcal{R}$ we have that (since $T \geq 0$):

$$\begin{aligned}0 &\leq \langle T(x+\lambda y), x+\lambda y\rangle \\ &= \lambda^2\langle Ty, y\rangle + \lambda[\langle Tx, y\rangle + \langle Ty, x\rangle] + \langle Tx, x\rangle \\ &= \lambda^2\langle Ty, y\rangle + \lambda[\langle Tx, y\rangle + \langle y, Tx\rangle] + \langle Tx, x\rangle \text{ since } T \text{ is self-adjoint} \\ &= \lambda^2\langle Ty, y\rangle + \lambda[\langle Tx, y\rangle + \overline{\langle Tx, y\rangle}] + \langle Tx, x\rangle \\ &= \lambda^2\langle Ty, y\rangle + 2\lambda\langle Tx, y\rangle + \langle Tx, x\rangle.\end{aligned}$$

Note that $\lambda^2\langle Ty,y\rangle + 2\lambda\langle Tx,y\rangle + \langle Tx,x\rangle$ is a quadratic expression in λ with coefficients in $\mathcal{R}$; and since this is ≥ 0 for all $\lambda \in \mathcal{R}$, it follows that

$$\langle Tx,y\rangle^2 - \langle Tx,x\rangle\langle Ty,y\rangle \leq 0.$$

Hence $\langle Tx,y\rangle^2 \leq \langle Tx,x\rangle\langle Ty,y\rangle$, from which we get

$$\left|\langle Tx,y\rangle^2\right| \leq |\langle Tx,x\rangle\langle Ty,y\rangle|\,, \text{ or } |\langle Tx,y\rangle|^2 \leq |\langle Tx,x\rangle|\,|\langle Ty,y\rangle|\,.$$

Now assume that $\langle Tx,y\rangle \in \mathcal{C}\setminus\mathcal{R}$; and write $\langle Tx,y\rangle = \alpha + i\beta$, $\beta \neq 0$. Then

$$\langle Tx,y\rangle = \sqrt{\alpha^2+\beta^2}\left[\frac{\alpha}{\sqrt{\alpha^2+\beta^2}} + i\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right].$$

Let

$$x_1 = x\left[\frac{\alpha}{\sqrt{\alpha^2+\beta^2}} - i\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right].$$

Then $\langle Tx_1,x_1\rangle = \langle Tx,x\rangle$ and

$$\langle Tx_1,y\rangle = \left[\frac{\alpha}{\sqrt{\alpha^2+\beta^2}} - i\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right]\langle Tx,y\rangle = \sqrt{\alpha^2+\beta^2} = |\langle Tx,y\rangle|_o$$

is in $\mathcal{R}$, where $|\cdot|_o$ denotes the ordinary modulus in $\mathcal{C}$. By the above, it follows that $\langle Tx_1,y\rangle^2 \leq \langle Tx_1,x_1\rangle\langle Ty,y\rangle$. Hence

$$|\langle Tx,y\rangle|_o^2 \leq \langle Tx,x\rangle\langle Ty,y\rangle = |\langle Tx,x\rangle\langle Ty,y\rangle|_o.$$

It follows that $|\langle Tx,y\rangle|^2 \leq |\langle Tx,x\rangle\langle Ty,y\rangle| = |\langle Tx,x\rangle||\langle Ty,y\rangle|$. □

Theorem 2.6. *For $T \in \mathcal{A}_1$, the following are equivalent:*

1. $T \geq 0$.
2. *T is self-adjoint; and all of its eigenvalues are in $\mathcal{R}$ and non-negative.*
3. *There exists $S \geq 0$ in $\mathcal{A}_1$ such that $T = S^2$.*
4. *There exists $S \in \mathcal{A}_1$ such that $T = S^*S$.*
5. *There exists $M \in \mathcal{A}_1$ such that $T = MM^*$.*

Proof. (1) $\Rightarrow$ (2): This follows from Lemma 2.2.

(2) $\Rightarrow$ (3): Assume (2) is true. Since T is compact and self-adjoint, then by Theorem 10 in [1] there exist $(\lambda_n)_{n\in\mathbb{N}} \in c_0(\mathcal{R})$ and an orthonormal sequence $(y_n)_{n\in\mathbb{N}}$ of elements $y_n \in c_0(\mathcal{C})$ such that

$$T = \sum_{n=1}^{\infty} \lambda_n \frac{\langle\cdot,y_n\rangle}{\langle y_n,y_n\rangle} y_n.$$

For each $n \in \mathbb{N}$, we have that λ_n is an eigenvalue of T [1]; and hence $\lambda_n \in \mathcal{R}$ and $\lambda_n \geq 0$ for all $n \in \mathbb{N}$. Let $S : c_0(\mathcal{C}) \to c_0(\mathcal{C})$ be given by

$$S = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \frac{\langle\cdot,y_n\rangle}{\langle y_n,y_n\rangle} y_n.$$

Then S is compact and self-adjoint, by Theorem 8 in [1]; and hence $S \in \mathcal{A}_1$.

We show that $S \geq 0$ and $S^2 = T$. For all $x \in c_0(\mathcal{C})$, we have that

$$\langle Sx, x\rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\frac{\langle x, y_n\rangle}{\langle y_n, y_n\rangle}\langle y_n, x\rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\frac{\langle x, y_n\rangle\overline{\langle x, y_n\rangle}}{\langle y_n, y_n\rangle} \geq 0.$$

Hence $S \geq 0$. Also, for all $x \in c_0(\mathcal{C})$, we have that

$$S^2x = S(Sx) = S\left(\sum_{n=1}^{\infty} \sqrt{\lambda_n}\frac{\langle x, y_n\rangle}{\langle y_n, y_n\rangle}y_n\right) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\frac{\langle x, y_n\rangle}{\langle y_n, y_n\rangle}S\left(y_n\right)$$

$$= \sum_{n=1}^{\infty} \sqrt{\lambda_n}\frac{\langle x, y_n\rangle}{\langle y_n, y_n\rangle}\left(\sqrt{\lambda_n}y_n\right) = \sum_{n=1}^{\infty} \lambda_n\frac{\langle x, y_n\rangle}{\langle y_n, y_n\rangle}y_n = Tx.$$

Hence $S^2 = T$.

(3) ⇒ (4): Assume there exists $S \geq 0$ in $\mathcal{A}_1$ such that $T = S^2$. Then S is self-adjoint by Lemma 2.2. Thus, $S = S^*$ and hence $T = S^2 = SS = S^*S$.

(4) ⇒ (5): Assume there exists $S \in \mathcal{A}_1$ such that $T = S^*S$. Let $M = S^*$. Then $M \in \mathcal{A}_1$ and $M^* = S$. Thus, $T = S^*S = MM^*$.

(5) ⇒ (1): This follows from Lemma 2.4. □

Remark 2.7. *Let T and S be as in Theorem 2.6: $T \geq 0$ and $S \geq 0$ in $\mathcal{A}_1$ such that $T = S^2$. Then S is unique. We say that S is the positive square root of T and write $S = \sqrt{T}$.*

Proof. Let $M \geq 0$ in $\mathcal{A}_1$ be such that $M^2 = S^2 = T$. We will show that $M = S$. Since $S \geq 0$ and $M \geq 0$, there exist $(\eta_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}} \in c_0(\mathcal{R})$ and orthonormal sequences $(y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}$ of elements $y_n, z_n \in c_0(\mathcal{C})$ such that

$$S = \sum_{n=1}^{\infty} \eta_n\frac{\langle \cdot, y_n\rangle}{\langle y_n, y_n\rangle}y_n \text{ and } M = \sum_{m=1}^{\infty} \mu_m\frac{\langle \cdot, z_m\rangle}{\langle z_m, z_m\rangle}z_m$$

with $\eta_n > 0$ for all n and $\mu_m > 0$ for all m. Then

$$T = \sum_{n=1}^{\infty} \eta_n^2\frac{\langle \cdot, y_n\rangle}{\langle y_n, y_n\rangle}y_n = \sum_{m=1}^{\infty} \mu_m^2\frac{\langle \cdot, z_m\rangle}{\langle z_m, z_m\rangle}z_m.$$

Note that $Tz_1 = \mu_1^2 z_1$. Hence z_1 is an eigenvector of T with eigenvalue μ_1^2. Thus, $\mu_1^2 = \eta_n^2$ for some n, by Proposition 6 in [1]. Without loss of generality, we may assume that $\mu_1^2 = \eta_1^2$ and hence $\mu_1 = \eta_1$. Let $\lambda_1 = \mu_1^2 = \eta_1^2$ and let n_1 be the dimension of the eigenspace E_1 of T corresponding to λ_1. Again, without loss of generality, we may assume that $E_1 = [y_1, y_2, \ldots, y_{n_1}] = [z_1, z_2, \ldots, z_{n_1}]$.

Continuing inductively, we get

$$T = \sum_{l=1}^{\infty} \lambda_l\left(\sum_{j=1}^{n_l} \frac{\langle \cdot, y_j^{(l)}\rangle}{\langle y_j^{(l)}, y_j^{(l)}\rangle}y_j^{(l)}\right) = \sum_{l=1}^{\infty} \lambda_l\left(\sum_{j=1}^{n_l} \frac{\langle \cdot, z_j^{(l)}\rangle}{\langle z_j^{(l)}, z_j^{(l)}\rangle}z_j^{(l)}\right),$$

where $\lambda_l \geq 0$ for $l = 1, 2, \ldots$ and $\lambda_l \neq \lambda_k$ for $l \neq k$; and the corresponding eigenspace $E_l = \left[y_1^{(l)}, y_2^{(l)}, \ldots, y_{n_l}^{(l)}\right] = \left[z_1^{(l)}, z_2^{(l)}, \ldots, z_{n_l}^{(l)}\right]$. It follows that

$$S = \sum_{l=1}^{\infty} \sqrt{\lambda_l}\left(\sum_{j=1}^{n_l} \frac{\langle \cdot, y_j^{(l)}\rangle}{\langle y_j^{(l)}, y_j^{(l)}\rangle}y_j^{(l)}\right) = \sum_{l=1}^{\infty} \sqrt{\lambda_l}\left(\sum_{j=1}^{n_l} \frac{\langle \cdot, z_j^{(l)}\rangle}{\langle z_j^{(l)}, z_j^{(l)}\rangle}z_j^{(l)}\right) = M.$$

□

Proposition 2.8. *Let $T \geq 0$ in $\mathcal{A}_1$ and let $S = \sqrt{T}$. Then*

$$\|S\| = \|T\|^{1/2}.$$

Proof. Since $T \geq 0$, there exist $(\lambda_n)_{n\in\mathbb{N}} \in c_0(\mathcal{R})$ and an orthonormal sequence $(y_n)_{n\in\mathbb{N}}$ of elements $y_n \in c_0(\mathcal{C})$ such that

$$T = \sum_{n=1}^{\infty} \lambda_n \frac{\langle \cdot, y_n\rangle}{\langle y_n, y_n\rangle} y_n,$$

where $\lambda_n > 0$ for all n. It follows that

$$S = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \frac{\langle \cdot, y_n\rangle}{\langle y_n, y_n\rangle} y_n.$$

By Remark 4 in [1], we have that $\|T\| = \|(\lambda_n)\| = \max_{n\in\mathbb{N}} |\lambda_n|$. Similarly,

$$\|S\| = \|(\sqrt{\lambda_n})\| = \max_{n\in\mathbb{N}} \left\{|\lambda_n|^{1/2}\right\} = \left[\max_{n\in\mathbb{N}} \{|\lambda_n|\}\right]^{1/2} = \|(\lambda_n)\|^{1/2} = \|T\|^{1/2}.$$

□

Proposition 2.9. *Let $T \geq 0$ in $\mathcal{A}_1$ and $x \in c_0(\mathcal{C})$ be given. Then $\langle Tx, x\rangle = 0$ if and only if $Tx = 0$.*

Proof. If $Tx = 0$ then $\langle Tx, x\rangle = 0$ by definition of the inner product. Now assume $\langle Tx, x\rangle = 0$. Then, since $T \geq 0$, there exists $S \in \mathcal{A}_1$ such that $T = S^*S$, by Theorem 2.6. Thus, $\langle S^*Sx, x\rangle = 0$, and hence $\langle Sx, Sx\rangle = 0$, from which we get $Sx = 0$. It follows that $Tx = S^*Sx = S^*0 = 0$. □

Corollary 2.10. *Let $T \geq 0$ in $\mathcal{A}_1$. Then $\langle Tx, x\rangle = 0$ for all $x \in c_0(\mathcal{C})$ if and only if $T = 0$.*

Proposition 2.11. *Let $T \geq 0$ in $\mathcal{A}_1$, let $S = \sqrt{T}$, and let $R \in \mathcal{A}_1$ be given. Then $TR = RT \Leftrightarrow SR = RS$.*

Proof. ($\Leftarrow$): Assume $SR = RS$. Then

$$RT = RS^2 = (RS)S = (SR)S = S(RS) = S(SR) = S^2R = TR.$$

($\Rightarrow$): Assume that $TR = RT$. We show that $SR = RS$. Write T and S as in the proof of Remark 2.7:

$$T = \sum_{l=1}^{\infty} \lambda_l \left(\sum_{j=1}^{n_l} \frac{\langle \cdot, y_j^{(l)}\rangle}{\langle y_j^{(l)}, y_j^{(l)}\rangle} y_j^{(l)} \right)$$
$$S = \sum_{l=1}^{\infty} \sqrt{\lambda_l} \left(\sum_{j=1}^{n_l} \frac{\langle \cdot, y_j^{(l)}\rangle}{\langle y_j^{(l)}, y_j^{(l)}\rangle} y_j^{(l)} \right).$$

Now let $x \in c_0(\mathcal{C})$ be given. Then from $TRx = RTx$, we get

$$\sum_{l=1}^{\infty} \lambda_l \left(\sum_{j=1}^{n_l} \frac{\langle Rx, y_j^{(l)}\rangle}{\langle y_j^{(l)}, y_j^{(l)}\rangle} y_j^{(l)} \right) = \sum_{l=1}^{\infty} \lambda_l \left(\sum_{j=1}^{n_l} \frac{\langle x, y_j^{(l)}\rangle}{\langle y_j^{(l)}, y_j^{(l)}\rangle} Ry_j^{(l)} \right). \tag{2.1}$$

But from $TRy_j^{(l)} = RTy_j^{(l)}$, we get that $TRy_j^{(l)} = \lambda_l Ry_j^{(l)}$, which shows that $Ry_j^{(l)} \in E_l$, where $E_l = \left[y_1^{(l)}, y_2^{(l)}, \ldots, y_{n_l}^{(l)}\right]$ is the eigenspace of T corresponding to the eigenvalue λ_l. It follows then from Equation (2.1) that

$$\sum_{j=1}^{n_l} \frac{\langle Rx, y_j^{(l)}\rangle}{\langle y_j^{(l)}, y_j^{(l)}\rangle} y_j^{(l)} = \sum_{j=1}^{n_l} \frac{\langle x, y_j^{(l)}\rangle}{\langle y_j^{(l)}, y_j^{(l)}\rangle} Ry_j^{(l)} \tag{2.2}$$

for each $l \in \mathbb{N}$. Hence

$$SRx = \sum_{l=1}^{\infty} \sqrt{\lambda_l} \left(\sum_{j=1}^{n_l} \frac{\langle Rx, y_j^{(l)} \rangle}{\langle y_j^{(l)}, y_j^{(l)} \rangle} y_j^{(l)} \right) = \sum_{l=1}^{\infty} \sqrt{\lambda_l} \left(\sum_{j=1}^{n_l} \frac{\langle x, y_j^{(l)} \rangle}{\langle y_j^{(l)}, y_j^{(l)} \rangle} R y_j^{(l)} \right)$$
$$= R \left(\sum_{l=1}^{\infty} \sqrt{\lambda_l} \left(\sum_{j=1}^{n_l} \frac{\langle x, y_j^{(l)} \rangle}{\langle y_j^{(l)}, y_j^{(l)} \rangle} y_j^{(l)} \right) \right) = RSx,$$

where in the second equality we made use of Equation (2.2). This is true for all $x \in c_0(\mathcal{C})$; hence $SR = RS$. □

Proposition 2.12. *Let $S, T \in \mathcal{A}_1$ be positive. Then $ST \geq 0 \Leftrightarrow ST = TS$.*

Proof. ($\Rightarrow$): Assume that $ST \geq 0$. Then ST is self-adjoint by Lemma 2.2. It follows that

$$ST = (ST)^* = T^* S^* = TS,$$

since T and S are both positive and hence self-adjoint.

($\Leftarrow$): Assume $ST = TS$. Let $N = \sqrt{T}$. Applying Proposition 2.11, we have that $NS = SN$. Now let $x \in c_0(\mathcal{C})$ be given. Then

$$\langle STx, x \rangle = \langle S(NN)x, x \rangle = \langle (SN)Nx, x \rangle = \langle (NS)Nx, x \rangle$$
$$= \langle N(SN)x, x \rangle = \langle SNx, N^*x \rangle = \langle S(Nx), Nx \rangle \geq 0,$$

since $S \geq 0$. Hence $ST \geq 0$. □

Proposition 2.13. *Let $T \in \mathcal{A}_2$ be given. Then there exist unique positive operators $A, B \in \mathcal{A}_2$ such that $T = A - B$ and $AB = BA = 0$.*

Proof. Since T is compact and self-adjoint, there exist $(\lambda_n)_{n\in\mathbb{N}} \in c_0(\mathcal{R})$ and an orthonormal sequence $(y_n)_{n\in\mathbb{N}}$ of elements $y_n \in c_0(\mathcal{C})$ such that

$$T = \sum_{n=1}^{\infty} \lambda_n \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n.$$

Thus,

$$T = \sum_{\substack{n=1 \\ \lambda_n > 0}}^{\infty} \lambda_n \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n + \sum_{\substack{n=1 \\ \lambda_n < 0}}^{\infty} \lambda_n \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n$$
$$= \sum_{\substack{n=1 \\ \lambda_n > 0}}^{\infty} \lambda_n \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n - \sum_{\substack{n=1 \\ \lambda_n < 0}}^{\infty} (-\lambda_n) \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n$$
$$= A - B$$

where

$$A = \sum_{\substack{n=1 \\ \lambda_n > 0}}^{\infty} \lambda_n \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n \text{ and } B = \sum_{\substack{n=1 \\ \lambda_n < 0}}^{\infty} (-\lambda_n) \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n,$$

are both positive by Theorem 2.6 since they are both self-adjoint and have positive eigenvalues. That $AB = BA = 0$ then follows from the fact that the sequence $(y_n)_{n\in\mathbb{N}}$ is orthonormal: Let $x \in c_0(\mathcal{C})$ be given. Then

$$\begin{aligned} ABx &= \sum_{\substack{n=1 \\ \lambda_n < 0}}^{\infty} (-\lambda_n)\frac{\langle x, y_n\rangle}{\langle y_n, y_n\rangle}A(y_n) \\ &= \sum_{\substack{n=1 \\ \lambda_n < 0}}^{\infty} (-\lambda_n)\frac{\langle x, y_n\rangle}{\langle y_n, y_n\rangle}\left(\sum_{\substack{l=1 \\ \lambda_l > 0}}^{\infty} \lambda_l \frac{\langle y_n, y_l\rangle}{\langle y_l, y_l\rangle} y_l\right) = 0. \end{aligned}$$

Hence $AB = 0$. A similar calculation as above or application of Proposition 2.12 show that $BA = 0$ too.

Finally, to show the uniqueness of A and B, assume that $T = A_1 - B_1$ with A_1 and B_1 positive operators in $\mathcal{A}_2$ and $A_1B_1 = B_1A_1 = 0$; we will show that $A_1 = A$ and $B_1 = B$. Since $A_1 \geq 0$ and since $B_1 \geq 0$, then there exist $(\alpha_l)_{l\in\mathbb{N}}, (\beta_j)_{j\in\mathbb{N}} \in c_0(\mathcal{R})$ and orthonormal sequences $(x_l)_{l\in\mathbb{N}}$ and $(z_j)_{j\in\mathbb{N}}$ of elements $x_l, z_j \in c_0(\mathcal{C})$ such that $\alpha_l > 0$ for all $l \in \mathbb{N}$, $\beta_j > 0$ for all $j \in \mathbb{N}$,

$$A_1 = \sum_{l=1}^{\infty} \alpha_l \frac{\langle \cdot, x_l\rangle}{\langle x_l, x_l\rangle} x_l \text{ and } B_1 = \sum_{j=1}^{\infty} \beta_j \frac{\langle \cdot, z_j\rangle}{\langle z_j, z_j\rangle} z_j.$$

Fix $l_0 \in \mathbb{N}$. Then

$$\begin{aligned} Tx_{l_0} &= (A_1 - B_1)\, x_{l_0} = A_1 x_{l_0} - B_1 x_{l_0} = \alpha_{l_0} x_{l_0} - B_1\left(\frac{1}{\alpha_{l_0}} A_1 x_{l_0}\right) \\ &= \alpha_{l_0} x_{l_0} - B_1 A_1 \left(\frac{1}{\alpha_{l_0}} x_{l_0}\right) = \alpha_{l_0} x_{l_0}, \text{ since } B_1A_1 = 0. \end{aligned}$$

This shows that α_{l_0} is an eigenvalue of T; and hence α_{l_0} is equal to some $\lambda_n > 0$ by Proposition 6 in [1]. Similarly we show that, for each $j \in \mathbb{N}$, $-\beta_j$ is equal to some $\lambda_n < 0$. It follows that

$$\{\alpha_l : l \in \mathbb{N}\} = \{\lambda_n : n \in \mathbb{N}, \lambda_n > 0\} \text{ and } \{-\beta_j : j \in \mathbb{N}\} = \{\lambda_n : n \in \mathbb{N}, \lambda_n < 0\}.$$

Using an argument similar to that of the proof of Remark 2.7, it then follows that $A_1 = A$ and $B_1 = B$. □

Remark 2.14. *Let T, A and B be as in Proposition 2.13 above. Then $\|T\| = \max\{\|A\|, \|B\|\}$.*

Proof. As in the proof of Proposition 2.13 above, write

$$T = \sum_{\substack{n=1 \\ \lambda_n > 0}}^{\infty} \lambda_n \frac{\langle \cdot, y_n\rangle}{\langle y_n, y_n\rangle} y_n - \sum_{\substack{n=1 \\ \lambda_n < 0}}^{\infty} (-\lambda_n)\frac{\langle \cdot, y_n\rangle}{\langle y_n, y_n\rangle} y_n = A - B.$$

Then using the fact that

$$\|T\| = \max_{n\in\mathbb{N}} |\lambda_n|, \|A\| = \max_{\substack{n \in \mathbb{N} \\ \lambda_n > 0}} |\lambda_n|, \text{ and } \|B\| = \max_{\substack{n \in \mathbb{N} \\ \lambda_n < 0}} |-\lambda_n| = \max_{\substack{n \in \mathbb{N} \\ \lambda_n < 0}} |\lambda_n|,$$

it follows that $\|T\| = \max\{\|A\|, \|B\|\}$. □

Proposition 2.15. *The set $\mathcal{P} := \{T \in \mathcal{A}_2 : T \geq 0\}$ is closed in $\mathcal{A}_2$.*

Proof. Let $T \in \overline{\mathcal{P}}$. Then there exists a sequence $\{T_n\}$ in $\mathcal{P}$ such that $\lim_{n\to\infty} T_n = T$. Since

$$\langle Tx, y\rangle = \lim_{n\to\infty} \langle T_n x, y\rangle = \lim_{n\to\infty} \langle x, T_n y\rangle = \langle x, Ty\rangle\,,$$

for all $x, y \in c_0(\mathcal{C})$, T is self-adjoint. That T is compact follows from the fact that the space of compact operators is closed in $\mathcal{L}(c_0)$. Hence $T \in \mathcal{A}_2$. To show that $T \in \mathcal{P}$, it remains to show that $T \geq 0$. So let $x \in c_0(\mathcal{C})$ be given. Then

$$\langle Tx, x\rangle = \lim_{n\to\infty} \langle T_n x, x\rangle \geq 0,$$

since $\langle T_n x, x\rangle \geq 0$ for all $n \in \mathbb{N}$ $(T_n \geq 0)$. □

Remark 2.16. *Given $a = (a_1, a_2, \ldots) \in c_0$, then M_a is the operator defined by*

$$M_a(\cdot) = \sum_{j=1}^{\infty} a_j \langle \cdot, e_j\rangle e_j.$$

Note that the operator $\Phi : c_0 \to \{M_a : a \in c_0\}$ defined by $\Phi(a) = M_a$ is a linear isometry. Moreover,

$$M_b \circ M_a(x) = M_b\left(\sum_{j=1}^{\infty} a_j \langle x, e_j\rangle e_j\right) = \sum_{j=1}^{\infty} a_j \langle x, e_j\rangle M_b(e_j)$$
$$= \sum_{j=1}^{\infty} a_j \langle x, e_j\rangle b_j e_j = \sum_{j=1}^{\infty} a_j b_j \langle x, e_j\rangle e_j.$$

So, if we define $ab = (a_1b_1, a_2b_2, \ldots)$, then $M_b \circ M_a = M_{ab}$.

Using Theorem 2.6, we readily obtain the following result.

Proposition 2.17. *Let $a = (a_j)_{j\in\mathbb{N}}$ be given. Then $M_a \geq 0$ if and only if $a_j \in \mathcal{R}$ and $a_j \geq 0$ for all $j \in \mathbb{N}$.*

Remark 2.18. *By virtue of Proposition 2.17, we say, for $a = (a_j)_{j\in\mathbb{N}}$ in c_0 that a is positive and write $a \geq 0$ if $a_j \in \mathcal{R}$ and $a_j \geq 0$ for all $j \in \mathbb{N}$. Then it follows from our work on positive operators above that*

1. *$a \geq 0$ in $c_0 \Rightarrow$ there exists a unique $b \geq 0$ in c_0 such that $a = bb$; and*

2. *$a \in c_0(\mathcal{R}) \Rightarrow$ there exist unique $b, c \geq 0$ in $c_0(\mathcal{R})$ such that $a = b - c$ and $bc = cb = \mathbf{0}$.*

The proof of (1) and (2) follows from the facts that $a \geq 0$ if and only if M_a is positive and $a \in c_0(\mathcal{R})$ if and only if $M_a \in \mathcal{A}_2$, and from using Theorem 2.6 and Proposition 2.13 and their proofs.

3. PARTIAL ORDER ON $\mathcal{A}_2$

In this section we introduce a relation on $\mathcal{A}_2$, we show it is a partial order and we study some of its properties.

Definition 3.1. *For $S, T \in \mathcal{A}_2$, we say that $S \geq T$ (or $T \leq S$) if $S - T \geq 0$ in the sense of Definition 2.1.*

Proposition 3.2. *The relation $\geq$ in Definition 3.1 defines a partial order on $\mathcal{A}_2$.*

Proof. The reflexivity and transitivity of $\geq$ are straightforward. To show that $\geq$ is antisymmetric, let $S, T \in \mathcal{A}_2$ be such that $S \geq T$ and $T \geq S$. Then $S - T \geq 0$ and $T - S \geq 0$. Thus, for all $x \in c_0(\mathcal{C})$ we have that $\langle (S-T)x, x\rangle \geq 0$ and $\langle (T-S)x, x\rangle \geq 0$, from which we get

$$\langle (S-T)x, x\rangle = 0 \text{ for all } x \in c_0(\mathcal{C}).$$

Thus, by Corollary 2.10, $S - T = 0$ and hence $S = T$.

That the order is not total is shown by the following example. □

Example 3.3. *Let $S, T \in \mathcal{A}_2$ be the operators given by the matrix representations*

$$[S] = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \textit{and} \quad [T] = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then

$$[S-T] = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & -1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \textit{and} \quad [T-S] = \begin{bmatrix} -1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Since both $S - T$ and $T - S$ have a negative eigenvalue (-1), it follows from Theorem 2.6 that neither $S - T \geq 0$ nor $T - S \geq 0$ and hence neither $S \geq T$ nor $T \geq S$.

The following result follows immediately from Lemma 2.3 and Definition 3.1; so we state it without proof.

Proposition 3.4. *If $S \geq T$ and $U \geq V$ in $\mathcal{A}_2$ and if $\alpha \geq 0$ in $\mathcal{R}$ then $S + U \geq T + V$, $\alpha S \geq \alpha T$, and $-T \geq -S$.*

However, the following example shows that, for $R, S, T \in \mathcal{A}_2$,

$$R \geq 0 \text{ and } S \geq T \nRightarrow SR \geq TR.$$

Example 3.5. *Let $R, S, T \in \mathcal{A}_2$ be the operators given by their matrix representations:*

$$[R] = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad [S] = \begin{bmatrix} 2 & -1 & 0 & \cdots \\ -1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad [T] = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then $R \geq 0$ by Theorem 2.6. Moreover, $S - T$, given by the matrix representation

$$[S-T] = \begin{bmatrix} 1 & -1 & 0 & \cdots \\ -1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

is positive since, for all $x \in c_0(\mathcal{C})$, we have that

$$\langle (S-T)x, x\rangle = \overline{x_1}(x_1 - x_2) + \overline{x_2}(x_2 - x_1) = |x_1|_o^2 - \overline{x_1}x_2 - \overline{x_2}x_1 + |x_2|_o^2$$

$$= |x_1|_o^2 - 2\mathcal{R}\left(\overline{x_1}x_2\right) + |x_2|_o^2$$
$$\geq |x_1|_o^2 - 2|x_1|_o|x_2|_o + |x_2|_o^2 = (|x_1|_o - |x_2|_o)^2 \geq 0,$$

where, for $z = \alpha + i\beta \in \mathcal{C}$, $\mathcal{R}(z) = \alpha$ denotes the $\mathcal{R}$-part of the $\mathcal{C}$-number z. However,

$$[SR] = \begin{bmatrix} 0 & -1 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \text{ and } [TR] = 0.$$

Thus,

$$[SR - TR] = \begin{bmatrix} 0 & -1 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and hence $SR - TR \ngeq 0$ since it is not self-adjoint. It follows that $SR \ngeq TR$.

Proposition 3.6. *Let $S, T \in \mathcal{A}_2$ be given. Then $S \geq T$ if and only if $\langle Sx, x\rangle \geq \langle Tx, x\rangle$ for all $x \in c_0(\mathcal{C})$.*

Proof. First note that, since S, T and $S - T$ are self-adjoint (being elements of $\mathcal{A}_2$), we have that $\langle Sx, x\rangle$, $\langle Tx, x\rangle$ and $\langle (S-T)x, x\rangle$ are elements of $\mathcal{R}$ for all $x \in c_0(\mathcal{C})$. Thus,

$$\begin{aligned} S \geq T &\Leftrightarrow S - T \geq 0 \\ &\Leftrightarrow \langle (S-T)x, x\rangle \geq 0 \text{ for all } x \in c_0(\mathcal{C}) \\ &\Leftrightarrow \langle Sx, x\rangle - \langle Tx, x\rangle \geq 0 \text{ for all } x \in c_0(\mathcal{C}) \\ &\Leftrightarrow \langle Sx, x\rangle \geq \langle Tx, x\rangle \text{ for all } x \in c_0(\mathcal{C}). \end{aligned}$$

□

Proposition 3.7. *Let $S, T \in \mathcal{A}_2$ be such that $S \geq T \geq 0$. Then $\|S\| \geq \|T\|$.*

Proof. Since $S, T \in \mathcal{A}_2$, there exist $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}} \in c_0(\mathcal{R})$ and orthonormal sequences $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ of elements $y_n, z_n \in c_0(\mathcal{C})$ such that

$$S = \sum_{n=1}^{\infty} \alpha_n \frac{\langle \cdot, y_n\rangle}{\langle y_n, y_n\rangle} y_n \text{ and } T = \sum_{j=1}^{\infty} \beta_j \frac{\langle \cdot, z_j\rangle}{\langle z_j, z_j\rangle} z_j,$$

with $\alpha_n > 0$ for all $n \in \mathbb{N}$, $\beta_j > 0$ for all $j \in \mathbb{N}$,

$$\|S\| = \max_{n\in\mathbb{N}} |\alpha_n|, \text{ and } \|T\| = \max_{j\in\mathbb{N}} |\beta_j|.$$

Fix $j \in \mathbb{N}$. Since $S \geq T$ we have by Proposition 3.6 that $\langle Sz_j, z_j\rangle \geq \langle Tz_j, z_j\rangle$, and hence

$$\sum_{n=1}^{\infty} \alpha_n \frac{\langle z_j, y_n\rangle}{\langle y_n, y_n\rangle} \langle y_n, z_j\rangle \geq \beta_j \langle z_j, z_j\rangle;$$

that is,

$$\sum_{n=1}^{\infty} \alpha_n \frac{|\langle z_j, y_n\rangle|_o^2}{\langle y_n, y_n\rangle} \geq \beta_j \langle z_j, z_j\rangle.$$

It follows that

$$\begin{aligned}
|\beta_j| &= |\beta_j\langle z_j, z_j\rangle| \\
&\le \left|\sum_{n=1}^{\infty} \alpha_n \frac{|\langle z_j, y_n\rangle|_o^2}{\langle y_n, y_n\rangle}\right| = \max_{n\in\mathbb{N}} \left|\alpha_n \frac{|\langle z_j, y_n\rangle|_o^2}{\langle y_n, y_n\rangle}\right| \\
&= \max_{n\in\mathbb{N}} |\alpha_n| \frac{|\langle z_j, y_n\rangle|^2}{|\langle y_n, y_n\rangle|} = \max_{n\in\mathbb{N}} |\alpha_n|\, |\langle z_j, y_n\rangle|^2 \\
&\le \max_{n\in\mathbb{N}} |\alpha_n|\, |\langle z_j, z_j\rangle|\, |\langle y_n, y_n\rangle| \text{ (Cauchy-Schwartz Inequality)} \\
&= \max_{n\in\mathbb{N}} |\alpha_n| = \|S\|.
\end{aligned}$$

Thus, $|\beta_j| \le \|S\|$ for all $j \in \mathbb{N}$; and hence $\|T\| = \max_{j\in\mathbb{N}} |\beta_j| \le \|S\|$. □

Corollary 3.8. *Let $S, T \in \mathcal{A}_2$ be such that $S \le T \le 0$. Then $\|S\| \ge \|T\|$.*

Proof. Since $S \le T \le 0$, it follows from Proposition 3.4 that $-S \ge -T \ge 0$. Hence, by Proposition 3.7, we obtain that $\|-S\| \ge \|-T\|$; that is, $\|S\| \ge \|T\|$. □

Proposition 3.9. *Let $S \ge T$ in $\mathcal{A}_2$ and let $R \in \mathcal{A}_1$ be given. Then*

$$R^*SR \ge R^*TR.$$

Proof. First note that R^*SR and R^*TR are both self-adjoint since S and T are. Thus, $R^*SR, R^*TR \in \mathcal{A}_2$. Now let $x \in c_0(\mathcal{C})$ be given. Then

$$\langle (R^*SR - R^*TR)x, x\rangle = \langle R^*(S-T)Rx, x\rangle = \langle (S-T)Rx, Rx\rangle \ge 0$$

since $S - T \ge 0$. Thus $R^*SR - R^*TR \ge 0$, and hence $R^*SR \ge R^*TR$. □

Remark 3.10. *As a follow-up to Remark 2.18, we can introduce a partial order on $c_0(\mathcal{R})$ (which is isometrically isomorphic to $\mathcal{A}_2$ [1]) as follows: for $a = (a_j)_{j\in\mathbb{N}}$ and $b = (b_j)_{j\in\mathbb{N}}$ in $c_0(\mathcal{R})$, we say that $a \ge b$ if $a - b \ge 0$; that is, if $a_j - b_j \ge 0$ for all $j \in \mathbb{N}$ (or equivalently $a_j \ge b_j$ for all $j \in \mathbb{N}$.) Then $a \ge b$ in $c_0(\mathcal{R})$ if and only if $M_a \ge M_b$ in $\mathcal{A}_2$.*

We finish the paper with the following result which gives equivalent conditions for two normal projections $P_1, P_2 \in \mathcal{A}_2$ to be related by the order relation defined above (Definition 3.1).

Theorem 3.11. *Let $P_1, P_2 \in \mathcal{A}_2$ be normal projections and let $M_1 = R(P_1)$ and $M_2 = R(P_2)$. Then the following are equivalent.*

(1) $P_2 \ge P_1$;

(2) $M_2 \supseteq M_1$;

(3) $P_2P_1 = P_1$;

(4) $P_1P_2 = P_1$.

Proof. (1) ⇒ (2): Assume that $P_2 \ge P_1$. Then $\langle P_2x, x\rangle \ge \langle P_1x, x\rangle$ for all $x \in c_0$. Since P_1 and P_2 are normal projections (hence idempotent and self-adjoint), it follows that

$$\langle P_2x, P_2x\rangle = \langle P_2x, x\rangle \ge \langle P_1x, x\rangle = \langle P_1x, P_1x\rangle \text{ for all } x \in c_0.$$

Now let $x \in M_1$ be given. Then $P_1x = x$ and hence it follows that

$$\langle x, x\rangle = \langle P_1x, P_1x\rangle \le \langle P_2x, P_2x\rangle \le \langle x, x\rangle;$$

and hence

$$\langle x, x\rangle = \langle P_2x, P_2x\rangle.$$

Using the Pythagorian Theorem, it follows that

$$\langle x - P_2x, x - P_2x\rangle = 0; \text{ and hence } P_2x = x.$$

This shows that $x \in M_2$. Thus, $M_1 \subseteq M_2$.

(2) $\Rightarrow$ (3): Assume that $M_1 \subseteq M_2$. Let $x \in c_0$ be given; then $P_1x \in M_1$ and hence $P_1x \in M_2$. It follows that

$$P_2P_1x = P_2(P_1x) = P_1x.$$

Since this is true for all $x \in c_0$, it follows that

$$P_2P_1 = P_1.$$

(3) $\Leftrightarrow$ (4): This follows from taking adjoints of the left- and right-hand sides of the last equation above.

(4) $\Rightarrow$ (1): Assume that $P_1P_2 = P_1$. Then $P_2P_1 = P_1$ too. Let $x \in c_0$ be given. Then

$$\langle P_2x, x\rangle - \langle P_1x, x\rangle = \langle P_2x, x\rangle - \langle P_2P_1x, x\rangle = \langle P_2(I - P_1)x, x\rangle.$$

Since P_1 and P_2 commute, so do P_2 and $I - P_1$. Let $P = P_2(I - P_1)$; we show that $P^2 = P$ and $P^* = P$ and hence P itself is a normal projection. Thus,

$$\begin{aligned} P^2 &= (P_2(I - P_1))(P_2(I - P_1)) = P_2(I - P_1)^2 P_2 = P_2(I - P_1)P_2 = P_2^2(I - P_1) \\ &= P_2(I - P_1) = P; \end{aligned}$$

and

$$(P_2(I - P_1))^* = (I - P_1)^* P_2^* = (I - P_1)P_2 = P_2(I - P_1) = P.$$

Thus, it follows that P is a normal projection. Therefore,

$$\langle P_2x, x\rangle - \langle P_1x, x\rangle = \langle Px, x\rangle = \langle Px, Px\rangle \geq 0; \text{ and hence } \langle P_2x, x\rangle \geq \langle P_1x, x\rangle.$$

Since the last equation holds for all $x \in c_0$, it follows that $P_2 \geq P_1$. □

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