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RESEARCH ARTICLES

A Local Mean Value Theorem for Functions on Non-Archimedean Field Extensions of the Real Numbers*

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Abstract—In this paper, we review the definition and properties of locally uniformly differentiable functions on $\mathcal{N}$, a non-Archimedean field extension of the real numbers that is real closed and Cauchy complete in the topology induced by the order. Then we define and study n-times locally uniform differentiable functions at a point or on a subset of $\mathcal{N}$. In particular, we study the properties of twice locally uniformly differentiable functions and we formulate and prove a local mean value theorem for such functions.

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1. INTRODUCTION

We start this section by reviewing some basic terminology and facts about non-Archimedean fields. So let F be an ordered non-Archimedean field extension of $\mathbb{R}$. We introduce the following terminology.

Definition 1.1 ($\sim$, $\approx$, $\ll$, S_F, λ). *For $x, y \in F^* := F \setminus \{0\}$, we say that x is of the same order as y and write $x \sim y$ if there exist $n, m \in \mathbb{N}$ such that $n|x| > |y|$ and $m|y| > |x|$, where $|\cdot|$ denotes the ordinary absolute value on F: $|x| = \max\{x, -x\}$. For nonnegative $x, y \in F$, we say that x is infinitely smaller than y and write $x \ll y$ if $nx < y$ for all $n \in \mathbb{N}$, and we say that x is infinitely small if $x \ll 1$ and x is finite if $x \sim 1$; finally, we say that x is approximately equal to y and write $x \approx y$ if $x \sim y$ and $|x - y| \ll |x|$. We also set $\lambda(x) = [x]$, the class of x under the equivalence relation $\sim$.*

The set of equivalence classes S_F (under the relation $\sim$) is naturally endowed with an addition via $[x] + [y] = [x \cdot y]$ and an order via $[x] < [y]$ if $|y| \ll |x|$ (or $|x| \gg |y|$), both of which are readily checked to be well-defined. Note that we use $+$ instead of $\cdot$ for the operation in S_F because, for the fields discussed in this paper, S_F is isomorphic to an additive subgroup of $\mathbb{R}$. It follows that $(S_F, +, <)$ is an ordered group, often referred to as the Hahn group or skeleton group, whose neutral element is [1], the class of 1. It follows from the above that the projection λ from F^* to S_F is a valuation.

The theorem of Hahn [2] provides a complete classification of non-Archimedean extensions of $\mathbb{R}$ in terms of their skeleton groups. In fact, invoking the axiom of choice it is shown that the elements of any such ordered field F can be written as (generalized) formal power series (also called Hahn series) over its skeleton group S_F with real coefficients, and the set of appearing exponents forms a well-ordered subset of S_F. That is, for all $x \in F$, we have

$$x = \sum_{q \in S_F} a_q d^q; \tag{1.1}$$

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with $a_q \in \mathbb{R}$ for all q, d a positive infinitely small element of F, and the support of x, given by

$$\operatorname{supp}(x) := \{q \in S_F : a_q \neq 0\},$$

forming a well-ordered subset of S_F. With the representation of elements of F as in Eq. (1.1) it follows that, for $x \neq 0$ in F,

$$\lambda(x) = \min\left(\operatorname{supp}(x)\right),$$

which exists since $\operatorname{supp}(x)$ is well-ordered. Moreover, we set $\lambda(0) = \infty$.

Addition, multiplication and order on the Hahn series are defined as follows. Given $x = \sum_{q\in\operatorname{supp}(x)} a_q d^q$ and $y = \sum_{t\in\operatorname{supp}(y)} b_t d^t$, then

$$x + y = \sum_{r\in\operatorname{supp}(x)\cup\operatorname{supp}(y)} (a_r + b_r)d^r; \text{ and}$$

$$x \cdot y = \sum_{r\in\operatorname{supp}(x)\oplus\operatorname{supp}(y)} \left(\sum_{\substack{q \in \operatorname{supp}(x),\, t \in \operatorname{supp}(y) \\ q+t=r}} a_q \cdot b_t \right) d^r. \tag{1.2}$$

Note that, since $\operatorname{supp}(x)$ and $\operatorname{supp}(y)$ are well-ordered, only finitely many terms contribute to the sum

$$\sum_{\substack{q \in \operatorname{supp}(x),\, t \in \operatorname{supp}(y) \\ q+t=r}} a_q \cdot b_t,$$

in Eq. (1.2), for each $r \in \operatorname{supp}(x) \oplus \operatorname{supp}(y)$.

Given a nonzero $x = \sum_{q\in\operatorname{supp}(x)} a_q d^q$, then $x > 0$ if and only if $a_{\lambda(x)} > 0$.

From general properties of formal power series fields [6, 8], it follows that if S_F is divisible then F is real closed; that is, every positive element of F is a square in F and every polynomial of odd degree over F has at least one root in F. For a general overview of the algebraic properties of formal power series fields, we refer to the comprehensive overview by Ribenboim [9], and for an overview of the related valuation theory the book by Krull [3]. A thorough and complete treatment of ordered structures can also be found in [7].

Throughout this paper, we will denote by $\mathcal{N}$ any totally ordered non-Archimedean field extension of $\mathbb{R}$ that is real closed and complete in the order topology and whose skeleton group $S_{\mathcal{N}}$ is Archimedean, i.e. a subgroup of $\mathbb{R}$. The coefficient a_q of the qth power in the Hahn representation of a given x will be denoted by $x[q]$, and hence the number d is given by $d[1] = 1$ and $d[q] = 0$ for $q \neq 1$. It is easy to check that, for $q \in S_{\mathcal{N}}$, $0 < d^q \ll 1$ if and only if $q > 0$, and $d^q \gg 1$ if and only if $q < 0$; moreover, $x \approx x[\lambda(x)]d^{\lambda(x)}$ for all $x \neq 0$.

The smallest such field $\mathcal{N}$ is the Levi-Civita field $\mathcal{R}$, first introduced in [4, 5]. In this case $S_{\mathcal{R}} = \mathbb{Q}$, and for any element $x \in \mathcal{R}$, $\operatorname{supp}(x)$ is a left-finite subset of $\mathbb{Q}$, i.e. below any rational bound r there are only finitely many exponents in the Hahn representation of x. The Levi-Civita field $\mathcal{R}$ is of particular interest because of its practical usefulness. Since the supports of the elements of $\mathcal{R}$ are left-finite, it is possible to represent these numbers on a computer [1]. Having infinitely small numbers allows for many computational applications similar to those obtained with the numerical system employed by Sergeyev in [12–14]. One such application is the computation of derivatives of real functions representable on a computer [16], where both the accuracy of formula manipulators and the speed of classical numerical methods are achieved. For a review of the Levi-Civita field $\mathcal{R}$, see [15, 17] and references therein.

In the wider context of valuation theory, it is interesting to note that the topology induced by the order on $\mathcal{N}$ is the same as that introduced via the valuation λ, as shown in Remark 1.2 below. It follows therefore that the field $\mathcal{N}$ is just a special case of the class of fields discussed in [11].

Remark 1.2. *The mapping $\Lambda : \mathcal{N} \times \mathcal{N} \to \mathbb{R}$, given by $\Lambda(x, y) = \exp(-\lambda(x - y))$, is an ultrametric distance (and hence a metric); the valuation topology it induces is equivalent to the order topology (we will use τ_v to denote either one of the two topologies in this paper). For if A is an open set in the order topology and $a \in A$, then there exists $r > 0$ in $\mathcal{N}$ such that, for all $x \in \mathcal{N}$, $|x - a| < r \Rightarrow x \in A$. Let $l = \exp(-\lambda(r))$, then we also have that, for all $x \in \mathcal{N}$, $\Lambda(x, a) < l \Rightarrow x \in A$; and hence A is open with respect to the valuation topology. The other direction of the equivalence of the topologies follows analogously.*

It follows from Remark 1.2 that $\mathcal{N}$ which is complete in the order topology is also complete in the valuation topology induced by the ultrametric Λ.

Remark 1.3. *Contrary to the field $\mathbb{R}^*$ of Nonstandard Analysis [10, 20], the field $\mathcal{N}$ is an ordered field extension of the field of real numbers $\mathbb{R}$; and the embedding of $\mathbb{R}$ in $\mathcal{N}$ is compatible with the orders in $\mathbb{R}$ and $\mathcal{N}$. While in Nonstandard Analysis there is a generally valid transfer principle that allows the transformation of known results of conventional analysis, here all relevant calculus theorems are developed separately. Moreover, besides being non-Archimedeanly valued, the fact that the field $\mathcal{N}$ has a total order (which is also non-Archimedean) gives the field a richer structure, thus opening up new possibilities of study, like monotonicity, which are not available in other non-Archimedean valued fields like the p-adic fields for example [11]. This makes $\mathcal{N}$ an outstanding example, worthy to be studied in detail in its own right.*

The following results were proved in [19]; they show that the topological structure of $\mathcal{N}$ is different from that of $\mathbb{R}$ or $\mathbb{C}$, and that makes doing Calculus on the field more difficult.

- $(\mathcal{N}, \tau_v)$ is a totally disconnected topological space. It is Hausdorff and nowhere locally compact. There are no countable bases. The topology induced to $\mathbb{R}$ is the discrete topology. As an immediate consequence of the fact that $(\mathcal{N}, \tau_v)$ is totally disconnected, it follows that, for any $x_0 \in \mathcal{N}$, the connected component of x_0 is $\{x_0\}$; moreover, the topology is zero-dimensional, that is, there is a base of clopen sets for the topology.

- If we view $\mathcal{N}$ as an infinite dimensional vector space over $\mathbb{R}$ then τ_v is not a vector topology; that is, $(\mathcal{N}, \tau_v)$ is not a linear topological space.

- If A is compact in $(\mathcal{N}, \tau_v)$ then A is closed and bounded and it has an empty interior in $(\mathcal{N}, \tau_v)$, that is,
$$\mathrm{int}(A) := \{a \in A : \exists r > 0 \text{ in } \mathcal{N} \ni (a - r, a + r) \subset A\} = \emptyset.$$
The converse is not true: the set $A = [0, 1] \cap \mathbb{Q}$ is a (countably infinite) closed and bounded subset of $\mathcal{N}$ with an empty interior; but A is not compact in $(\mathcal{N}, \tau_v)$ [19].

- Given a sequence (x_n) of elements of $\mathcal{N}$, the series $\sum_{n=1}^{\infty} x_n$ converges if and only if the sequence (x_n) converges to zero.

In [19] we studied the properties of locally uniformly differentiable functions on $\mathcal{N}$. In particular, we showed that this class of functions is closed under addition, multiplication and composition of functions. Then we stated and proved local versions of the inverse function theorem and the intermediate value theorem for $\mathcal{N}$-valued locally uniformly differentiable functions on $\mathcal{N}$. The stronger condition (local uniform differentiability) on the function than that of the real case was needed for the proofs of both theorems because of the total disconnectedness of the field $\mathcal{N}$ in the order topology. In this paper which is a continuation of [19] and complements it, we generalize the definition of local uniform differentiability to any order. Then we study the properties of n-times locally uniformly differentiable functions and we formulate and prove a local mean value theorem for $\mathcal{N}$-valued functions that are twice locally uniformly differentiable at a point of $\mathcal{N}$.

2. LOCAL UNIFORM DIFFERENTIABILITY

Definition 2.1. *Let $A \subseteq \mathcal{N}$ and let $f : A \to \mathcal{N}$. We say that is uniformly differentiable (UD) on A if f is differentiable on A and for every $\epsilon > 0$ in $\mathcal{N}$ there exists $\delta > 0$ in $\mathcal{N}$ such that, whenever $x, y \in A$ with $0 < |y - x| < \delta$, we have that $|f(y) - f(x) - f'(x)(y - x)| < \epsilon|y - x|$.*

Remark 2.2. *The domain A as well as the range of f in Definition 2.1 above (and throughout the rest of this paper) are subsets of $\mathcal{N}$ and hence they may have numbers that are finite, infinitely small or infinitely large in absolute value. The same is true about ϵ and δ in Definition 2.1 and throughout the rest of the paper: they are positive elements of $\mathcal{N}$ which may be finite, infinitely small or infinitely large numbers.*

Definition 2.3. *Let $A \subseteq \mathcal{N}$, let $f : A \to \mathcal{N}$, and let $x_0 \in A$ be given. We say that f is locally uniformly differentiable (LUD) at x_0 if there exists a neighborhood U of x_0 in A such that f is uniformly differentiable on U. Moreover, we say that f is locally uniformly differentiable on A if f is locally uniformly differentiable at every point in A.*

For a review of the basic properties of LUD functions on $\mathcal{N}$, we refer the reader to [19]. In particular, we note here that the class of LUD functions is closed under addition, multiplication, and composition, and, as we will prove in the following proposition, is a subset of the C^1 functions.

Proposition 2.4. *Let $f : A \to \mathcal{N}$ be LUD at $x_0 \in A$. Then f is C^1 at x_0.*

Proof. Let U be a neighborhood of x_0 in A such that f is uniformly differentiable on U and let $\epsilon > 0$ in $\mathcal{N}$ be given. Then there exists $\delta > 0$ in $\mathcal{N}$ such that, for every $x, y \in U$ with $0 < |y - x| < \delta$, we have that

$$|f(y) - f(x) - f'(x)(y - x)| < \frac{\epsilon}{2}|y - x|.$$

It follows that, for all $x \in U$ satisfying $0 < |x - x_0| < \delta$, we have that

$$\begin{aligned} |f'(x) - f'(x_0)| &\leq |f'(x) - \frac{f(x_0) - f(x)}{x_0 - x}| + |\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)| \\ &= |\frac{f(x_0) - f(x)}{x_0 - x} - f'(x)| + |\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

The question then arises on how to extend the concept of local uniform differentiability to higher orders. Naturally we would want to define this in such a way that 'LUD^n' implies 'LUD^{n-1}', but as was shown in [19], the property of a function's being LUD is neither passed on to its derivatives, nor inherited from its derivatives. This difficulty was solved in [19] by defining a function to be LUD^n when the function as well as its first $(n - 1)$ derivatives are LUD. However, as we shall see, this definition does not imply the function locally has the mean value property. Thus we instead consider the definition of LUD in the sense that it gives us a (local) uniform bound for the remainder of the first order Taylor polynomial. In this light, a natural extension of the concept to higher orders becomes apparent.

Definition 2.5. *Let $A \subset \mathcal{N}$, let $f : A \to \mathcal{N}$, and let $n \in \mathbb{N} \cup \{0\}$ be given. Then we say that f is UD^n on A if f is n times differentiable on A and for every $\epsilon > 0$ in $\mathcal{N}$ there exists $\delta > 0$ in $\mathcal{N}$ such that, whenever $x, y \in A$ with $0 < |y - x| < \delta$, we have that*

$$|f(y) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(y - x)^k| < \epsilon|y - x|^n.$$

Definition 2.6. *Let $A \subseteq \mathcal{N}$, let $f : A \to \mathcal{N}$, let $x_0 \in A$, and let $n \in \mathbb{N} \cup \{0\}$ be given. We say that f is LUD^n at x_0 if there exists a neighborhood U of x_0 in A such that f is UD^n on U. Moreover, we say that f is LUD^n on A if f is LUD^n at every point in A.*

Remark 2.7. *For $n = 0$, UD^n means uniformly continuous; and hence LUD^n means locally uniformly continuous.*

Definition 2.8. *Let $A \subseteq \mathcal{N}$, let $f : A \to \mathcal{N}$, and let $x_0 \in A$ be given. We say that f is LUD^∞ at x_0 if f is LUD^n at x_0 for every $n \in \mathbb{N}$. Moreover, we say that f is LUD^∞ on A if f is LUD^∞ at every point in A.*

Thus, LUD^n gives us a (local) uniform bound for the remainder of the nth order Taylor polynomial. It is worth noting (although we will not prove it here) that in the real case LUD^n is equivalent[1] to C^n, and so for that case there is no distinction between our definition and that which was presented in [19]. Moreover, as we will see (in Proposition 2.13 below), this definition of LUD^n implies LUD^{n-1} (for $n \in \mathbb{N}$) without having to include it explicitly as part of the definition, and thus is arguably a more natural way of extending the concept of LUD to higher orders.

Remark 2.9. *For functions on $\mathcal{N}$, the concept of LUD^n presented in this paper is distinct from the concept of LUD^n presented in [19]. In particular, in the next example we show there exist functions that are LUD with LUD derivatives that are not LUD^2.*

Example 2.10. *Let $f : (-1, 1) \to \mathcal{N}$ be given by*

$$f(x) = \begin{cases} d^{2\lambda(x)} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

We will show that f and f' are LUD. Indeed, it suffices to prove that f is LUD on (-1, 1) with derivative $f'(x) = 0$, as all constant functions are trivially LUD.

For $x_0 \neq 0$, let $U = \{x \in \mathcal{N} \mid x \approx x_0\}$. Then f is constant on U and thus is LUD on U with derivative $f'(x) = 0$. For $x_0 = 0$, let $U = (-d, d)$, let $\epsilon > 0$ in $\mathcal{N}$ be given, let $\delta = d\epsilon$, and let $x, y \in U$ be such that $0 < |y - x| < \delta$. We have the following 3 cases.

Case 1: Assume $x, y \neq 0$ and $x \sim y$. Then $\lambda(y) = \lambda(x)$ and so

$$|f(y) - f(x) - 0(y - x)| = 0 < \epsilon|y - x|.$$

Case 2: Assume $x, y \neq 0$ but $x \not\sim y$. Then, without loss of generality, we may assume that $|x| < |y|$ and hence $\lambda(x) > \lambda(y)$. Thus

$$\begin{aligned} |f(y) - f(x) - 0(y - x)| &= |d^{2\lambda(y)} - d^{2\lambda(x)}| \sim d^{2\lambda(y)} \sim y^2 \\ &\sim (y - x)^2 \ll d^{-1/2}\delta|y - x| = d^{1/2}\epsilon|y - x| \\ &\ll \epsilon|y - x|. \end{aligned}$$

Case 3: Assume either x or y is zero. Without loss of generality, we may assume $x = 0$. Then

$$\begin{aligned} |f(y) - f(x) - 0(y - x)| = d^{2\lambda(y)} &\sim y^2 = (y - x)^2 \ll d^{-1/2}\delta|y - x| = d^{1/2}\epsilon|y - x| \\ &\ll \epsilon|y - x|, \end{aligned}$$

which completes the proof that f is LUD on (-1, 1) with derivative $f'(x) = 0$. As we will see however, f is not LUD^2 at 0 in the sense of Definition 2.6. That is, for any neighborhood U of 0, there exists $\epsilon > 0$ in $\mathcal{N}$ such that for every $\delta > 0$ in $\mathcal{N}$ there exist $x, y \in U$ with $0 < |y - x| < \delta$ such that $|f(y) - f(x)| \geq \epsilon(y - x)^2$.

Let U be a given neighborhood of 0, let $\epsilon = 1/2$, let $\delta > 0$ be given in $\mathcal{N}$, and let $N \in \mathbb{N}$ be such that $d^N \in U$ and $d^N < \delta$. Then

$$|f(d^N) - f(0)| = d^{2N} > \epsilon\left(d^N - 0\right)^2.$$

Hence f is not LUD^2 at 0.

[1] This is a fairly simple consequence of the Lagrange form of the $(n-1)$th Taylor remainder.

Notation 2.11. *Throughout the rest of the paper, we will use $B(x_0, r)$ to denote the open interval $(x_0 - r, x_0 + r)$, for $x_0 \in \mathcal{N}$ and $r > 0$ in $\mathcal{N}$.*

Lemma 2.12. *Let $f : A \to \mathcal{N}$ be LUD^n at $x_0 \in A$. Then $f^{(n)}$ is locally bounded at x_0; that is, there exist a neighborhood U of x_0 in A and an $M > 0$ in $\mathcal{N}$ such that, for every $x \in U$, we have that $|f^{(n)}(x)| \le M$.*

Proof. Let U_0 be a neighborhood of x_0 such that f is UD^n on U_0. Hence there exists $\delta_1 > 0$ in $\mathcal{N}$ such that, for every $x, y \in U_0$ satisfying $0 < |y - x| < \delta_1$, we have that

$$|f(y) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(y - x)^k| < |y - x|^n.$$

For each $k \in \{0, 1, \ldots, n-1\}$, $f^{(k)}$ is continuous at x_0, and so there exist a neighborhood V_k of x_0 and a number $M_k > 0$ in $\mathcal{N}$ such that for every $x \in V_k$ we have that $|f^{(k)}(x)| \le M_k$. Let $\delta > 0$ in $\mathcal{N}$ be such that $B(x_0, \delta) \subseteq B(x_0, \delta_1) \cap U_0 \cap \left(\cap_{k=0}^{n-1} V_k\right)$ and let $U = B(x_0, \delta)$. Now let

$$M = n! \left(\frac{2}{\delta}\right)^n \left(\left(\frac{\delta}{2}\right)^n + M_0 + \sum_{k=0}^{n-1} \frac{M_k}{k!} \left(\frac{\delta}{2}\right)^k \right),$$

and let $x \in U$ be given. Choose

$$y = \begin{cases} x + \frac{\delta}{2} & \text{if } x + \frac{\delta}{2} \in B(x_0, \delta) \\ x - \frac{\delta}{2} & \text{otherwise.} \end{cases}$$

Then it follows that $y \in U \subset U_0$ and $|y - x| = \delta/2 < \delta_1$. Thus,

$$\begin{aligned} |f^{(n)}(x)| &\le \frac{n!}{|y - x|^n} \left(|f(y) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(y - x)^k| + |f(y)| + \sum_{k=0}^{n-1} |\frac{f^{(k)}(x)}{k!}(y - x)^k| \right) \\ &< \frac{n!}{|y - x|^n} \left(|y - x|^n + M_0 + \sum_{k=0}^{n-1} \frac{M_k}{k!} |y - x|^k \right) \\ &= n! \left(\frac{2}{\delta}\right)^n \left(\left(\frac{\delta}{2}\right)^n + M_0 + \sum_{k=0}^{n-1} \frac{M_k}{k!} \left(\frac{\delta}{2}\right)^k \right) = M. \end{aligned}$$

□

Proposition 2.13. *Let $f : A \to \mathcal{N}$ be LUD^n at $x_0 \in A$ for some $n \in \mathbb{N}$. Then f is LUD^{n-1} at x_0.*

Proof. Let U_1 be a neighborhood of x_0 such that f is UD^n on U_1. By Lemma 2.12, there exists a neighborhood U_2 of x_0 such that $f^{(n)}$ is locally bounded by some $M > 0$ on U_2. Let $U = U_1 \cap U_2$, and let $\epsilon > 0$ in $\mathcal{N}$ be given. As f is UD^n on U, there exists $\delta_1 > 0$ in $\mathcal{N}$ such that, for every $x, y \in U$ satisfying $0 < |y - x| < \delta_1$, we have that

$$|f(y) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(y - x)^k| < \frac{\epsilon}{2}|y - x|^n.$$

Let $\delta = \min\{\delta_1, 1, n!\epsilon/(2M)\}$. Then, for every $x, y \in U$ satisfying $0 < |y - x| < \delta$, we have that

$$\begin{aligned} |f(y) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!}(y - x)^k| &\le |f(y) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(y - x)^k| \\ &\quad + |\frac{f^{(n)}(x)}{n!}(y - x)^n| \end{aligned}$$

$$
\begin{aligned}
&< \frac{\epsilon}{2}|y-x|^n + |\frac{f^{(n)}(x)}{n!}|\delta|y-x|^{n-1} \\
&\le \frac{\epsilon}{2}|y-x||y-x|^{n-1} + \frac{M}{n!}\frac{n!\epsilon}{2M}|y-x|^{n-1} \\
&< \epsilon|y-x|^{n-1},
\end{aligned}
$$

where in the last inequality we used the fact that $|y-x| < \delta \le 1$. □

Proposition 2.14. *Let $f : A \to \mathcal{N}$ be LUD^2 at $x_0 \in A$. Then f is C^2 at x_0.*

Proof. Let U be a neighborhood of x_0 in A such that f is UD^2 on U and let $\epsilon > 0$ in $\mathcal{N}$ be given. Then there exists $\delta_1 > 0$ in $\mathcal{N}$ such that, for every $x, y \in U$ satisfying $0 < |y-x| < \delta_1$, we have that

$$|f(y) - f(x) - f'(x)(y-x) - \frac{1}{2}f''(x)(y-x)^2| < \frac{\epsilon}{6}(y-x)^2.$$

As f' is differentiable at x_0, there exists $\delta_2 > 0$ in $\mathcal{N}$ such that, for every $x \in A$ satisfying $0 < |x - x_0| < \delta_2$, we have that

$$|\frac{f'(x) - f'(x_0)}{x - x_0} - f''(x_0)| < \frac{\epsilon}{6}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then it follows that $\delta > 0$ in $\mathcal{N}$ and if $|x - x_0| < \delta$ then we have that

$$
\begin{aligned}
|f''(x) - f''(x_0)| &\le 2|\frac{1}{2}f''(x) + \frac{f'(x)}{x_0 - x} + \frac{f(x) - f(x_0)}{(x_0 - x)^2}| \\
&\quad + 2|\frac{f'(x) - f'(x_0)}{x - x_0} - f''(x_0)| \\
&\quad + 2|\frac{1}{2}f''(x_0) + \frac{f'(x_0)}{x - x_0} + \frac{f(x_0) - f(x)}{(x - x_0)^2}| \\
&= 2|\frac{f(x_0) - f(x)}{(x_0 - x)^2} - \frac{f'(x)}{x_0 - x} - \frac{1}{2}f''(x)| \\
&\quad + 2|\frac{f'(x) - f'(x_0)}{x - x_0} - f''(x_0)| \\
&\quad + 2|\frac{f(x) - f(x_0)}{(x - x_0)^2} - \frac{f'(x_0)}{x - x_0} - \frac{1}{2}f''(x_0)| \\
&< 2\frac{\epsilon}{6} + 2\frac{\epsilon}{6} + 2\frac{\epsilon}{6} = \epsilon.
\end{aligned}
$$

□

Remark 2.15. *Example 2.10 shows that the converse of Proposition 2.14 is not true. Thus, Proposition 2.14 and Example 2.10 show that that the class of LUD^2 functions is a proper subset of the class of C^2 functions. However, this is still large enough to include all polynomial functions as Corollary 2.18 below will show.*

Similar to LUD functions, we find that the class of LUD^n functions are closed under addition, multiplication, and composition. As this paper will focus primarily on functions that are LUD^2, we present the proofs for the $n = 2$ case here. The proofs for the general case are similar.

Proposition 2.16. *Let $f, g : A \to \mathcal{N}$ be LUD^2 at $x_0 \in A$ and let $\alpha \in \mathcal{N}$ be given. Then $(f + \alpha g)$ is LUD^2 at x_0.*

Proof. Without loss of generality, we may assume $\alpha \neq 0$. Let U_f and U_g be neighborhoods of x_0 in A such that f and g are UD^2 on U_f and U_g respectively, let $U = U_f \cap U_g$, and let $\epsilon > 0$ in $\mathcal{N}$ be given. Then there exists $\delta > 0$ in $\mathcal{N}$ such that, for every $x, y \in U$ with $0 < |y - x| < \delta$, we have that

$$|f(y) - f(x) - f'(x)(y-x) - \frac{1}{2}f''(x)(y-x)^2| < \frac{\epsilon}{2}(y-x)^2$$

and

$$|g(y) - g(x) - g'(x)(y-x) - \frac{1}{2}g''(x)(y-x)^2| < \frac{\epsilon}{2|\alpha|}(y-x)^2.$$

Hence, for every $x, y \in U$ with $0 < |y - x| < \delta$, we have that

$$\begin{aligned}
&\left|(f+\alpha g)(y) - (f+\alpha g)(x) - (f+\alpha g)'(x)(y-x) - \frac{1}{2}(f+\alpha g)''(x)(y-x)^2\right| \\
&= \left|\left[f(y) - f(x) - f'(x)(y-x) - \frac{1}{2}f''(x)(y-x)^2\right]\right. \\
&\qquad \left. + \alpha\left[g(y) - g(x) - g'(x)(y-x) - \frac{1}{2}g''(x)(y-x)^2\right]\right| \\
&\le |f(y) - f(x) - f'(x)(y-x) - \frac{1}{2}f''(x)(y-x)^2| \\
&\qquad + |\alpha||g(y) - g(x) - g'(x)(y-x) - \frac{1}{2}g''(x)(y-x)^2| \\
&< \frac{\epsilon}{2}(y-x)^2 + |\alpha|\frac{\epsilon}{2|\alpha|}(y-x)^2 \\
&= \epsilon(y-x)^2.
\end{aligned}$$

□

Proposition 2.17. *Let $f, g : A \to \mathcal{N}$ be LUD^2 at $x_0 \in A$. Then fg is LUD^2 at x_0.*

Proof. Let U_f and U_g be neighborhoods of x_0 such that f and g are UD^2 on U_f and U_g respectively. By Proposition 2.13, g is LUD and so there exists a neighborhood U_0 of x_0 on which g is uniformly differentiable. Finally, as f and g are C^2 at x_0, there exists a neighborhood U_1 of x_0 such that $|f(x) - f(x_0)| < 1$, $|g(x) - g(x_0)| < 1$, $|f'(x) - f'(x_0)| < 1$, and $|f''(x) - f''(x_0)| < 1$ on U_1. Let $U = U_f \cap U_g \cap U_0 \cap U_1$ and let $\epsilon > 0$ in $\mathcal{N}$ be given. Then there exist $\delta_f, \delta_g, \delta_0, \delta_1 > 0$ in $\mathcal{N}$ such that for every $x, y \in U$ we have that

$$|f(y) - f(x) - f'(x)(y-x) - \frac{1}{2}f''(x)(y-x)^2| < \frac{\epsilon}{4(|g(x_0)|+1)}(y-x)^2$$

if $|y - x| < \delta_f$;

$$|g(y) - g(x) - g'(x)(y-x) - \frac{1}{2}g''(x)(y-x)^2| < \frac{\epsilon}{4(|f(x_0)|+1)}(y-x)^2$$

if $|y - x| < \delta_g$;

$$|g(y) - g(x) - g'(x)(y-x)| < \frac{\epsilon}{4(|f'(x_0)|+1)}|y-x|$$

if $|y - x| < \delta_0$; and

$$|g(y) - g(x)| < \frac{\epsilon}{2(|f''(x_0)|+1)}$$

if $|y - x| < \delta_1$.

Let $\delta = \min\{\delta_f, \delta_g, \delta_0, \delta_1\}$. Then it follows that, for every $x, y \in U$ with $0 < |y - x| < \delta$, we have that

$$
\begin{aligned}
&|(fg)(y) - (fg)(x) - (fg)'(x)(y-x) - \frac{1}{2}(fg)''(x)(y-x)^2| \\
&= \Big| f(y)g(y) - f(x)g(x) - \Big(f'(x)g(x) + f(x)g'(x)\Big)(y-x) \\
&\qquad - \frac{1}{2}\Big(f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)\Big)(y-x)^2 \Big| \\
&\le |f(y)g(y) - f(x)g(y) - f'(x)g(y)(y-x) - \frac{1}{2}f''(x)g(y)(y-x)^2| \\
&\qquad + |f(x)g(y) - f(x)g(x) - f(x)g'(x)(y-x) - \frac{1}{2}f(x)g''(x)(y-x)^2| \\
&\qquad + |f'(x)g(y)(y-x) - f'(x)g(x)(y-x) - f'(x)g'(x)(y-x)^2| \\
&\qquad + \frac{1}{2}|f''(x)g(y) - f''(x)g(x)|(y-x)^2 \\
&= |g(y)||f(y) - f(x) - f'(x)(y-x) - \frac{1}{2}f''(x)(y-x)^2| \\
&\qquad + |f(x)||g(y) - g(x) - g'(x)(y-x) - \frac{1}{2}g''(x)(y-x)^2| \\
&\qquad + |f'(x)||y-x||g(y) - g(x) - g'(x)(y-x)| \\
&\qquad + \frac{1}{2}|f''(x)||g(y) - g(x)|(y-x)^2 \\
&< \frac{|g(y)|}{4(|g(x_0)|+1)}\epsilon(y-x)^2 + \frac{|f(x)|}{4(|f(x_0)|+1)}\epsilon(y-x)^2 \\
&\qquad + \frac{|f'(x)|}{4(|f'(x_0)|+1)}\epsilon(y-x)^2 + \frac{|f''(x)|}{4(|f''(x_0)|+1)}\epsilon(y-x)^2 \\
&< \epsilon(y-x)^2.
\end{aligned}
$$

□

Corollary 2.18. *All polynomials are LUD*^{2} *on* $\mathcal{N}$.

Proof. Using Proposition 2.16 and Proposition 2.17, it suffices to show that the function $f(x) = x$ is LUD^2 on $\mathcal{N}$. But that follows readily from the fact that, for all $x, y \in \mathcal{N}$, we have that

$$|f(y) - f(x) - f'(x)(y-x) - \frac{1}{2}f''(x)(y-x)^2| = |y - x - (y-x)| = 0.$$

□

Proposition 2.19. *Let* $g : A \to B$ *be LUD*^{2} *at* $x_0 \in A$ *and* $f : B \to \mathcal{N}$ *be LUD*^{2} *at* $g(x_0) \in B$. *Then* $f \circ g : A \to \mathcal{N}$ *is LUD*^{2} *at* x_0.

Proof. Let U_f be a neighborhood of $g(x_0)$ in B such that f is UD^2 on U_f and let U_g be a neighborhood of x_0 in A such that g is UD^2 on U_g. The function g^2 is also LUD^2 at x_0 by Proposition 2.17, so there is a neighborhood U_1 of x_0 in A such that g^2 is UD^2 on U_1, and as f is C^2 at $g(x_0)$ and g is C^2 at x_0, there exists a neighborhood U_2 of x_0 such that $|g(x) - g(x_0)| < 1$, $|f'(g(x)) - f'(g(x_0))| < 1$, and $|f''(g(x))g(x) - f''(g(x_0))g(x_0)| < 1$ for all $x \in U_2$. Let $U = g^{-1}(U_f) \cap U_g \cap U_1 \cap U_2$ and let $\epsilon > 0$ in $\mathcal{N}$ be given. Then g and g^2 are UD^2 on U, and f is UD^2 on $g(U)$. Thus, there exist $\delta_1, \delta_2, \delta_3 > 0$ in $\mathcal{N}$ such that

$$|f(y) - f(x) - f'(x)(y-x) - \frac{1}{2}f''(x)(y-x)^2| < \frac{\epsilon}{3\left(1 + |g'(x_0)|\right)^2}(y-x)^2$$

for every $x, y \in g(U)$ satisfying $0 < |y-x| < \delta_1$;

$$|g(y) - g(x) - g'(x)(y-x) - \frac{1}{2}g''(x)(y-x)^2| < \frac{\epsilon}{3\Big(|f'\big(g(x_0)\big)| + |f''\big(g(x_0)\big)g(x_0)| + 2\Big)}(y-x)^2$$

for every $x, y \in U$ satisfying $0 < |y-x| < \delta_2$; and

$$|g^2(y) - g^2(x) - 2g(x)g'(x)(y-x) - \Big(g'(x)^2 + g(x)g''(x)\Big)(y-x)^2| < \frac{\epsilon}{3\Big(|f''\big(g(x_0)\big)| + 1\Big)}(y-x)^2$$

for every $x, y \in U$ satisfying $0 < |y-x| < \delta_3$. Moreover, as g is uniformly differentiable on U, it is also uniformly continuous on U, and so there exists $\delta_4 > 0$ in $\mathcal{N}$ such that for every $x, y \in U$ with $0 < |y-x| < \delta_4$ we have that $|g(y) - g(x)| < \delta_1$. Finally, since g is LUD at x_0, there exists $\delta_5 > 0$ in $\mathcal{N}$ such that for every $x, y \in U$ with $0 < |y-x| < \delta_5$ we have that

$$|g(y) - g(x)| < \big(1 + |g'(x_0)|\big)\,|y-x|.$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$. Then for every $x, y \in U$ with $0 < |y-x| < \delta$ we have that

$$\begin{aligned}
&\left| f\big(g(y)\big) - f\big(g(x)\big) - g'(x)f'\big(g(x)\big)(y-x) - \frac{1}{2}\Big(g''(x)f'\big(g(x)\big) + g'(x)^2 f''\big(g(x)\big)\Big)(y-x)^2 \right| \\
&\le |f\big(g(y)\big) - f\big(g(x)\big) - f'\big(g(x)\big)\Big(g(y) - g(x)\Big) - \frac{1}{2}f''\big(g(x)\big)\Big(g(y) - g(x)\Big)^2| \\
&\quad + |f'\big(g(x)\big) - f''\big(g(x)\big)g(x)||g(y) - g(x) - g'(x)(y-x) - \frac{1}{2}g''(x)(y-x)^2| \\
&\quad + \frac{1}{2}|f''\big(g(x)\big)||g(y)^2 - g(x)^2 - 2g(x)g'(x)(y-x) - \Big(g'(x)^2 + g(x)g''(x)\Big)(y-x)^2| \\
&< \frac{\epsilon}{3\left(1 + |g'(x_0)|\right)^2}\left(g(y) - g(x)\right)^2 + \frac{|f'\big(g(x)\big) - f''\big(g(x)\big)g(x)|}{3\Big(|f'\big(g(x_0)\big)| + |f''\big(g(x_0)\big)g(x_0)| + 2\Big)}\epsilon(y-x)^2 \\
&\quad + \frac{|f''\big(g(x)\big)|}{3\Big(|f''\big(g(x_0)\big)| + 1\Big)}\epsilon(y-x)^2 \\
&< \epsilon(y-x)^2.
\end{aligned}$$

□

3. THE INTERMEDIATE VALUE PROPERTY

A noteworthy obstacle in our finding sufficient conditions for the mean value property is that we do not know whether the derivative of a function f has the intermediate value property. In the real case we can get the intermediate value property of the derivative as a result of the intermediate value theorem or Darboux's theorem, but these theorems do not hold for functions over $\mathcal{N}$. In [19] it was proved that f has the intermediate value property in the neighborhood of a point if f is LUD and has non-zero derivative at that point. However, neither local uniform differentiability nor the intermediate value property are (in general) passed on from a function to its derivative, so this still does not give us the property for f'. We could, of course, require that f' be LUD with non-zero derivative, and just apply the local intermediate value theorem to f', however, as we will see in the next section, this isn't enough to give f the mean value property. We thus consider higher orders of local uniform differentiability, and find that, indeed, if f is LUD^2 with non-zero second derivative then f' will locally have the intermediate value property. But before we prove this central result, we first establish the following lemmas.

Lemma 3.1. *Let $f : A \to \mathcal{N}$ be LUD^2 at $x_0 \in A$. Then for every $\epsilon > 0$ in $\mathcal{N}$, there exists $\delta > 0$ in $\mathcal{N}$ such that, for every $x, y \in B(x_0, \delta)$, we have that*

$$|f(y) - f(x) - f'(x)(y-x) - \frac{1}{2}f''(x_0)(y-x)^2| < \epsilon(y-x)^2.$$

Proof. Let $U \subseteq A$ be a neighborhood of x_0 such that f is UD^2 on U, let $\delta_0 > 0$ in $\mathcal{N}$ be such that $B(x_0, \delta_0) \subseteq U$, and let $\epsilon > 0$ in $\mathcal{N}$ be given. Then there exists $\delta_1 > 0$ in $\mathcal{N}$ such that, whenever $x, y \in U$ and $0 < |y-x| < \delta_1$, we have that

$$|f(y) - f(x) - f'(x)(y-x) - \frac{1}{2}f''(x)(y-x)^2| < \frac{\epsilon}{2}(y-x)^2.$$

By Proposition 2.14, we have that f is C^2 at x_0; hence there exists $\delta_2 > 0$ in $\mathcal{N}$ such that if $|x - x_0| < \delta_2$ then $|f''(x) - f''(x_0)| < \epsilon$. Let $\delta = \min\{\delta_0, \delta_1, \delta_2\}$. Then it follows that if $x, y \in B(x_0, \delta)$ then

$$\begin{aligned}
&|f(y) - f(x) - f'(x)(y-x) - \frac{1}{2}f''(x_0)(y-x)^2| \\
&\leq |f(y) - f(x) - f'(x)(y-x) - \frac{1}{2}f''(x)(y-x)^2| + \frac{1}{2}|f''(x) - f''(x_0)|(y-x)^2 \\
&< \frac{\epsilon}{2}(y-x)^2 + \frac{\epsilon}{2}(y-x)^2 = \epsilon(y-x)^2.
\end{aligned}$$

□

Lemma 3.2. *Let $f : A \to \mathcal{N}$ be LUD^2 at $x_0 \in A$. Then for every $\epsilon > 0$ in $\mathcal{N}$, there exists $\delta > 0$ in $\mathcal{N}$ such that, for every $x, y \in B(x_0, \delta)$, we have that*

$$|f'(y) - f'(x) - f''(x_0)(y-x)| < \epsilon|y-x|.$$

Proof. Let $\epsilon > 0$ in $\mathcal{N}$ be given. By Lemma 3.1 , there exists $\delta > 0$ in $\mathcal{N}$ such that, for every $x, y \in B(x_0, \delta)$, we have that

$$|f(y) - f(x) - f'(x)(y-x) - \frac{1}{2}f''(x_0)(y-x)^2| < \frac{\epsilon}{2}(y-x)^2.$$

Thus, for $x, y \in B(x_0, \delta)$, we have that

$$\begin{aligned}
|f'(y) - f'(x) - f''(x_0)(y-x)| &\leq |\frac{f(x)-f(y)}{y-x} + f'(y) - \frac{1}{2}f''(x_0)(y-x)| \\
&\quad + |\frac{f(y)-f(x)}{y-x} - f'(x) - \frac{1}{2}f''(x_0)(y-x)| \\
&= |\frac{f(x)-f(y)}{x-y} - f'(y) - \frac{1}{2}f''(x_0)(x-y)| \\
&\quad + |\frac{f(y)-f(x)}{y-x} - f'(x) - \frac{1}{2}f''(x_0)(y-x)| \\
&< \frac{\epsilon}{2}|x-y| + \frac{\epsilon}{2}|y-x| = \epsilon|y-x|.
\end{aligned}$$

□

The following lemma was proved in [18] (Lemma 4.1) .

Lemma 3.3. *Let $\delta_1 > 0$ in $\mathcal{N}$ be given and let $\phi : B(0, \delta_1) \to \mathcal{N}$ be such that $|\phi(t)| \leq c|t|$ for every $t \in B(0, \delta_1)$, where $0 < c \ll 1$. For $m \in \mathbb{N}$ let $\phi^{[m]} = \underbrace{\phi \circ \cdots \circ \phi}_{m \text{ times}}$ and set $\phi^{[0]}$ to be the identity map.*

Let $\delta \in \mathcal{N}$ be such that $0 < \delta \leq (1-c)\delta_1$ and let $\psi(t) = \sum_{i=0}^{\infty} \phi^{[m]}(t)$, for every $t \in B(0, \delta)$. Then *(i)* $|\psi(t)| \le \frac{|t|}{1-c}$; *and*

(ii) $\psi(t) - \phi(\psi(t)) = t$.

Lemma 3.4. *Let $A \subset \mathcal{N}$ be open and let $f : A \to \mathcal{N}$ be LUD^2 on A with $f''(x_0) \neq 0$ for some $x_0 \in A$ and with $f'(x_0) = y_0$. Then there exist $\delta, \eta > 0$ in $\mathcal{N}$ and a function F defined on $B(y_0, \eta)$ such that*

(i) $B(x_0, \delta) \subseteq A$,

(ii) $f'|_{B(x_0,\delta)}$ *is injective,*

(iii) $B(y_0, \eta) \subseteq f'(B(x_0, \delta))$ *and* $F(B(y_0, \eta) \subseteq B(x_0, \delta)$, *and*

(iv) $f'(F(x)) = x$ *for every* $x \in B(y_0, \eta)$.

Proof. Without loss of generality, we may assume that $x_0 = 0$ and $y_0 = 0$, for if this is not the case, then we can replace $f(x)$ with $\tilde{f}(x) = f(x + x_0) - y_0 x$. As f is LUD^2 at x_0 and the function $g(x) = y_0 x$ is LUD^2 at 0, we have that $\tilde{f}$ is LUD^2 at 0 with $\tilde{f}'(0) = f'(x_0) - y_0 = 0$ and $\tilde{f}''(0) = f''(x_0) \neq 0$. Moreover, without loss of generality, we may assume $f''(x_0) > 0$, for if $f''(x_0) < 0$ we could apply this proof to $(-f)$ and get the desired result.

Let $B(x_0, \omega_1)$ be a neighborhood of x_0 where f is UD^2. By Proposition 2.14, f is C^2, and so there exists $\omega_2 > 0$ in $\mathcal{N}$ such that $f''(x) \ge \frac{1}{2} f''(0) > 0$ for every $x \in B(x_0, \omega_2)$. Let $\omega = \min\{\omega_1, \omega_2\}$ and let $L = f''(0)$. Let $\phi(x) = \frac{1}{2}x^2 - \frac{1}{L} f(x)$. It follows that $\phi'(x) = x - \frac{1}{L} f'(x)$ and $\phi''(x) = 1 - \frac{1}{L} f''(x)$; so $\phi'(0) = \phi''(0) = 0$. Let $c \in \mathcal{N}$ be such that $0 < c \ll 1$. As ϕ is LUD^2 at 0, then by Lemma 3.2, there exists $\delta_0 > 0$ in $\mathcal{N}$ such that for every $s, t \in B(0, \delta_0)$ we have that

$$|\phi'(s) - \phi'(t) - \phi''(0)(s - t)| < c|s - t|.$$

As A is open, we may choose δ_0 such that $B(0, \delta_0) \subseteq A$. Thus,

$$|\phi'(s) - \phi'(t)| < c|s - t|. \tag{3.1}$$

Let $s, t \in B(0, \delta_0)$ be such that $f'(s) = f'(t)$. Then

$$|\phi'(s) - \phi'(t)| = |s - t| \le c|s - t|.$$

As $c \ll 1$, it follows that $s = t$, and thus $f'|_{B(0,\delta_0)}$ is injective. By Lemma 3.2, there exists $\delta_f > 0$ in $\mathcal{N}$ such that for every $s, t \in B(0, \delta_f)$ we have that

$$|f'(s) - f'(t) - L(s - t)| < \frac{L}{2}|s - t|.$$

Let $\delta = \min\{(1 - c)\delta_0, \omega, \delta_f\}$. Then $B(0, \delta) \subseteq B(0, \delta_0) \subseteq A$ and thus $f'|_{B(0,\delta)}$ is injective. This shows (i) and (ii).

By Equation (3.1) with $t = 0$, we have that $|\phi'(s)| < c|s|$ for every $s \in B(0, \delta)$, and so we have a function ψ with properties of that in Lemma 3.3. Let $\eta = L(1 - c)\delta$ and define $F(x) = \psi(\frac{x}{L})$ for every $x \in B(0, \eta)$. Thus for every $x \in B(0, \eta)$ we have that

$$|F(x)| = |\psi\left(\frac{x}{L}\right)| \le \frac{|x|}{L(1 - c)} < \frac{\eta}{L(1 - c)} = \delta.$$

Thus $F(B(0, \eta)) \subseteq B(0, \delta)$. Furthermore, for every $x \in B(0, \delta)$, we have that

$$x - \phi'(x) = \frac{f'(x)}{L}.$$

Let $x \in B(0, \eta)$. Then

$$\frac{|x|}{L} < (1 - c)\delta < \delta.$$

Thus $\frac{x}{L} \in B(0,\delta)$ and hence

$$\frac{x}{L} - \phi'\left(\frac{x}{L}\right) = \frac{1}{L} f'\left(\frac{x}{L}\right).$$

Moreover, we have by Lemma 3.3 that

$$\psi\left(\frac{x}{L}\right) - \phi'\left(\psi\left(\frac{x}{L}\right)\right) = \frac{x}{L}$$

and thus

$$\frac{1}{L} f'\left(\psi\left(\frac{x}{L}\right)\right) = \frac{x}{L}.$$

It follows that for every $x \in B(0,\eta)$,

$$f'(F(x)) = f'\left(\psi\left(\frac{x}{L}\right)\right) = x$$

and hence $B(0,\eta) \subseteq f'(B(0,\delta))$, as $F(x) \in B(0,\delta)$ for every $x \in B(0,\eta)$. This shows (iii) and (iv). □

Theorem 3.5. *Let $A \subset \mathcal{N}$ be open and let $f : A \to \mathcal{N}$ be LUD^2 on A with $f''(x_0) \neq 0$ for some $x_0 \in A$. Then there exists a neighborhood U of x_0 such that*

(i) *$f'|_U$ is injective and*

(ii) *$f'(U)$ is open.*

Proof. By Lemma 3.4, there exists a neighborhood U_0 of x_0 such that f' is injective on U_0. As f is C^2 and $f''(x_0) \neq 0$, there exists a neighborhood U_1 of x_0 such that $f''(x) \neq 0$ for every $x \in U_1$. Let $U = U_0 \cap U_1$. Then U is a neighborhood of x_0 and $f'|_U$ is injective.

Let $x \in U$ and let $y = f'(x)$. Lemma 3.4 applied to $f'|_U$ at x gives a δ, η, and F as stated in the lemma, for which $B(y,\eta) \subseteq f'(B(x,\delta)) \subseteq f'(U)$. As this holds for every $x \in U$, we have that $f'(U)$ is open. □

Theorem 3.6. *Let $f : A \to \mathcal{N}$ be LUD^2 on A and let $x_0 \in A$ be such that $f''(x_0) \neq 0$. Then there exists a neighborhood U of x_0 such that f' has the intermediate value property on U. That is, for every $a, b \in U$ with $a < b$, if c is between $f'(a)$ and $f'(b)$, then there exists $x \in (a,b)$ such that $f'(x) = c$.*

Proof. Without loss of generality, we may assume $f''(x_0) > 0$. By Lemma 3.2, there exists $\delta > 0$ in $\mathcal{N}$ such that for every $x, y \in B(x_0,\delta)$,

$$|f'(y) - f'(x) - f''(x_0)(y-x)| < \frac{f''(x_0)}{2}|y-x|$$

and thus

$$\frac{f'(y) - f'(x)}{y-x} > f''(x_0) - \frac{f''(x_0)}{2} = \frac{f''(x_0)}{2} > 0.$$

Hence f' is strictly increasing on $B(x_0,\delta)$. Applying Theorem 3.5 to f gives a neighborhood $U_0 \subseteq B(x_0,\delta)$ of x_0 such that $f'(U_0)$ is open. Let $\epsilon > 0$ in $\mathcal{N}$ be such that $B(f'(x_0),\epsilon) \subseteq f'(U_0)$ and let $U = f'^{-1}(B(f'(x_0),\epsilon))$, which is an open neighborhood of x_0. Let $a, b \in U$ be such that $a < b$ and let $c \in (f'(a), f'(b))$ be given. As $f'(a), f'(b) \in B(f'(x_0),\epsilon)$ and $B(f'(x_0),\epsilon)$ is a convex set, we have that $c \in B(f'(x_0),\epsilon)$. Thus there exists $x \in U = f'^{-1}(B(f'(x_0),\epsilon))$ such that $f'(x) = c$. As f' is strictly increasing on U, it follows that $x \in (a,b)$. □

4. THE LOCAL MEAN VALUE THEOREM

Now that we have sufficient conditions for which a function has the local intermediate value property, our next goal is to try to find sufficient conditions for which a function has the local mean value property. That is, conditions on a function f to determine whether in some neighborhood of a point we have that for every $a < b$, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{4.1}$$

As LUD functions were used to prove the inverse function theorem and local intermediate value theorem, one might wonder whether LUD is a sufficient condition for the mean value property as well.[2] Unfortunately, as the next example will illustrate, this condition is not enough to give the local mean value property. Moreover, we will even see that requiring all the derivatives of f to be LUD is still not enough to give the mean value property in any neighborhood.

Example 4.1. *Let $f : (-1, 1) \to \mathcal{N}$ be defined by*

$$f(x) = \begin{cases} d^{2\lambda(x)} + dx^2 & x \neq 0 \\ 0 & x = 0. \end{cases}$$

From our analysis in Example 2.10, it is evident that f is LUD with derivative $f'(x) = 2dx$. Moreover, as f' is a polynomial, it is clear that $f^{(k)}$ is LUD for every $k \in \mathbb{N}$. Now let U be a neighborhood of 0 and let $N \in \mathbb{N}$ be such that $d^N \in U$. Then for every $c \in (0, d^N)$ we have that

$$f'(c) = 2dc < 2d^{N+1} < d^N < (1 + d)d^N = \frac{f(d^N)}{d^N}.$$

Thus there is no $c \in (0, d^N)$ such that $f'(c) = \frac{f(d^N)}{d^N}$, and so f does not have the mean value property in any neighborhood of 0.

Another curious property to note about the above example is that, as $f''(0) \neq 0$, f' has the intermediate value property in some neighborhood of 0. Indeed, $f'(c)$ was always strictly less than $f(d^N)/d^N$ in the interval $(0, d^N)$, and so we are unable to use the intermediate value property to satisfy Equation (4.1).

Theorem 4.2 (the local mean value theorem)**.** *Let $f : A \to \mathcal{N}$ be LUD^2 at $x_0 \in A$ and assume that $f''(x_0) \neq 0$. Then there exists a neighborhood U of x_0 such that f has the mean value property on U. That is, for every $a, b \in U$ with $a < b$, there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. We may assume without loss of generality that $f''(x_0) > 0$. Let U_1 be a neighborhood of $x_0 \in A$ as in Definition 2.6. By Proposition 2.14, f'' is continuous at x_0. Thus there exists $\delta_1 > 0$ such that for every $x \in U_1 \cap B(x_0, \delta_1)$ we have that

$$|f''(x) - f''(x_0)| < \frac{1}{4} f''(x_0).$$

As f is UD^2 on U_1, there exists $\delta_2 > 0$ in $\mathcal{N}$ such that, for every $x, y \in U_1$ with $0 < |y - x| < \delta_2$, we have that

$$|f(y) - f(x) - f'(x)(y - x) - \frac{1}{2} f''(x)(y - x)^2| < \frac{1}{4} f''(x_0)(y - x)^2.$$

[2]Note that the value of the derivative doesn't actually matter for the mean value property because we can always add an appropriate linear factor to f to make the derivative non-zero. This linear factor will then cancel out in Equation (4.1).

Let $\delta = \min\{\delta_1, \delta_2\}$. Then it follows that, for every $x, y \in U_1$ satisfying $0 < |y - x| < \delta$, we have that

$$\begin{aligned} f(y) - f(x) - f'(x)(y-x) &> \left(\frac{1}{2}f''(x) - \frac{1}{4}f''(x_0)\right)(y-x)^2 \\ &> \frac{1}{8}f''(x_0)(y-x)^2 > 0. \end{aligned} \tag{4.2}$$

Applying Theorem 3.6 to f at x_0 gives a neighborhood U_2 of x_0 such that f' has the intermediate value property in U_2. Let $U = U_1 \cap U_2 \cap B(x_0, \delta)$ and let $a, b \in U$ with $a < b$ be given. By Equation (4.2) we have that $f(b) > f(a) + f'(a)(b - a)$, and thus

$$f'(a) < \frac{f(b) - f(a)}{b - a}.$$

Similarly, we have that $f(a) > f(b) + f'(b)(a - b)$, and thus

$$f'(b) > \frac{f(b) - f(a)}{b - a}$$

Thus, by Theorem 3.6, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

□

As in the real case, the mean value property can be used to prove other important results. In particular, while L'Hôpital's rule does not hold for differentiable functions on $\mathcal{N}$, we will prove the result under similar conditions to those of the local mean value theorem. To do this we first prove the local equivalent of the Cauchy mean value theorem (Lemma 4.3). The proof is obtained from the mean value property the same way as in the real case.

Lemma 4.3. *Let $f, g : A \to \mathcal{N}$ be LUD^2 on A and $x_0 \in A$ be such that $f''(x_0) \neq 0$ and $g''(x_0) \neq 0$. Then there exists a neighborhood U of x_0 such that for every $a, b \in U$ with $a < b$, there exists $c \in (a, b)$ such that*

$$f'(c)\left(g(b) - g(a)\right) = g'(c)\left(f(b) - f(a)\right).$$

Theorem 4.4. *Let $f, g : A \to \mathcal{N}$ be LUD^2 on A and let $a \in A$ be such that $f''(a) \neq 0$ and $g''(a) \neq 0$. Furthermore, suppose that $f(a) = g(a) = 0$, that $\lim_{x\to a} f'(x)/g'(x)$ exists, and that $g'(x) \neq 0$ for every $x \in A \setminus \{a\}$. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Proof. By Lemma 4.3 there exists a neighborhood U of a such that for every $x \in U$, there exists c between x and a such that

$$f'(c)(g(x) - g(a)) = g'(c)(f(x) - f(a)).$$

Let $\delta_1 > 0$ in $\mathcal{N}$ be such that $B(a, \delta_1) \subseteq U$, let $L = \lim_{x\to a} f'(x)/g'(x)$, and let $\epsilon > 0$ in $\mathcal{N}$ be given. Then there exists $\delta_2 > 0$ in $\mathcal{N}$ such that for all $x \in B(a, \delta_2)$ we have that

$$|\frac{f'(x)}{g'(x)} - L| < \epsilon.$$

Let $\delta = \min\{\delta_1, \delta_2\}$, and let $x \in B(a, \delta)$ be given such that $g(x) \neq 0$. Then, there exists $c \in B(a, \delta)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x)}{g(x)}.$$

It follows that

$$|\frac{f(x)}{g(x)} - L| = |\frac{f'(c)}{g'(c)} - L| < \epsilon.$$

□

Remark 4.5. *As it is apparent from this paper and [18, 19], the concept of local uniform differentiability is a very useful one for doing calculus on the non-Archimedean field $\mathcal{N}$ since the ordinary differentiability is not strong enough; and therefore, it is worthwhile looking at other possible applications of this new differentiability concept. For example, we are currently investigating under what conditions an LUD^∞ function at a point $x_0 \in \mathcal{N}$ will have a convergent Taylor series at x_0, as well as other pure and computational applications of the results in this paper.*

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