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RESEARCH ARTICLES

On a Lebesgue-like Measure on the Levi-Civita Space $\mathcal{R}^j$

M. Restrepo Borrero[1*] and Khodr Shamseddine[2]**

[1]*Department of Mathematics, Universidad Nacional de Colombia, Colombia*

[2]*Department of Physics and Astronomy and Department of Mathematics, University of Manitoba, Winnipeg, Manitoba R3T 2N2, Canada*

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Abstract—In a previous paper [2], we developed a new Lebesgue-like measure on the Levi-Civita field $\mathcal{R}$ that proved to be a strict improvement over the previously defined S-measure defined in [9, 13]. Nevertheless, we were only at first able to define such a measure for the one dimensional case leaving the case for higher dimensions as an open-ended question to be further researched. In another paper [15], the authors developed a generalization of the S-measure into higher dimensions using simplexes as their basic building blocks instead of boxes as simplexes proved to be more suitable for the topological structure of the Levi-Civita field $\mathcal{R}$. However, the resulting measure naturally inherited the same limitations that the original S-measure on $\mathcal{R}$ had. In this new paper, we expand the same characterization given in [2] for the one-dimensional S-measurable sets to the S-measurable sets in $\mathcal{R}^j$ as defined in [15] and develop our own generalization to higher dimensions for the measure given in [2].

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1. INTRODUCTION

In this section, we will introduce the reader to preliminary results about the Levi-Civita field $\mathcal{R}$ as well as previous work on measure theory on $\mathcal{R}$ and $\mathcal{R}^j$, $j > 1$ in $\mathbb{N}$.

1.1. The Levi-Civita Field $\mathcal{R}$

We recall that the elements of the Levi-Civita field $\mathcal{R}$ are functions from $\mathbb{Q}$ to $\mathbb{R}$ with left-finite support (denoted by supp). That is, for every $q \in \mathbb{Q}$ there are only finitely many elements in the support that are smaller than q. For the further discussion, it is convenient to introduce the following terminology.

Definition 1.1. *(λ, $=_r$, $\sim$, $\approx$) We define $\lambda : \mathcal{R} \to \mathbb{Q}$ by*

$$\lambda(x) = \begin{cases} \min(supp(x)) & if\ x \neq 0 \\ \infty & if\ x = 0. \end{cases}$$

The minimum exists because of the left-finiteness of $supp(x)$ when $x \neq 0$. Moreover, we denote the value of x at $q \in \mathbb{Q}$ with brackets like $x[q]$.

Given $x, y \in \mathcal{R}$ and $r \in \mathbb{Q}$, we say that $x =_r y$ if $x[q] = y[q]$ for all $q \leq r$.

Given $x, y \neq 0$ in $\mathcal{R}$, we say $x \sim y$ if $\lambda(x) = \lambda(y)$; and we say $x \approx y$ if $\lambda(x) = \lambda(y)$ and $x[\lambda(x)] = y[\lambda(y)]$.

*E-mail: marestrepob@unal.edu.co

**E-mail: khodr.shamseddine@umanitoba.ca

At this point, these definitions may feel somewhat arbitrary; but after having introduced an order on $\mathcal{R}$, we will see that λ describes orders of magnitude, $\sim$ corresponds to agreement of the order of magnitude, while $\approx$ corresponds to agreement up to infinitely small relative error.

The set $\mathcal{R}$ is endowed with formal power series multiplication and componentwise addition, which make it into a field [8] in which we can isomorphically embed the field of real numbers $\mathbb{R}$ as a subfield via the map $E : \mathbb{R} \to \mathcal{R}$ defined by

$$E(x)[q] = \begin{cases} x & \text{if } q = 0 \\ 0 & \text{else.} \end{cases} \tag{1.1}$$

Definition 1.2. *(Order in $\mathcal{R}$) Let $x, y \in \mathcal{R}$ be given. Then we say that $x > y$ (or $y < x$) if $x \neq y$ and $(x - y)[\lambda(x - y)] > 0$; and we say $x \geq y$ (or $y \leq x$) if $x = y$ or $x > y$.*

It follows that the relation $\geq$ (or $\leq$) defines a total order on $\mathcal{R}$ which makes it into an ordered field. Note that, given $a < b$ in $\mathcal{R}$, we define the $\mathcal{R}$-interval $[a, b] = \{x \in \mathcal{R} : a \leq x \leq b\}$, with the obvious adjustments in the definitions of the intervals $[a, b)$, $(a, b]$, and (a, b). Moreover, the embedding E in Equation (1.1) of $\mathbb{R}$ into $\mathcal{R}$ is compatible with the order.

The order leads to the definition of an ordinary absolute value on $\mathcal{R}$:

$$|x| = \max\{x, -x\} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0; \end{cases}$$

which induces the same topology on $\mathcal{R}$ (called the order topology or valuation topology) as that induced by the ultrametric absolute value $|\cdot|_u : \mathcal{R} \to \mathbb{R}$, given by

$$|x|_u = \begin{cases} e^{-\lambda(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

as was shown in [14].

We note in passing here that $|\cdot|_u$ is a non-Archimedean valuation on $\mathcal{R}$; and hence $(\mathcal{R}, |\cdot|_u)$ is a non-Archimedean valued field. Moreover, $|.|_u$ induces a metric Δ on $\mathcal{R}$ given by $\Delta(x, y) = |y - x|_u$ which satisfies the strong triangle inequality and is thus an ultrametric, making $(\mathcal{R}, \Delta)$ an utrametric space.

Definition 1.3. *(The Number d) Let d be the element of $\mathcal{R}$ given by $d[1] = 1$ and $d[t] = 0$ for $t \neq 1$.*

Remark 1.4. *Given $q \in \mathbb{Q}$, then it can be shown [2] that*

$$d^q[t] = \begin{cases} 1 & \textit{if } t = q \\ 0 & \textit{otherwise.} \end{cases}$$

It is easy to check that d^q is infinitely small if $q > 0$ and infinitely large if $q < 0$ in $\mathbb{Q}$. Moreover, for all $x \in \mathcal{R}$, the elements of $supp(x)$ can be arranged in ascending order, say $supp(x) = \{q_1, q_2, \ldots\}$ with $q_j < q_{j+1}$ for all j; and x can be written as $x = \sum_j x[q_j]d^{q_j}$, where the series converges in the order (valuation) topology [1].

Altogether, it follows that $\mathcal{R}$ is a non-Archimedean (valued and ordered) field extension of $\mathbb{R}$. For a detailed study of this field, we refer the reader to the survey paper [10] and the references therein. In particular, it is shown that $\mathcal{R}$ is complete with respect to the natural (valuation) topology or, equivalently, with respect to the ultrametric Δ.

It follows therefore that $\mathcal{R}$ is just a special case of the class of fields discussed in [7]. For a general overview of the algebraic properties of formal power series fields, we refer to the comprehensive overview by Ribenboim [6], and for an overview of the related valuation theory, to the book by Krull [4]. A thorough and complete treatment of ordered structures can also be found in [5]. A more comprehensive survey of all non-Archimedean fields can be found in [3].

Besides being the smallest non-Archimedean ordered field extension of the real numbers that is both complete in the order topology and real closed, the Levi-Civita field $\mathcal{R}$ is of particular interest because of its practical usefulness. Because of the left-finiteness of the supports of the Levi-Civita numbers, those numbers can be used on a computer, thus allowing for many useful computational applications. One such application is the computation of derivatives of real functions representable on a computer [11], where both the accuracy of formula manipulators and the speed of classical numerical methods are achieved.

The following result is not special to $\mathcal{R}$ but it holds in any non-Archimedean valued field; its proof can be found in [8, 12].

Proposition 1.5. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{R}$. Then $\{a_n\}$ is a Cauchy sequence in the valuation topology if and only if $\lim\limits_{n\to\infty}(a_{n+1}-a_n)=0$.*

Since $\mathcal{R}$ is Cauchy complete, we readily obtain the following result.

Corollary 1.6. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{R}$. Then $\{a_n\}$ converges in $\mathcal{R}$ if and only if $\lim\limits_{n\to\infty}(a_{n+1}-a_n)=0$.*

Corollary 1.7. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{R}$. Then $\sum\limits_{n\in\mathbb{N}} a_n$ converges in $\mathcal{R}$ if and only if $\lim\limits_{n\to\infty} a_n = 0$.*

Moreover, thanks to the non-Archimedean (ultrametric) nature of $\mathcal{R}$, the order of limits, including double infinite sums, can be interchanged more conveniently than in $\mathbb{R}$.

1.2. The S-Measure on $\mathcal{R}$

In [9, 13], we developed a measure and integration theory on $\mathcal{R}$ that uses the $\mathcal{R}$-analytic functions (functions given locally by power series) as the building blocks for measurable functions instead of the step functions used in the real case. We will refer to that measure by the S-measure henceforth in this paper.

Notation 1.8. *Let $a<b$ in $\mathcal{R}$ be given. Then by $l(I(a,b))$ we will denote the length of the interval $I(a,b)$, that is*

$$l(I(a,b)) = \text{ length of } I(a,b) = b-a.$$

Definition 1.9. *Let $A\subset\mathcal{R}$ be given. Then we say that A is S-measurable if for every $\epsilon>0$ in $\mathcal{R}$, there exist a sequence of pairwise disjoint intervals $\{I_n\}_{n=1}^{\infty}$ and a sequence of pairwise disjoint intervals $\{J_n\}_{n=1}^{\infty}$ such that $\bigcup\limits_{n=1}^{\infty} I_n \subset A \subset \bigcup\limits_{n=1}^{\infty} J_n$, $\sum\limits_{n=1}^{\infty} l(I_n)$ and $\sum_{n=1}^{\infty} l(J_n)$ converge in $\mathcal{R}$, and $\sum\limits_{n=1}^{\infty} l(J_n) - \sum\limits_{n=1}^{\infty} l(I_n) \le \epsilon$.*

Given an S-measurable set A, then for every $k\in\mathbb{N}$, we can select a sequence of pairwise disjoint intervals $\left\{I_n^k\right\}_{n=1}^{\infty}$ and a sequence of pairwise disjoint intervals $\left\{J_n^k\right\}_{n=1}^{\infty}$ such that $\sum\limits_{n=1}^{\infty} l\left(I_n^k\right)$ and $\sum\limits_{n=1}^{\infty} l\left(J_n^k\right)$ converge in $\mathcal{R}$ for all k,

$$\bigcup_{n=1}^{\infty} I_n^k \subset \bigcup_{n=1}^{\infty} I_n^{k+1} \subset A \subset \bigcup_{n=1}^{\infty} J_n^{k+1} \subset \bigcup_{n=1}^{\infty} J_n^k \text{ and } \sum_{n=1}^{\infty} l\left(J_n^k\right) - \sum_{n=1}^{\infty} l\left(I_n^k\right) \le d^k$$

for all $k \in \mathbb{N}$. Since $\mathcal{R}$ is Cauchy complete in the order (valuation) topology, it follows that $\lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(I_n^k\right)$ and $\lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(J_n^k\right)$ both exist and they are equal. We call the common value of the limits the S-measure of A and we denote it by $M_s(A)$. Thus,

$$M_s(A) = \lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(I_n^k\right) = \lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(J_n^k\right).$$

Contrary to the real case,

$$\sup\left\{\sum_{n=1}^{\infty} l(I_n) : I_n\text{'s are pairwise disjoint intervals and } \bigcup_{n=1}^{\infty} I_n \subset A\right\}$$

and

$$\inf\left\{\sum_{n=1}^{\infty} l(J_n) : J_n\text{'s are pairwise disjoint intervals and } A \subset \bigcup_{n=1}^{\infty} J_n\right\}$$

need not exist for a given set $A \subset \mathcal{R}$. However, as shown in [13], if A is S-measurable then both the supremum and infimum exist and they are equal to $M_s(A)$. This shows that the definition of S-measurable sets in Definition 2.2 is a good generalization of that of the Lebesgue measurable sets of real analysis that corrects for the lack of suprema and infima in non-Archimedean ordered fields.

It follows directly from the definition that $M_s(A) \geq 0$ for any S-measurable set $A \subset \mathcal{R}$ and that any interval $I(a, b)$ is S-measurable with S-measure $M_s(I(a, b)) = l(I(a, b)) = b - a$. It also follows that if A is a countable union of pairwise disjoint intervals $(I_n(a_n, b_n))$ such that $\sum_{n=1}^{\infty}(b_n - a_n)$ converges then A is S-measurable with $M_s(A) = \sum_{n=1}^{\infty}(b_n - a_n)$. Moreover, if $B \subset A \subset \mathcal{R}$ and if A and B are S-measurable, then $M_s(B) \leq M_s(A)$.

In [13] we show that the S-measure defined on $\mathcal{R}$ above satisfies many of the nice properties of the Lebesgue measure on $\mathbb{R}$. For example, we show that any subset of an S-measurable set of S-measure 0 is itself S-measurable and has S-measure 0. We also show that any countable unions of S-measurable sets whose S-measures form a null sequence is S-measurable and the S-measure of the union is less than or equal to the sum of the S-measures of the original sets; moreover, the S-measure of the union is equal to the sum of the S-measures of the original sets if the latter are pairwise disjoint. Furthermore, we show that any finite intersection of S-measurable sets is also S-measurable and that the sum of the S-measures of two S-measurable sets is equal to the sum of the S-measures of their union and intersection.

However, the S-measure on $\mathcal{R}$ has its shortcomings. For example, the complement of an S-measurable set in an S-measurable set need not be S-measurable: $[0, 1]$ and $[0, 1] \cap \mathbb{Q}$ are both S-measurable with S-measures 1 and 0, respectively; but the complement of $[0, 1] \cap \mathbb{Q}$ in $[0, 1]$ is not S-measurable. On the other hand, if $B \subset A \subset \mathcal{R}$ and if A, B and $A \setminus B$ are all S-measurable, then $M_s(A) = M_s(B) + M_s(A \setminus B)$.

The example of $[0, 1] \setminus [0, 1] \cap \mathbb{Q}$ above shows that the axiom of choice is not needed here to construct a set that is not S-measurable, as there are many simple examples of such sets. Indeed, any uncountable real subset of $\mathcal{R}$, like $[0, 1] \cap \mathbb{R}$ for example, is not S-measurable. This ease of finding subsets of $\mathcal{R}$ that are not S-measurable may seem surprising; however, through closer inspection and the following characterization (proved in [2]), it becomes obvious that the family of S-measurable sets is simply too narrow, thus the need for a new measure on $\mathcal{R}$ that will extend the family of S-measurable sets and will share more of the nice properties of the Lebesgue measure on $\mathbb{R}$.

Theorem 1.10. *Let $A \subset \mathcal{R}$ be S-measurable. Then A can be written as a disjoint union $A = \left(\bigcup_{n=1}^{\infty} K_n\right) \cup S$, where K_n is an interval in $\mathcal{R}$ for each $n \in \mathbb{N}$ and where $\sum_{n=1}^{\infty} l(K_n) = M_s(A)$ and $M_s(S) = 0$.*

1.3. The L-Measure on $\mathcal{R}$

The effect of having too small a family of S-measurable sets impedes further progress into more significant results that the reader associates with the Lebesgue measure in $\mathbb{R}$. So we introduced in [2] a new definition that enlarged the pool of measurable sets while still circumventing the fact that not all bounded sets in $\mathcal{R}$ have an infimum or a supremum. We first introduced the notion of an outer measure on $\mathcal{R}$ and showed some key properties the outer measure has.

Definition 1.11. *Let $A \subset \mathcal{R}$ be given. Then we say that A is outer measurable if*

$$\inf \left\{ \sum_{n=1}^{\infty} l(S_n) : S_n\text{'s are intervals and } A \subseteq \bigcup_{n=1}^{\infty} S_n \right\}$$

exists in $\mathcal{R}$. If so, we call that number the outer measure of A and denote it by $M_u(A)$.

Then, we used the notion of outer measure and Caratheodory's criterion to define a new measure on $\mathcal{R}$ similarly to how the Lebesgue measure of real analysis is defined in terms of the outer measure on $\mathbb{R}$.

Definition 1.12. *Let $A \subset \mathcal{R}$ be an outer measurable set. Then we say that A is L-measurable if for every other outer measurable set $B \subset \mathcal{R}$ both $A \cap B$ and $A^c \cap B$ are outer measurable and*

$$M_u(B) = M_u(A \cap B) + M_u(A^c \cap B).$$

In this case, we define the L-measure of A to be $M(A) := M_u(A)$. The family of L-measurable sets in $\mathcal{R}$ will be denoted by $\mathcal{M}_L$.

As shown in [2], the L-measure proves to be a better generalization of the Lebesgue measure from $\mathbb{R}$ to $\mathcal{R}$ than the S-measure and it leads to a family of measurable sets in $\mathcal{R}$ that strictly contains the family of S-measurable sets from [13], and for which most of the classic results for Lebesgue measurable sets in $\mathbb{R}$ hold. We present here a summary of the key results for L-measurable sets and refer the reader to [2] for the proofs. We will prove the analogues of these results for the L-measure that we will develop on $\mathcal{R}^j$ in Section 4 below.

- If $a < b$ in $\mathcal{R}$ then $I(a, b) \in \mathcal{M}_L$ and $M(I(a, b)) = b - a$.

- If $C \subset \mathcal{R}$ is outer measurable with $M_u(C) = 0$ then $C \in \mathcal{M}_L$ with $M(C) = 0$. Consequently, if $A \in \mathcal{M}_L$ with $M(A) = 0$ and if $B \subset A$ then $B \in \mathcal{M}_L$ and $M(B) = 0$.

- If $\{J_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint intervals in $\mathcal{R}$ such that $\lim_{n\to\infty} l(J_n) = 0$ then $\bigcup_{n=1}^{\infty} J_n$ is L-measurable, and
$$M\left(\bigcup_{n=1}^{\infty} J_n\right) = \sum_{n=1}^{\infty} l(J_n).$$

- If $A \subset \mathcal{R}$ is S-measurable then A is L-measurable and $M(A) = M_s(A)$. The converse is not true.

- If $A, B \in \mathcal{M}_L$ then $A \cap B, A \cup B, A \cap B^c \in \mathcal{M}_L$. Moreover,
$$M(A \cup B) = M(A) + M(B) - M(A \cap B) \text{ and } M(A \cap B^c) = M(A) - M(A \cap B).$$

- If, for each $n \in \mathbb{N}$, $A_n \in \mathcal{M}_L$ and if $\lim_{N\to\infty} M\left(\bigcup_{n=1}^{N} A_n\right)$ exists in $\mathcal{R}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}_L$ and

$$M\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{N\to\infty} M\left(\bigcup_{n=1}^{N} A_n\right).$$

  If, in addition, the A_n's are mutually disjoint then

$$M\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} M\left(A_n\right).$$

- If, for each $n \in \mathbb{N}$, $A_n \in \mathcal{M}_L$ and if $\lim_{N\to\infty} M\left(\bigcap_{n=1}^{N} A_n\right)$ exists in $\mathcal{R}$ then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{M}_L$ and

$$M\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{N\to\infty} M\left(\bigcap_{n=1}^{N} A_n\right).$$

2. THE S-MEASURE ON $\mathcal{R}^j$

Notation 2.1. *We define the volume of a simplex $S \subseteq \mathcal{R}^j$ spanned by the vectors $(v_0, ..., v_j)$ by*

$$\frac{1}{j!}\left|\det(v_1 - v_0, ..., v_j - v_0)\right|$$

and denote it by $V(S)$.

In the following we give an adjusted version of the definition in [15] of a measurable set in $\mathcal{R}^j$.

Definition 2.2. *Let $A \subset \mathcal{R}^j$ be given. Then we say that A is S-measurable if for every $\epsilon > 0$ in $\mathcal{R}$, there exist two sequences of pairwise disjoint symplexes $\{I_n\}_{n=1}^{\infty}$ and $\{J_n\}_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} I_n \subset A \subset \bigcup_{n=1}^{\infty} J_n$, $\sum_{n=1}^{\infty} V(I_n)$ and $\sum_{n=1}^{\infty} V(J_n)$ converge in $\mathcal{R}$, and $\sum_{n=1}^{\infty} V(J_n) - \sum_{n=1}^{\infty} V(I_n) \leq \epsilon$.*

Given an S-measurable set A, then for every $k \in \mathbb{N}$, we can select a sequence of pairwise disjoint symplexes $\left\{I_n^k\right\}_{n=1}^{\infty}$ and a sequence of pairwise disjoint symplexes $\left\{J_n^k\right\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} V\left(I_n^k\right)$ and $\sum_{n=1}^{\infty} V\left(J_n^k\right)$ converge in $\mathcal{R}$ for all k,

$$\bigcup_{n=1}^{\infty} I_n^k \subset \bigcup_{n=1}^{\infty} I_n^{k+1} \subset A \subset \bigcup_{n=1}^{\infty} J_n^{k+1} \subset \bigcup_{n=1}^{\infty} J_n^k \text{ and } \sum_{n=1}^{\infty} V\left(J_n^k\right) - \sum_{n=1}^{\infty} V\left(I_n^k\right) \leq d^k$$

for all $k \in \mathbb{N}$. Since $\mathcal{R}$ is Cauchy complete in the order (valuation) topology, it follows that $\lim_{k\to\infty} \sum_{n=1}^{\infty} V\left(I_n^k\right)$ and $\lim_{k\to\infty} \sum_{n=1}^{\infty} V\left(J_n^k\right)$ both exist and they are equal. We call the common value of the limits the S-measure of A and we denote it by $M_s(A)$. Thus,

$$M_s(A) = \lim_{k\to\infty} \sum_{n=1}^{\infty} V\left(I_n^k\right) = \lim_{k\to\infty} \sum_{n=1}^{\infty} V\left(J_n^k\right).$$

It follows directly from the definition that $M_s(A) \geq 0$ for any S-measurable set $A \subset \mathcal{R}^j$ and that any symplex S is S-measurable with S-measure $M_s(S) = V(S)$. It also follows that if A is a countable union of pairwise disjoint symplexes (S_n) such that $\sum_{n=1}^{\infty} V(S_n)$ converges then A is S-measurable with $M_s(A) = \sum_{n=1}^{\infty} V(S_n)$. Moreover, if $B \subset A \subset \mathcal{R}^j$ and if A and B are S-measurable, then $M_s(B) \leq M_s(A)$.

The following theorem is a generalization of Theorem 1.10 to the multi-dimensional case.

Theorem 2.3. *Let $A \subset \mathcal{R}^j$ be S-measurable. Then A can be written as a disjoint union $A = \left(\bigcup_{n=1}^{\infty} K_n\right) \cup S$, where K_n is a symplex in $\mathcal{R}^j$ for each $n \in \mathbb{N}$ and where $\sum_{n=1}^{\infty} V(K_n) = M_s(A)$ and $M_s(S) = 0$.*

Proof. Let $\epsilon > 0$ in $\mathcal{R}$ be given. Then there exist two sequences of pairwise disjoint symplexes $\{I_n\}_{n=1}^{\infty}$ and $\{J_n\}_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} I_n \subseteq A \subseteq \bigcup_{n=1}^{\infty} J_n$, $\sum_{n=1}^{\infty} V(I_n)$ and $\sum_{n=1}^{\infty} V(J_n)$ both converge in the order topology, and $\sum_{n=1}^{\infty} V(J_n) - \sum_{n=1}^{\infty} V(I_n) < \epsilon/2$.

We can rewrite the collection $\{I_n\}_{n=1}^{\infty}$ as $\bigcup_{m=1}^{\infty} \{I_n \cap J_m\}_{n=1}^{\infty}$ due to $I_n \cap J_m$ being a finite union of simplexes [15], the sum of whose volumes we will denote by $V(I_n \cap J_m)$. Since, for every $m \in \mathbb{N}$, we have that $\lim_{n\to\infty} V(I_n \cap J_m) = 0$, it follows that $\sum_{n=1}^{\infty} V(I_n \cap J_m)$ converges for every $m \in \mathbb{N}$. Thus, there exists $N_m \in \mathbb{N}$ such that $\sum_{n=N_m+1}^{\infty} V(I_n \cap J_m) < d^m \epsilon$. It follows that

$$
\begin{aligned}
\sum_{n=1}^{\infty} V(J_n) - \sum_{m=1}^{\infty} \sum_{n=1}^{N_m} V(I_n \cap J_m) &\leq \sum_{n=1}^{\infty} V(J_n) - \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} V(I_n \cap J_m) - d^m \epsilon\right] \\
&= \sum_{n=1}^{\infty} V(J_n) - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} V(I_n \cap J_m) + \sum_{m=1}^{\infty} d^m \epsilon \\
&= \sum_{n=1}^{\infty} V(J_n) - \sum_{n=1}^{\infty} V(I_n) + \sum_{m=1}^{\infty} d^m \epsilon \\
&< \frac{\epsilon}{2} + \frac{d}{1-d}\epsilon \\
&< \epsilon.
\end{aligned}
$$

Thus, we can replace the original collections of symplexes $\{I_n\}_{n=1}^{\infty}$ and $\{J_n\}_{n=1}^{\infty}$ with $\bigcup_{m=1}^{\infty} \{J_m \cap I_n\}_{n=1}^{N_m}$ and $\{J_n\}_{n=1}^{\infty}$ which can be easily re-written as $\{S_n\}_{n=1}^{\infty}$, $\{X_n\}_{n=1}^{\infty}$ where $S_n \subseteq X_n$ for each n. Moreover, since $X_n \setminus S_n$ is a finite disjoint union of symplexes [15], we can write $\{X_n\}_{n=1}^{\infty} = \{S_n\}_{n=1}^{\infty} \cup \{R_n\}_{n=1}^{\infty}$ where $\sum_{n=1}^{\infty} V(R_n) < \epsilon$.

Let $\epsilon = d$. As shown, we can find two sequences of pairwise disjoint symplexes $\{S_n^1\}_{n=1}^{\infty}$ and $\{R_n^1\}_{n=1}^{\infty}$ such that

$$\bigcup_{n=1}^{\infty} S_n^1 \subseteq A \subseteq \left(\bigcup_{n=1}^{\infty} S_n^1\right) \cup \left(\bigcup_{n=1}^{\infty} R_n^1\right) \text{ and } \sum_{n=1}^{\infty} V(R_n^1) < d.$$

Now, given an arbitrary $k \in \mathbb{N}$, assume that for every positive integer $m \leq k$ we have a pair of sequences of pairwise disjoint symplexes $\{S_n^m\}_{n=1}^{\infty}$ and $\{R_n^m\}_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} S_n^m \subseteq A \subseteq \left(\bigcup_{n=1}^{\infty} S_n^m\right) \cup$

$\left(\bigcup_{n=1}^{\infty} R_n^m\right)$, $\sum_{n=1}^{\infty} V(R_n^m) < d^m$, and $\{S_n^m\}_{n=1}^{\infty} \subseteq \{S_n^{m+1}\}_{n=1}^{\infty}$. Take now a pair of sequences of pairwise disjoint symplexes $\{I_n\}_{n=1}^{\infty}$ and $\{O_n\}_{n=1}^{\infty}$ such that

$$\bigcup_{n=1}^{\infty} I_n \subseteq A \subseteq \left(\bigcup_{n=1}^{\infty} I_n\right) \cup \left(\bigcup_{n=1}^{\infty} O_n\right) \text{ and } \sum_{n=1}^{\infty} V(O_n) < d^{k+1}.$$

Consider the collections of pairwise disjoint symplexes $\bigcup_{m=1}^{\infty} \{I_n \cap R_m^k\}$ and $\bigcup_{m=1}^{\infty} \{O_n \cap R_m^k\}$. We define

$$\{R_n^{k+1}\} := \bigcup_{m=1}^{\infty} \{O_n \cap R_m^k\} \text{ and } \{S_n^{k+1}\} := \{S_n^k\} \cup \left(\bigcup_{m=1}^{\infty} \{I_n \cap R_m^k\}\right).$$

Then $\{S_n^{k+1}\}$ and $\{R_n^{k+1}\}$ are pairwise disjoint collections of symplexes that satisfy

$$\bigcup_{n=1}^{\infty} S_n^{k+1} \subseteq A \subseteq \left(\bigcup_{n=1}^{\infty} S_n^{k+1}\right) \cup \left(\bigcup_{n=1}^{\infty} R_n^{k+1}\right)$$

and

$$\sum_{n=1}^{\infty} V(R_n^{k+1}) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} V(O_n \cap R_m^k) \leq \sum_{n=1}^{\infty} V(O_n) < d^{k+1}.$$

We define $\{S_n^{\infty}\} = \bigcup_{k=1}^{\infty} \{S_n^k\}$, which is a disjoint countable union of symplexes that are contained in A. It follows that $\{R_n^k\}$ is a sequence of covers of $A \setminus \bigcup_{n=1}^{\infty} S_n^{\infty}$ that satisfies the condition $\lim_{k\to\infty} \sum_{n=1}^{\infty} V(R_n^k) = 0$. Thus,

$$\sum_{n=1}^{\infty} V(S_n^{\infty}) \leq M_s(A) \leq \sum_{n=1}^{\infty} V(S_n^{\infty}) + \lim_{k\to\infty} \sum_{n=1}^{\infty} V(R_n^k) = \sum_{n=1}^{\infty} V(S_n^{\infty}).$$

We conclude that $A = \left(\bigcup_{n=1}^{\infty} K_n\right) \cup S$ where K_n is a symplex, $\sum_{n=1}^{\infty} V(K_n) = M_s(A)$ and $M_s(S) = 0$. $\square$

3. THE OUTER MEASURE ON $\mathcal{R}^j$

Notation 3.1. *Whenever a set A can be written as a finite disjoint union of simplexes $A = \bigcup_{n=1}^{N} S_n$ we write $V(A)$ instead of $\sum_{n=1}^{N} V(S_n)$.*

Definition 3.2. *We say that a set $X \subset \mathcal{R}^j$ is outer measurable if the set*

$$C_A := \left\{\sum_{n=1}^{\infty} V(S_n) : A \subseteq \bigcup_{n=1}^{\infty} S_n \text{ where } S_n \text{ is a simplex}\right\}$$

has an infimum. When this holds, we define

$$M_u(A) := \inf(C_A)$$

3.1. General Properties

Proposition 3.3. *Let $A \subset \mathcal{R}^j$ be outer measurable. Then there exists a sequence of sequences of mutually disjoint simplexes $\left(\{I_n^k\}_{n=1}^{\infty}\right)_{k=1}^{\infty}$ such that $\lim_{k\to\infty} \sum_{n=1}^{\infty} V(I_n^k) = M_u(A)$ and, for all $k \in \mathbb{N}$, we have*

$$A \subseteq \bigcup_{n=1}^{\infty} I_n^{k+1} \subseteq \bigcup_{n=1}^{\infty} I_n^k.$$

We say that such a sequence outer-converges to A.

Proof. We leave the proof as an exercise to the reader. □

Lemma 3.4. *Let A, B and C be outer measurable sets in $\mathcal{R}^j$ such that $A \subseteq B \cup C$. Then, $M_u(A) \leq M_u(B) + M_u(C)$.*

Proof. Let $\{I_n\}$, $\{J_n\}$ be arbitrary coverings of B and C respectively. Then, $\{I_n\} \cup \{J_n\}$ is a covering of A and hence

$$M_u(A) \leq \sum_{n=1}^{\infty} V(I_n) + \sum_{n=1}^{\infty} V(J_n)$$

It follows that

$$M_u(A) - \sum_{n=1}^{\infty} V(I_n) \leq \sum_{n=1}^{\infty} V(J_n)$$

and hence $M_u(A) - \sum_{n=1}^{\infty} V(I_n)$ is a lower bound for the set $C_C := \left\{\sum_{n=1}^{\infty} V(S_n) : C \subseteq \bigcup_{n=1}^{\infty} S_n\right\}$ and thus

$$M_u(A) - \sum_{n=1}^{\infty} V(I_n) \leq \inf(C_C) = M_u(C).$$

It follows that

$$M_u(A) - M_u(C) \leq \sum_{n=1}^{\infty} V(I_n),$$

Thus,

$$M_u(A) - M_u(C) \leq \inf(C_B) = M_u(B),$$

and the result follows. □

Proposition 3.5. *Let $A \subset \mathcal{R}^j$ be outer measurable, and let $T : \mathcal{R}^j \to \mathcal{R}^j$ be an affine transformation of the form $T(x) := Mx + r$ where M is a matrix and $r \in \mathcal{R}^j$ is fixed. Then $T(A)$ is outer measurable and has measure*

$$M_u(T(A)) = |\det(M)| \cdot M_u(A).$$

Proof. The result follows immediately from the fact that affine transformations map simplexes into simplexes and from the definition of the volume of a simplex and that of the outer measure. □

It turns out that sets of measure zero in this definition inherit one of the key properties present in the traditional Lebesgue measure for $\mathbb{R}^j$. Namely, we have the following result

Proposition 3.6. *Let $A, B \subseteq \mathcal{R}^j$ be outer measurable with $M_u(B) = 0$. Then, for any subset $C \subseteq B$ we have that $M_u(C) = 0$ and $M_u(A \setminus C) = M_u(A)$.*

Proof. It follows immediately from Definition 3.2 that $M_u(C) = 0$. To see that $M_u(A \setminus C) = M_u(A)$ it is enough to notice that if $\{J_n^k\}$ outer-converges to A then it outer-converges to $A \setminus C$, for if $\{S_n\}$ covers $A \setminus C$ and $\sum_{n=1}^{\infty} V(S_n) < M_u(A)$, we can find $\{I_n\}$ covering C such that $\sum_{n=1}^{\infty} V(I_n) + \sum_{n=1}^{\infty} V(S_n) < M_u(A)$. This is a contradiction, since $\{I_n\} \cup \{S_n\}$ covers A. □

3.2. Simplexes and the Outer Measure

We now introduce a series of results showing that simplexes behave particularly well with the definition of the outer measure.

Proposition 3.7. *Let $A \subseteq \mathcal{R}^j$ be outer measurable and let $I \subset \mathcal{R}^j$ be a simplex. Then $A \cap I$ is outer measurable.*

Proof. Let $\{J_n^k\}$ outer-converge to A. We define

$$I_n^k := I \cap J_n^k \,.$$

Clearly $V(I_n^k) \leq V(J_n^k)$, implying that for every k, the series $\sum_{n=1}^{\infty} V(I_n^k)$ converges. We show that $\lim_{k\to\infty} \sum_{n=1}^{\infty} V(I_n^k)$ exists and is equal to $M_u(A \cap I)$.

First we note that since, for every $k \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} (J_n^{k+1} \cap I^c) \subseteq \bigcup_{n=1}^{\infty} (J_n^k \cap I^c)$, we have that $\sum_{n=1}^{\infty} V(J_n^{k+1} \cap I^c) \leq \sum_{n=1}^{\infty} V(J_n^k \cap I^c)$. It follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} V(I_n^k) - \sum_{n=1}^{\infty} V(I_n^{k+1}) &= \sum_{n=1}^{\infty} V(J_n^k \cap I) - \sum_{n=1}^{\infty} V(J_n^{k+1} \cap I) \\
&= \sum_{n=1}^{\infty} (V(J_n^k) - V(J_n^k \cap I^c)) - \sum_{n=1}^{\infty} (V(J_n^{k+1}) - V(J_n^{k+1} \cap I^c)) \\
&= \sum_{n=1}^{\infty} V(J_n^k) - \sum_{n=1}^{\infty} V(J_n^{k+1}) + \sum_{n=1}^{\infty} V(J_n^{k+1} \cap I^c) - \sum_{n=1}^{\infty} V(J_n^k \cap I^c) \\
&\leq \sum_{n=1}^{\infty} V(J_n^k) - \sum_{n=1}^{\infty} V(J_n^{k+1}).
\end{aligned}$$

And thus $x := \lim_{k\to\infty} \sum_{n=1}^{\infty} V(I_n^k)$ exists.

Suppose now that (J_n) is a covering of $A \cap I$ by mutually disjoint simplexes. By way of contradiction, suppose $\sum_{n=1}^{\infty} V(J_n) < x$. Then

$$\begin{aligned}
M_u(A) &= \lim_{k\to\infty} \sum_{n=1}^{\infty} V(J_n^k) \\
&= \lim_{k\to\infty} \sum_{n=1}^{\infty} V(I \cap J_n^k) + \lim_{k\to\infty} \sum_{n=1}^{\infty} V(I^c \cap J_n^k) \\
&= x + \lim_{k\to\infty} \sum_{n=1}^{\infty} V(I^c \cap J_n^k)
\end{aligned}$$

$$> \sum_{n=1}^{\infty} V(J_n) + \lim_{k\to\infty} \sum_{n=1}^{\infty} V(I^c \cap J_n^k).$$

It follows that $M_u(A) > \sum_{n=1}^{\infty} V(J_n) + \sum_{n=1}^{\infty} V(I^c \cap J_n^k)$ for k big enough. This is a contradiction, since $\{J_n\} \cup \{I^c \cap J_n^k\}$ is a covering of A. We conclude that $A \cap I$ is outer measurable. □

Remark 3.8. *Using a similar argument, one can prove that $A \cap I^c$ is also outer measurable for any simplex I and outer measurable set A in $\mathcal{R}^j$.*

Corollary 3.9. *Let A be outer measurable in $\mathcal{R}^j$ and, for each $n \in \mathbb{N}$, let I_n be a simplex in $\mathcal{R}^j$. Then*

$$A \cap \bigcup_{n=1}^{N} I_n \text{ and } A \setminus \bigcup_{n=1}^{N} I_n$$

are outer measurable.

Proposition 3.10. *Let A be outer measurable and I, J two disjoint simplexes in $\mathcal{R}^j$. Then $(A \cap I) \cup (A \cap J)$ is outer measurable. Moreover,*

$$M_u((A \cap I) \cup (A \cap J)) = M_u(A \cap I) + M_u(A \cap J).$$

Proof. Since $A \cap I$ and $A \cap J$ are outer measurable, there exist two sequences of sequences of simplexes $\{I_n^k\}$, $\{J_n^k\}$ outer-converging to $A \cap I$ and $A \cap J$, respectively. Now, the sequence of sequences $\{I_n^k\} \cup \{J_n^k\}$ is a covering of $(A \cap I) \cup (A \cap J)$ that satisfies

$$\lim_{k\to\infty} \sum_{X \in \{I_n^k\} \cup \{J_n^k\}} V(X) = \lim_{k\to\infty} \left[\sum_{n=1}^{\infty} V(I_n^k) + \sum_{n=1}^{\infty} V(J_n^k) \right] = M_u(A \cap I) + M_u(A \cap J).$$

Now let $\{S_n\}$ be a covering of $(A \cap I) \cup (A \cap J)$. Without loss of generality, we may assume that $S_n = S_n \cap I$ or $S_n = S_n \cap J$. We now may subdivide $\{S_n\}$ into $\{S_n \cap I\} \cup \{S_n \cap J\} := \{I_n\} \cup \{J_n\}$. It follows that $\{I_n\}$ covers $A \cap I$ and $\{J_n\}$ covers $A \cap J$. Thus

$$\sum_{n=1}^{\infty} V(S_n) = \sum_{n=1}^{\infty} V(I_n) + \sum_{n=1}^{\infty} V(J_n) \geq M_u(A \cap I) + M_u(A \cap J).$$

We conclude that $(A \cap I) \cup (A \cap J)$ is outer measurable and that

$$M_u((A \cap I) \cup (A \cap J)) = M_u(A \cap I) + M_u(A \cap J).$$

□

Corollary 3.11. *Let $A \subset \mathcal{R}^j$ be outer measurable, let $N \in \mathbb{N}$ be given and, for each $n \in \{1, \ldots, N\}$, let J_n be a simplex in $\mathcal{R}^j$ such that $J_n \cap J_m = \emptyset$ if $m \neq n$. Then*

$$M_u\left(A \cap \left(\bigcup_{n=1}^{N} J_n\right)\right) = \sum_{n=1}^{N} M_u(A \cap J_n).$$

Proposition 3.12. *Let $A \subset \mathcal{R}^j$ be measurable and, for each $n \in \mathbb{N}$, let J_n be a simplex in $\mathcal{R}^j$ such that $J_n \cap J_m = \emptyset$ if $m \neq n$ and $\lim_{n\to\infty} V(J_n) = 0$. Then*

$$M_u\left(A \cap \left(\bigcup_{n=1}^{\infty} J_n\right)\right) = \sum_{n=1}^{\infty} M_u(A \cap J_n).$$

Proof. Since $J_n \cap A \subseteq J_n$, we have that $M_u(A \cap J_n) \leq V(J_n)$ for all $n \in \mathbb{N}$. Thus, $\sum\limits_{n=1}^{\infty} M_u(A \cap J_n)$ converges. Let $\{J_{n,m}^k\}_{m=1}^{\infty}$ be a sequence that converges to $A \cap J_n$. The covering $\bigcup\limits_{n=1}^{\infty} \{J_{n,m}^k\}$ satisfies

$$\lim_{k\to\infty} \sum_{X \in \bigcup\limits_{n=1}^{\infty} \{J_{n,m}^k\}} V(X) = \lim_{k\to\infty} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} V(J_{n,m}^k) = \sum_{n=1}^{\infty} \left(\lim_{k\to\infty} \sum_{m=1}^{\infty} V(J_{n,m}^k) \right) = \sum_{n=1}^{\infty} M_u(A \cap J_n).$$

Suppose now that $\{S_n\}$ is another covering of $A \cap \left(\bigcup\limits_{n=1}^{\infty} J_n \right)$. Then, for every natural number N, we have

$$\sum_{n=1}^{\infty} V(S_n) \geq \sum_{n=1}^{N} M_u(A \cap J_n) = M_u \left(A \cap \left(\bigcup_{n=1}^{N} J_n \right) \right).$$

Hence $\sum\limits_{n=1}^{\infty} V(S_n) \geq \sum\limits_{n=1}^{\infty} M_u(A \cap J_n)$.

We conclude that $M_u \left(A \cap \left(\bigcup\limits_{n=1}^{\infty} J_n \right) \right) = \sum\limits_{n=1}^{\infty} M_u(A \cap J_n)$. □

Proposition 3.13. *Let A be outer measurable and I a simplex in $\mathcal{R}^j$ such that $I \cap A = \emptyset$. Then $A \cup I$ is outer measurable and*

$$M_u(A \cup I) = M_u(A) + V(I).$$

Proof. Let $\{J_n^k\}$ be a sequence that outer-converges to A. Without loss of generality, suppose $J_n^k = J_n^k \cap I^c$. For each $k \in \mathbb{N}$, let

$$I_0^k := I \text{ and } I_n^k := J_n^k \text{ for } n \geq 1.$$

Then we have that the sequence $\{I_n^k\}$ satisfies that $A \cup I \subseteq \bigcup\limits_{n=0}^{\infty} I_n^{k+1} \subseteq \bigcup\limits_{n=0}^{\infty} I_n^k$ and that $\lim\limits_{k\to\infty} \sum\limits_{n=0}^{\infty} V(I_n^k) = M_u(A) + V(I)$.

Let $\{S_n\}$ be a covering of $A \cup I$. We can subdivide $\{S_n\}$ into $\{S_n \cap I\}$ and $\{S_n \cap I^c\}$ coverings of I and A respectively. Thus,

$$\begin{aligned}
\sum_{n=0}^{\infty} V(S_n) &= \sum_{n=0}^{\infty} (V(S_n \cap I) + V(S_n \cap I^c)) \\
&= \sum_{n=0}^{\infty} V(S_n \cap I) + \sum_{n=0}^{\infty} V(S_n \cap I^c) \\
&\geq V(I) + M_u(A).
\end{aligned}$$

We conclude that $M_u(A \cup I) = M_u(A) + V(I)$. □

Corollary 3.14. *Let $A \subset \mathcal{R}^j$ be outer measurable, let $N \in \mathbb{N}$ be given and, for each $n \in \{1, \ldots, N\}$, let I_n be a simplex in $\mathcal{R}^j$ such that $A \cap I_n = \emptyset$ for all $n \in \{1, \ldots, N\}$ and $I_n \cap J_m = \emptyset$ for $m \neq n$ in $\{1, \ldots, N\}$. Then*

$$M_u \left(A \cup \left(\bigcup_{n=1}^{N} I_n \right) \right) = M_u(A) + \sum_{n=1}^{N} V(I_n).$$

Corollary 3.15. *Let $A \subset \mathcal{R}^j$ be outer measurable, let $N \in \mathbb{N}$ be given and, for each $n \in \{1, \ldots, N\}$, let I_n be a simplex in $\mathcal{R}^j$. Then $A \cup \left(\bigcup_{n=1}^{N} I_n\right)$ is outer measurable.*

Proof. It is enough to see that $A \cup I_n = (A \cap I_n^c) \cup I_n$, for each $n \in \{1, \ldots, N\}$; moreover, $A \cap I_n^c$ is outer measurable and $(A \cap I_n^c) \cap I_n = \emptyset$ for each $n \in \{1, \ldots, N\}$. □

Proposition 3.16. *Let $A \subset \mathcal{R}^j$ be outer measurable and, for each $n \in \mathbb{N}$, let I_n be a simplex in $\mathcal{R}^j$ such that $A \cap I_n = \emptyset$ for all $n \in \mathbb{N}$, $I_n \cap J_m = \emptyset$ for $m \neq n$ in $\mathbb{N}$ and $\lim_{n\to\infty} V(I_n) = 0$. Then*

$$M_u\left(A \cup \left(\bigcup_{n=1}^{\infty} I_n\right)\right) = M_u(A) + \sum_{n=1}^{\infty} V(I_n).$$

Proof. Let $\{J_n^k\}$ be a sequence outer-converging to A. Then, $\{J_n^k\} \cup \{I_n\}$ covers $A \cup \left(\bigcup_{n=1}^{\infty} I_n\right)$ and

$$\lim_{k\to\infty} \sum_{X \in \{J_n^k\} \cup \{I_n\}_{n=1}^{\infty}} V(X) = \lim_{k\to\infty} \sum_{n=1}^{\infty} V(J_n^k) + \sum_{n=1}^{\infty} V(I_n) = M_u(A) + \sum_{n=1}^{\infty} V(I_n).$$

Now, let $\{S_n\}$ be a covering of $\{A\} \cup \{I_n\}_{n=1}^{\infty}$. It follows that, for every natural number N,

$$M_u\left(A \cup \left(\bigcup_{n=1}^{N} I_n\right)\right) = M_u(A) + \sum_{n=1}^{N} V(I_n) \leq \sum_{n=1}^{\infty} V(S_n).$$

Hence

$$M_u(A) + \sum_{n=1}^{\infty} V(I_n) \leq \sum_{n=1}^{\infty} V(S_n).$$

□

Proposition 3.17. *Let $A \subset \mathcal{R}^j$ be outer measurable and, for each $n \in \mathbb{N}$, let I_n be a simplex in $\mathcal{R}^j$ such that $\lim_{n\to\infty} V(I_n) = 0$. Then $A \cup \left(\bigcup_{n=1}^{\infty} I_n\right)$ is outer measurable.*

Proof. Without loss of generality, suppose $\{I_n\}_{n=1}^{\infty}$ is a mutually disjoint collection that's arranged in the order of decreasing volume. For each $m \in \mathbb{N}$ there exists some natural number N_m and some covering $\{S_n^m\}$ of $A_m := A \cup \left(\bigcup_{n=1}^{N_m} I_n\right)$ such that

$$\sum_{n=N_m+1}^{\infty} V(I_n) < d^m$$

and

$$\sum_{n=1}^{\infty} V(S_n^m) - M_u(A_m) < d^m.$$

Then, $C_m := \{S_n^m\} \cup \{I_n\}_{n=N_m+1}^{\infty}$ is a sequence of coverings for $A \cup \left(\bigcup_{n=1}^{\infty} I_n\right)$ that satisfies

$$\left|\sum_{X \in C_{m+1}} V(X) - \sum_{X \in C_m} V(X)\right| = \left|\sum_{n=N_{m+1}+1}^{\infty} V(I_n) + \sum_{n=1}^{\infty} V(S_n^{m+1}) - \sum_{n=N_m+1}^{\infty} V(I_n) - \sum_{n=1}^{\infty} V(S_n^m)\right|$$

$$
\begin{aligned}
&\le \left|\sum_{n=N_{m+1}+1}^{\infty} V(I_n)\right| + \left|\sum_{n=N_m+1}^{\infty} V(I_n)\right| + \left|\sum_{n=1}^{\infty} V(S_n^{m+1}) - \sum_{n=1}^{\infty} V(S_n^m)\right| \\
&< 2d^m + \left|\sum_{n=1}^{\infty} V(S_n^{m+1}) - \sum_{n=1}^{\infty} V(S_n^m)\right| \\
&\le 2d^m + \left|\sum_{n=1}^{\infty} V(S_n^{m+1}) - M_u(A_{m+1})\right| + |M_u(A_{m+1}) - M_u(A_m)| + \left|M_u(A_m) - \sum_{n=1}^{\infty} V(S_n^m)\right| \\
&< 4d^m + |M_u(A_{m+1}) - M_u(A_m)| \\
&\le 4d^m + \sum_{n=N_m+1}^{\infty} V(I_n) \\
&< 5d^m.
\end{aligned}
$$

Thus $x := \lim_{m\to\infty} \sum_{X\in C_m} V(X)$ exists since $\mathcal{R}$ is Cauchy-completee. Suppose now that $\{S_n\}$ is a covering of $A \cup \left(\bigcup_{n=1}^{\infty} I_n\right)$ such that $\sum_{n=1}^{\infty} V(S_n) < x$. We choose k such that $d^k + \sum_{n=1}^{\infty} V(S_n) < x$ and $m > k$ such that $d^k + \sum_{n=1}^{\infty} V(S_n) < \sum_{X\in C_m} V(X)$. It follows that

$$
\begin{aligned}
M_u(A_m) &> \sum_{n=1}^{\infty} V(S_n^m) - d^m \\
&= \sum_{X\in C_m} V(X) - \sum_{n=N_m+1}^{\infty} V(I_n) - d^m \\
&> \sum_{X\in C_m} V(X) - 2d^m \\
&> \sum_{X\in C_m} V(X) - d^k \\
&> \sum_{n=1}^{\infty} V(S_n),
\end{aligned}
$$

which is a contradiction, since $\{S_n\}$ covers A_m. We conclude that $A \cup \left(\bigcup_{n=1}^{\infty} I_n\right)$ is outer measurable. □

Lemma 3.18. *Let A be outer measurable and let $\{I_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint simplexes in $\mathcal{R}^j$ whose volumes form a vanishing sequence. Then the set*

$$A \setminus \bigcup_{n=1}^{\infty} I_n$$

is outer measurable and

$$M_u\left(A \setminus \bigcup_{n=1}^{\infty} I_n\right) = \lim_{N\to\infty} M_u\left(A \setminus \bigcup_{n=1}^{N} I_n\right).$$

Proof. We define

$$A_k := A \setminus \bigcup_{n=1}^{k} I_n$$

and

$$A_\infty := A \setminus \bigcup_{n=1}^{\infty} I_n.$$

Clearly, $A_{k+1} \subseteq A_k$ and thus $M_u(A_{k+1}) \leq M_u(A_k)$. We note that

$$A_k = A_{k+1} \cup (A \cap I_{k+1}).$$

Hence

$$M_u(A_k) - M_u(A_{k+1}) \leq M_u(A \cap I_{k+1}) \leq V(I_{k+1}).$$

Thus, the sequence $M_u(A_k)$ is Cauchy and therefore convergent. We define

$$x := \lim_{N\to\infty} M_u\left(A \setminus \bigcup_{n=1}^{N} I_n\right).$$

Since each A_k is outer measurable, there exists a covering $\{S_n^k\}$ of A_k such that $\sum\limits_{n=1}^{\infty} V(S_n^k) - M_u(A_k) < d^k$. Now, the sequence $\{S_n^k\}$ is a sequence of coverings of A_∞ that satisfies

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} V(S_n^k) = x.$$

It remains to show that x is a lower bound for the sum of volumes of any countable collection of simplexes covering A_∞. We proceed by way of contradiction and suppose that $\{J_n\}$ is a covering of A_∞ such that

$$\sum_{n=1}^{\infty} V(J_n) < x.$$

We now take $N \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} V(J_n) + \sum_{n=N}^{\infty} V(I_n) < x \leq M_u(A_N),$$

which yields a contradiction given that $\{J_n\} \cup \{I_n\}_{n=N}^{\infty}$ covers A_N. □

Proposition 3.19. *Let A and B be outer measurable sets in $\mathcal{R}^j$ such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$ and $B \subseteq (\bigcup_{n=1}^{\infty} I_n)^c$, where the I_n's are pairwise disjoint simplexes in $\mathcal{R}^j$ with $\lim\limits_{n\to\infty} V(I_n) = 0$. Then, $A \cup B$ is outer measurable, and*

$$M_u(A \cup B) = M_u(A) + M_u(B)$$

Proof. Let $\{S_k\}_{k=1}^{\infty}$ be a simplex covering of $A \cup B$ in $\mathcal{R}^j$. Then, for every $M \in \mathbb{N}$, we have that

$$A \subseteq \bigcup_{k=1}^{\infty} S_k \cap \bigcup_{n=1}^{\infty} I_n = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} S_k \cap I_n$$

and

$$B \subseteq \bigcup_{k=1}^{\infty} S_k \setminus \bigcup_{n=1}^{\infty} I_n \subseteq \bigcup_{k=1}^{\infty} S_k \setminus \bigcup_{n=1}^{M} I_n.$$

Thus,

$$\sum_{k=1}^{\infty} V(S_k) = \sum_{k=1}^{\infty} \left(V\left(S_k \cap \bigcup_{n=1}^{M} I_n\right) + V\left(S_k \setminus \bigcup_{n=1}^{M} I_n\right)\right)$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty}\left(\sum_{n=1}^{M} V\left(S_k \cap I_n\right) + V\left(S_k \setminus \bigcup_{n=1}^{M} I_n\right)\right)\\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} V\left(S_k \cap I_n\right) + \sum_{k=1}^{\infty} V\left(S_k \setminus \bigcup_{n=1}^{M} I_n\right) - \sum_{k=1}^{\infty}\sum_{n=M+1}^{\infty} V\left(S_k \cap I_n\right)\\
&\geq M_u(A) + M_u(B) - \sum_{k=1}^{\infty}\sum_{n=M+1}^{\infty} V\left(S_k \cap I_n\right).
\end{aligned}
$$

Taking the limit as $M \to \infty$ yields the inequeality

$$\sum_{k=1}^{\infty} V(S_k) \geq M_u(A) + M_u(B).$$

Now we simply note that if $\{A_n^k\}$ and $\{B_n^k\}$ outer-converge to A and B, respectively, then $\{A_n^k\} \cup \{B_n^k\}$ outer-converges to $A \cup B$. □

Theorem 3.20. *Let A and B be outer measurable in $\mathcal{R}^j$. Then $A \cup B$ is outer measurable.*

Proof. Let $\{I_n^k\}$ be a sequence that outer-converges to A. We define

$$B_k := B \cap \bigcap_{n=1}^{\infty}\left(I_n^k\right)^c$$

where $\left(I_n^k\right)^c$ denotes the complement of I_n^k. Since each B_k is outer measurable, there exists a covering $\{J_n^k\}$ of B_k such that $\sum_{n=1}^{\infty} V(J_n^k) - M_u(B_k) < d^k$. We note that, since

$$\bigcup_{n=1}^{\infty} I_n^{k+1} \subseteq \bigcup_{n=1}^{\infty} I_n^k,$$

then

$$\bigcap_{n=1}^{\infty}\left(I_n^k\right)^c \subseteq \bigcap_{n=1}^{\infty}\left(I_n^{k+1}\right)^c$$

and hence $B_k \subseteq B_{k+1}$. Moreover,

$$
\begin{aligned}
B_{k+1} &= B_k \cup \left(B_{k+1} \setminus B_k\right)\\
&= B_k \cup \left(B \cap \left(\bigcup_{n=1}^{\infty} I_n^k \setminus \bigcup_{n=1}^{\infty} I_n^{k+1}\right)\right)\\
&= B_k \cup \left(\bigcup_{n=1}^{\infty} I_n^k \cap \left(B \setminus \bigcup_{n=1}^{\infty} I_n^{k+1}\right)\right)\\
&\subseteq B_k \cup \left(\bigcup_{n=1}^{\infty} I_n^k \setminus \bigcup_{n=1}^{\infty} I_n^{k+1}\right).
\end{aligned}
$$

Thus, $B_{k+1} \setminus B_k$ is outer measurable and $M_u(B_{k+1} \setminus B_k) \leq \sum_{n=1}^{\infty} V(I_n^k) - \sum_{n=1}^{\infty} V(I_n^{k+1})$. It now follows that the sequence $M_u(B_k)$ is Cauchy, and therefore

$$\lim_{k\to\infty}\sum_{n=1}^{\infty} V(J_n^k) = \lim_{k\to\infty} M_u(B_k) \in \mathcal{R}.$$

We define $B_\infty := \bigcup_{n=1}^{\infty} B_k$. Let $\{S_n\}$ be a cover of B_∞. It follows that $\{S_n\}$ covers B_k and hence $M_u(B_k) \leq \sum_{n=1}^{\infty} V(S_n)$. Thus, $\lim_{k\to\infty} M_u(B_k) =: x \leq \sum_{n=1}^{\infty} V(S_n)$.

Now, given that every $B_{k+1} \setminus B_k$ is outer measurable and $\lim_{k\to\infty} M_u(B_{k+1} \setminus B_k) = 0$, there exists $\{R_n^k\}$ covering of $B_{k+1} \setminus B_k$ such that $\sum_{n=1}^{\infty} V(R_n^k) - M_u(B_{k+1} \setminus B_k) < d^k$. It follows that $\{J_n^k\} \cup \bigcup_{m=k}^{\infty} \{R_n^m\}$ covers $B_\infty = B_k \cup \left(\bigcup_{m=k}^{\infty} (B_{m+1} \setminus B_m) \right)$ and

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} V(J_n^k) + \sum_{m=k}^{\infty} \sum_{n=1}^{\infty} V(R_n^m) = x.$$

We conclude that B_∞ is outer measurable and has measure x.

Now, $\{J_n^k\} \cup \{I_n^k\}$ covers $A \cup B$ and

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} V(J_n^k) + \sum_{n=1}^{\infty} V(I_n^m) = M_u(A) + x.$$

Finally, if $\{T_n\}$ covers $A \cup B$, then it does also cover $A \cup B_k$, which is measurable by Proposition 3.19. Hence

$$M_u(A) + M_u(B_k) = M_u(A \cup B_k) \leq \sum_{n=1}^{\infty} V(S_n),$$

implying that

$$M_u(A) + x = M_u(A) + \lim_{k\to\infty} M_u(B_k) \leq \sum_{n=1}^{\infty} V(S_n).$$

We conclude that $A \cup B$ is outer measurable. □

4. THE L-MEASURE ON $\mathcal{R}^j$: A LEBESGUE-LIKE MEASURE

With the last theorem from the previous section, we are ready to define a new family of measurable sets.

Definition 4.1. *We say that an outer measurable set $A \subseteq \mathcal{R}^j$ is L-measurable if for every other outer measurable set $B \subseteq \mathcal{R}^j$ both $A \cap B$ and $A^c \cap B$ are outer measurable, and*

$$M_u(B) = M_u(A \cap B) + M_u(A^c \cap B)$$

If so, we define the L-measure of A to be $M(A) := M_u(A)$. We call the family of L-measurable sets $\mathcal{M}_L$.

4.1. General Properties

Proposition 4.2. *Let $A, B \in \mathcal{M}_L$. Then $A \cap B, A \cup B, A \cap B^c \in \mathcal{M}_L$.*

Proof. Let X be outer measurable. By definition, the sets

- $A \cap X$
- $B \cap X$
- $A^c \cap X$

- $B^c \cap X$

are outer measurable, implying that

- $A \cap B \cap X$
- $(A \cup B) \cap X$
- $A^c \cap B^c \cap X$
- $(A^c \cup B^c) \cap X$
- $X \cap A \cap B^c$
- $X \cap A^c \cap B$
- $(A^c \cup B) \cap X$
- $(A \cup B^c) \cap X$

and

- $X \cap (A^c \cup B^c) \cap A$
- $X \cap (A^c \cup B^c) \cap A^c$
- $X \cap (A \cup B) \cap A$
- $X \cap (A \cup B) \cap A^c$
- $X \cap (A^c \cup B) \cap A$
- $X \cap (A^c \cup B) \cap A^c$

are outer measurable too.

Now, we simply check that

$$\begin{aligned} M_u(X) &= M_u(X \cap A) + M_u(X \cap A^c) \\ &= M_u(X \cap A \cap B) + M_u(X \cap A \cap B^c) + M_u(X \cap A^c) \\ &= M_u(X \cap A \cap B) + M_u(X \cap (A^c \cup B^c) \cap A) + M_u(X \cap (A^c \cup B^c) \cap A^c) \\ &= M_u(X \cap A \cap B) + M_u(X \cap (A^c \cup B^c)) \\ &= M_u(X \cap (A \cap B)) + M_u(X \cap (A \cap B)^c), \end{aligned}$$

$$\begin{aligned} M_u(X) &= M_u(X \cap A^c) + M_u(X \cap A) \\ &= M_u(X \cap A^c \cap B^c) + M_u(X \cap A^c \cap B) + M_u(X \cap A) \\ &= M_u(X \cap A^c \cap B^c) + M_u(X \cap (A \cup B) \cap A^c) + M_u(X \cap (A \cup B) \cap A) \\ &= M_u(X \cap A^c \cap B^c) + M_u(X \cap (A \cup B)) \\ &= M_u(X \cap (A \cup B)^c) + M_u(X \cap (A \cup B)), \end{aligned}$$

and

$$\begin{aligned} M_u(X) &= M_u(X \cap A) + M_u(X \cap A^c) \\ &= M_u(X \cap A \cap B^c) + M_u(X \cap A \cap B) + M_u(X \cap A^c) \\ &= M_u(X \cap A \cap B^c) + M_u(X \cap (A^c \cup B) \cap A) + M_u(X \cap (A^c \cup B) \cap A^c) \\ &= M_u(X \cap A \cap B^c) + M_u(X \cap (A^c \cup B)) \\ &= M_u(X \cap (A \cap B^c)) + M_u(X \cap (A \cap B^c)^c). \end{aligned}$$

Thus $A \cup B$, $A \cap B$ and $A \cap B^c$ are L-measurable.

□

This family of L-measurable sets naturally inherits some of the key properties of the Lebesgue measure in $\mathbb{R}^j$, as shown below.

Proposition 4.3. *Let $A, B \in \mathcal{M}_L$. Then*

$$M(A \cup B) = M(A) + M(B) - M(A \cap B).$$

Proof. We already know by Proposition 4.2 that $A \cup B, A \cap B \in \mathcal{M}_L$. It follows that

$$\begin{aligned} M(A \cup B) &= M_u(A \cup B) \\ &= M_u((A \cup B) \cap A) + M_u((A \cup B) \cap A^c)) \\ &= M_u(A) + M_u(B \cap A^c) \\ &= M_u(A) + M_u(B) - M_u(A \cap B) \\ &= M(A) + M(B) - M(A \cap B). \end{aligned}$$

□

The L-measure also proves to be a direct improvement over the S-measure by strictly expanding the family of measurable sets.

Proposition 4.4. *For each $n \in \mathbb{N}$ let J_n be a simplex in $\mathcal{R}^j$ such that $J_n \cap J_m = \emptyset$ for $m \neq n$ and $\lim_{n\to\infty} V(J_n) = 0$. Then*

$$M\left(\bigcup_{n=1}^{\infty} J_n\right) = \sum_{n=1}^{\infty} V(J_n).$$

Proof. The result follows directly from Lemma 3.18 and Proposition 3.19 and from the fact that

$$A = \left(A \cap \bigcup_{n=1}^{\infty} J_n\right) \cup \left(A \setminus \bigcup_{n=1}^{\infty} J_n\right).$$

□

Proposition 4.5. *Sets of outer measure zero are L-measurable.*

Proof. The result follows directly from Proposition 3.6 and from the fact that if $M_u(C) = 0$ and A is outer measurable, then

$$M_u(A) = M_u(A \setminus C) = M_u(A \setminus C) + 0 = M_u(A \setminus C) + M_u(A \cap C).$$

□

Corollary 4.6. *Let $K \subseteq \mathcal{R}^j$ be compact. Then K is L-measurable and its L-measure is equal to zero.*

Proof. Let $\epsilon > 0$ in $\mathcal{R}$ be given. Then, for each $x \in K$, take S_x to be an open symplex containing x such that $V(S_x)$ is infinitely smaller than ϵ. Clearly, $\{S_x\}_{x\in K}$ is an open cover of K. Thus, by compactness, we may extract a finite subcover $\{S_n\}_{n=1}^{N}$ of K that satisfies

$$\sum_{n=1}^{N} V(S_n) < \epsilon.$$

It follows that K is outer-measurable with $M_u(K) = 0$, and hence $K \in \mathcal{M}_L$ with $M(K) = 0$. □

4.2. Continuity of the Measure

One of the key results in probability theory is that of the continuity of the probability function. Despite not being able to get the full result due to the topology of the Levi-Civita field $\mathcal{R}$, we manage to get very close to it.

Lemma 4.7. *For each $n \in \mathbb{N}$ let $A_n \in \mathcal{M}_L$, with $M(A_n) \to 0$ when $n \to \infty$. Then, for any outer measurable set X in $\mathcal{R}^j$, we have that $X \cap \bigcup_{n=1}^{\infty} A_n$ is outer measurable, with*

$$M_u\left(X \cap \bigcup_{n=1}^{\infty} A_n\right) = \lim_{N\to\infty} M_u\left(X \cap \bigcup_{n=1}^{N} A_n\right).$$

Proof. We define $X_N := X \cap \bigcup_{n=1}^{N} A_n$ and $X_\infty := X \cap \bigcup_{n=1}^{\infty} A_n$. Then, since

$$X_{N+1} = X \cap \bigcup_{n=1}^{N+1} A_n = \left(X \cap \bigcup_{n=1}^{N} A_n\right) \cup (X \cap A_{N+1}) \subset X_N \cup A_{N+1},$$

we have that

$$M_u(X_{N+1}) \le M_u(X_N) + M_u(A_{N+1}).$$

Since the measures of A_n form a null sequence, it follows that the sequence $(M_u(X_N))_{n\in\mathbb{N}}$ is Cauchy and hence convergent in $\mathcal{R}$. We define

$$t := \lim_{N\to\infty} M_u(X_N).$$

Given that each X_m and A_k are outer measurable, we can find $\{J_n^m\}$ and $\{I_n^k\}$ covers of X_m and A_k, respectively, such that

$$\sum_{n=1}^{\infty} V(J_n^m) - M_u(X_m) < d^m$$

and

$$\sum_{n=1}^{\infty} V(I_n^k) - M_u(A_k) < d^k.$$

We define $\{S_n^k\} := \{J_n^k\} \cup \bigcup_{m=k+1}^{\infty} \{I_n^m\}$. Clearly, $\{S_n^k\}$ is a covering of X_∞ and

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} V(S_n^k) = \lim_{k\to\infty} \left(\sum_{n=1}^{\infty} V(J_n^k) + \sum_{m=k+1}^{\infty}\sum_{n=1}^{\infty} V(I_n^m)\right) = t.$$

Finally, if $\{P_n\}$ covers X_∞, then it covers X_N for all N. Thus,

$$M_u(X_N) \le \sum_{n=1}^{\infty} V(P_n)$$

and therefore

$$\lim_{N\to\infty} M_u(X_N) = t \le \sum_{n=1}^{\infty} V(P_n).$$

We conclude that $X_\infty = X \cap \bigcup_{n=1}^{\infty} A_n$ is outer measurable and

$$M_u\left(X \cap \bigcup_{n=1}^{\infty} A_n\right) = \lim_{N\to\infty} M_u\left(X \cap \bigcup_{n=1}^{N} A_n\right).$$

□

Corollary 4.8. *For each $n \in \mathbb{N}$, let $A_n \in \mathcal{M}_L$ be such that $\lim_{N\to\infty} M\left(\bigcup_{n=1}^{N} A_n\right) \in \mathcal{R}$. Then, for any outer measurable set X in $\mathcal{R}^j$, we have that $X \cap \bigcup_{n=1}^{\infty} A_n$ is outer measurable, with*

$$M_u\left(X \cap \bigcup_{n=1}^{\infty} A_n\right) = \lim_{N\to\infty} M_u\left(X \cap \bigcup_{n=1}^{N} A_n\right).$$

Proof. Since $\lim_{N\to\infty} M\left(\bigcup_{n=1}^{N} A_n\right) \in \mathcal{R}$, if we define $A_0 = \emptyset$, we get

$$M\left(\bigcup_{n=0}^{N} A_n\right) - M\left(\bigcup_{n=0}^{N-1} A_n\right) = M\left(\bigcup_{n=0}^{N} A_n \setminus \bigcup_{n=0}^{N-1} A_n\right) \underset{N\to\infty}{\rightarrow} 0.$$

Thus, defining $B_N := \bigcup_{n=0}^{N} A_n \setminus \bigcup_{n=0}^{N-1} A_n$ yields a sequence satisfying $\bigcup_{n=1}^{N} B_n = \bigcup_{n=1}^{N} A_n$, $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$ and $\lim_{n\to\infty} M(B_n) = 0$. The result then follows from Lemma 4.7. □

Corollary 4.9. *For each $n \in \mathbb{N}$, let $A_n \in \mathcal{M}_L$ be such that $\lim_{N\to\infty} M\left(\bigcup_{n=1}^{N} A_n\right) \in \mathcal{R}$. Then $\bigcup_{n=1}^{\infty} A_n$ is outer measurable, with*

$$M_u\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{N\to\infty} M_u\left(\bigcup_{n=1}^{N} A_n\right).$$

Proof. Without loss of generality, suppose that $M(A_m)$ forms a null sequence. Then, for each $m \in \mathbb{N}$, we can find a covering $\{J_n^m\}$ of A_m such that $\left(\sum_{n=1}^{\infty} V(J_n^m)\right)_{m\in\mathbb{N}}$ forms a null sequence. Thus, $\bigcup_{m,n=1}^{\infty} J_n^m$ is outer measurable and the result follows from Corollary 4.8, using $X = \bigcup_{m,n=1}^{\infty} J_n^m$. □

Lemma 4.10. *For each $n \in \mathbb{N}$, let $A_n \in \mathcal{M}_L$ be such that $\lim_{N\to\infty} M\left(\bigcap_{n=1}^{N} A_n\right) \in \mathcal{R}$. Then, for any outer measurable set X in $\mathcal{R}^j$, we have that $X \cap \bigcap_{n=1}^{\infty} A_n$ is outer measurable, with*

$$M_u\left(X \cap \bigcap_{n=1}^{\infty} A_n\right) = \lim_{N\to\infty} M_u\left(X \cap \bigcap_{n=1}^{N} A_n\right).$$

Proof. For each $N \in \mathbb{N}$, let $B_N := \bigcap_{n=1}^{N} A_n$, and let $B_\infty := \bigcap_{n=1}^{\infty} A_n$, $X_N = X \cap B_N$ and $X_\infty := X \cap B_\infty$. We notice that, since $M(B_N)$ is convergent, then

$$M_u(B_N \setminus B_{N+1}) = M_u(B_N) - M_u(B_{N+1})$$

is a null sequence. Now, $X_{N+1} \subseteq X_N \subseteq X_{N+1} \cup (B_N \setminus B_{N+1})$, and hence

$$M_u(X_N) \leq M_u(X_{N+1}) + M_u(B_N \setminus B_{N+1}).$$

Thus, the sequence $(M_u(X_N))_{N\in\mathbb{N}}$ is Cauchy and therefore convergent. Let

$$t := \lim_{N\to\infty} M_u(X_N).$$

For each $k \in \mathbb{N}$, let $\{J_n^k\}$ be a covering of X_k such that $\sum_{n=1}^{\infty} V(J_n^k) - M_u(X_k) < d^k$. Now, $\{J_n^k\}$ is a sequence of coverings of X_∞ satisfying $\lim_{k\to\infty} \sum_{n=1}^{\infty} V(J_n^k) = t$.

Suppose now, by way of contradiction, that there exists a covering $\{I_n\}$ of X_∞ such that $\sum_{n=1}^{\infty} V(I_n) < t$. Since for each $m \in \mathbb{N}$, $B_m \setminus B_{m+1} \in \mathcal{M}_L$ and since $\lim_{m\to\infty} M(B_m \setminus B_{m+1}) = 0$, then there exists some $N \in \mathbb{N}$ such that $\sum_{n=1}^{\infty} V(I_n) + \sum_{m=N}^{\infty} M(B_m \setminus B_{m+1}) < t$. Thus, we can find covers $\{J_n^m\}$ of $B_m \setminus B_{m+1}$ so that $\sum_{n=1}^{\infty} V(I_n) + \sum_{m=N}^{\infty} \sum_{n=1}^{\infty} V(J_n^m) < t$, which is a contradiction given that $\{I_n\} \cup \bigcup_{m=N}^{\infty} \{J_n^m\}$ covers X_N.

We conclude that $X_\infty = X \cap \bigcap_{n=1}^{\infty} A_n$ is outer measurable, and

$$M_u\left(X \cap \bigcap_{n=1}^{\infty} A_n\right) = \lim_{N\to\infty} M_u\left(X \cap \bigcap_{n=1}^{N} A_n\right).$$

□

Corollary 4.11. *For each $n \in \mathbb{N}$, let $A_n \in \mathcal{M}_L$ be such that $\lim_{N\to\infty} M\left(\bigcap_{n=1}^{N} A_n\right) \in \mathcal{R}$. Then $\bigcap_{n=1}^{\infty} A_n$ is outer measurable, with*

$$M_u\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{N\to\infty} M_u\left(\bigcap_{n=1}^{N} A_n\right).$$

Proof. The result follows immediately by taking $X = A_1$ in Lemma 4.10. □

Theorem 4.12. *For each $n \in \mathbb{N}$, let $A_n \in \mathcal{M}_L$ such that $\lim_{N\to\infty} M\left(\bigcup_{n=1}^{N} A_n\right) \in \mathcal{R}$. Then, $\bigcup_{n=1}^{\infty} A_n$ is L-measurable. Moreover,*

$$M\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{N\to\infty} M\left(\bigcup_{n=1}^{N} A_n\right).$$

Proof. We already know that $\bigcup_{n=1}^{\infty} A_n$ is outer measurable. Let $X \subset \mathcal{R}^j$ be outer measurable and let $\{I_n\}$ be a cover of X. Since for each $n \in \mathbb{N}$, $\bigcup_{n=1}^{N} A_n$ and $\bigcup_{n=1}^{\infty} I_n \setminus \bigcup_{n=1}^{N} A_n$ are L-measurable and, since the sequences of their measures are convergent, then

$$M_u(X) = \lim_{N\to\infty} M_u(X)$$

$$= \lim_{N\to\infty} \left[M_u\left(X \cap \bigcup_{n=1}^{N} A_n \right) + M_u\left(X \setminus \bigcup_{n=1}^{N} A_n \right) \right]$$
$$= \lim_{N\to\infty} \left[M_u\left(X \cap \bigcup_{n=1}^{N} A_n \right) + M_u\left(X \cap \bigcup_{n=1}^{\infty} I_n \setminus \bigcup_{n=1}^{N} A_n \right) \right]$$
$$= M_u\left(X \cap \bigcup_{n=1}^{\infty} A_n \right) + M_u\left(X \cap \bigcup_{n=1}^{\infty} I_n \setminus \bigcup_{n=1}^{\infty} A_n \right)$$
$$= M_u\left(X \cap \bigcup_{n=1}^{\infty} A_n \right) + M_u\left(X \setminus \bigcup_{n=1}^{\infty} A_n \right).$$

The equality

$$M\left(\bigcup_{n=1}^{\infty} A_n \right) = \lim_{N\to\infty} M\left(\bigcup_{n=1}^{N} A_n \right)$$

follows then from Corollary 4.9. □

Theorem 4.13. *For each $n \in \mathbb{N}$, let $A_n \in \mathcal{M}_L$ be such that $\lim_{N\to\infty} M\left(\bigcap_{n=1}^{N} A_n\right) \in \mathcal{R}$. Then $\bigcap_{n=1}^{\infty} A_n$ is L-measurable. Moreover,*

$$M\left(\bigcap_{n=1}^{\infty} A_n \right) = \lim_{N\to\infty} M\left(\bigcap_{n=1}^{N} A_n \right).$$

Proof. We already know that $\bigcap_{n=1}^{\infty} A_n$ is outer measurable. Let $X \subseteq \mathcal{R}^j$ be outer measurable and let $\{I_n\}$ be a cover of X. Since for each $n \in \mathbb{N}$, $\bigcap_{n=1}^{N} A_n$ and $\bigcup_{n=1}^{\infty} I_n \setminus \bigcap_{n=1}^{N} A_n$ are L-measurable and, since the sequences of their measures are convergent, then

$$M_u(X) = \lim_{N\to\infty} M_u(X)$$
$$= \lim_{N\to\infty} \left[M_u\left(X \cap \bigcap_{n=1}^{N} A_n \right) + M_u\left(X \setminus \bigcap_{n=1}^{N} A_n \right) \right]$$
$$= \lim_{N\to\infty} \left[M_u\left(X \cap \bigcap_{n=1}^{N} A_n \right) + M_u\left(X \cap \bigcup_{n=1}^{\infty} I_n \setminus \bigcap_{n=1}^{N} A_n \right) \right]$$
$$= M_u\left(X \cap \bigcap_{n=1}^{\infty} A_n \right) + M_u\left(X \cap \bigcup_{n=1}^{\infty} I_n \setminus \bigcap_{n=1}^{\infty} A_n \right)$$
$$= M_u\left(X \cap \bigcap_{n=1}^{\infty} A_n \right) + M_u\left(X \setminus \bigcap_{n=1}^{\infty} A_n \right).$$

The equality

$$M\left(\bigcap_{n=1}^{\infty} A_n \right) = \lim_{N\to\infty} M\left(\bigcap_{n=1}^{N} A_n \right)$$

follows then from Corollary 4.11. □

5. FUTURE WORK

Ongoing research aims at developing a Lebesgue-like integration theory for $\mathcal{R}$-valued functions on L-measurable subsets of $\mathcal{R}^j$, for $j \in \mathbb{N}$. We will develop the theory for $j = 1$ first, and then generalize it to $j > 1$. Considering the success of the measure theory developed in [2] for the case of $j = 1$ and in this paper for $j > 1$, we are hopeful that the sought after integral will satisfy most of the nice properties of the Lebesgue integral on $\mathbb{R}^j$.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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