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RESEARCH ARTICLES

On Integrable Delta Functions on the Levi-Civita Field[*1]

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Abstract—In this paper, we develop a theory of integrable delta functions on the Levi-Civita field $\mathcal{R}$ as well as on $\mathcal{R}^2$ and $\mathcal{R}^3$ with similar properties to the one-dimensional, two-dimensional and three-dimensional Dirac Delta functions and which reduce to them when restricted to points in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. First we review the recently developed Lebesgue-like measure and integration theory over $\mathcal{R}$, $\mathcal{R}^2$ and $\mathcal{R}^3$. Then we introduce delta functions on $\mathcal{R}$, $\mathcal{R}^2$ and $\mathcal{R}^3$ that are integrable in the context of the aforementioned integration theory; and we study their properties and some applications.

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1. INTRODUCTION

In various branches of physics, one encounters sources which are nearly instantaneous (if time is the independent variable) or almost localized (if the independent variable is a space coordinate). To avoid the cumbersome studies of the detailed functional dependencies of such sources, one would like to replace them with idealized sources that are truly instantaneous or localized. Typical examples of such sources are the concentrated forces and moments in solid mechanics, the point masses in the theory of the gravitational potential, and the point charges in electrostatics. The field of real numbers $\mathbb{R}$ does not permit a direct representation of the (improper) delta functions used for the description of impulsive (instantaneous) or concentrated (localized) sources. Of course, within the framework of distributions, these concepts can be accounted for in a rigorous fashion, but at the expense of the intuitive interpretation.

The existence of infinitely small numbers and infinitely large numbers in the non-Archimedean Levi-Civita field $\mathcal{R}$ allows us to have well-behaved delta functions. For example, the function $\delta : \mathcal{R} \to \mathcal{R}$, given by $\delta(x) = \frac{3}{4}d^{-3}\left(d^2 - x^2\right)$ if $|x| < d$ and 0 otherwise, where d is a positive infinitely small number, is a (one-dimensional) continuous (and piece-wise infinitely differentiable) delta function; it assumes an infinitely large value $(3/4d^{-1})$ at 0, it vanishes at all other real points and its integral is equal to one.

We recall that the elements of the Levi-Civita field $\mathcal{R}$ and its complex counterpart $\mathcal{C}$ are functions from $\mathbb{Q}$ to $\mathbb{R}$ and $\mathbb{C}$, respectively, with left-finite support (denoted by supp). That is, below every rational number q, there are only finitely many points where the given function does not vanish. For the further discussion, it is convenient to introduce the following terminology.

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Definition 1.1. *($\lambda, \sim, \approx, =_q$) For $x \neq 0$ in $\mathcal{R}$ or $\mathcal{C}$, we let $\lambda(x) = \min(supp(x))$, which exists because of the left-finiteness of $supp(x)$; and we let $\lambda(0) = +\infty$. Moreover, we denote the value of x at $q \in \mathbb{Q}$ with brackets like $x[q]$.*

Given $x, y \neq 0$ in $\mathcal{R}$ or $\mathcal{C}$, we say $x \sim y$ if $\lambda(x) = \lambda(y)$; and we say $x \approx y$ if $\lambda(x) = \lambda(y)$ and $x[\lambda(x)] = y[\lambda(y)]$. Finally, for any $q \in \mathbb{Q}$, we say $x =_q y$ if $x[p] = y[p]$ for all $p \leq q$ in $\mathbb{Q}$.

At this point, these definitions may feel somewhat arbitrary; but after having introduced an order on $\mathcal{R}$, we will see that λ describes orders of magnitude, the relation $\approx$ corresponds to agreement up to infinitely small relative error, while $\sim$ corresponds to agreement of order of magnitude.

The sets $\mathcal{R}$ and $\mathcal{C}$ are endowed with formal power series multiplication and componentwise addition, which make them into fields [6, 9] in which we can isomorphically embed $\mathbb{R}$ and $\mathbb{C}$ (respectively) as subfields via the map $\Pi : \mathbb{R}, \mathbb{C} \to \mathcal{R}, \mathcal{C}$ defined by

$$\Pi(x)[q] = \begin{cases} x & \text{if } q = 0 \\ 0 & \text{else} \end{cases} . \tag{1.1}$$

Definition 1.2. *(Order in $\mathcal{R}$) Let $x, y \in \mathcal{R}$ be given. Then we say that $x > y$ (or $y < x$) if $x \neq y$ and $(x-y)[\lambda(x-y)] > 0$; and we say $x \geq y$ (or $y \leq x$) if $x = y$ or $x > y$.*

It follows that the relation $\geq$ (or $\leq$) defines a total order on $\mathcal{R}$ which makes it into an ordered field. Note that, given $a < b$ in $\mathcal{R}$, we define the $\mathcal{R}$-interval $[a, b] = \{x \in \mathcal{R} : a \leq x \leq b\}$, with the obvious adjustments in the definitions of the intervals $[a, b)$, $(a, b]$, and (a, b). Moreover, the embedding Π in Equation (1.1) of $\mathbb{R}$ into $\mathcal{R}$ is compatible with the order.

The order leads to the definition of an ordinary absolute value on $\mathcal{R}$:

$$|x| = \max\{x, -x\} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0; \end{cases}$$

which induces the same topology on $\mathcal{R}$ (called the order topology or valuation topology) as that induced by the ultrametric absolute value:

$$|x|_u = \begin{cases} e^{-\lambda(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

as was shown in [13]. Moreover, two corresponding absolute values are defined on $\mathcal{C}$ in the natural way:

$$|x + iy| = \sqrt{x^2 + y^2}; \text{ and } |x + iy|_u = e^{-\lambda(x+iy)} = \max\{|x|_u, |y|_u\}.$$

Thus, $\mathcal{C}$ is topologically isomorphic to $\mathcal{R}^2$ provided with the product topology induced by $|\cdot|$ (or $|\cdot|_u$) in $\mathcal{R}$.

We note in passing here that $|\cdot|_u$ is a non-Archimedean valuation on $\mathcal{R}$ (resp. $\mathcal{C}$); that is, it satisfies the following properties

1. $|v|_u \geq 0$ for all $v \in \mathcal{R}$ (resp. $v \in \mathcal{C}$) and $|v|_u = 0$ if and only if $v = 0$;

2. $|vw|_u = |v|_u |w|_u$ for all $v, w \in \mathcal{R}$ (resp. $v, w \in \mathcal{C}$); and

3. $|v + w|_u \leq \max\{|v|_u, |w|_u\}$ for all $v, w \in \mathcal{R}$ (resp. $v, w \in \mathcal{C}$): the strong triangle inequality.

Thus, $(\mathcal{R}, |\cdot|_u)$ and $(\mathcal{C}, |\cdot|_u)$ are non-Archimedean valued fields.

Besides the usual order relations on $\mathcal{R}$, some other notations are convenient.

Definition 1.3. *($\ll, \gg$) Let $x, y \in \mathcal{R}$ be non-negative. We say x is infinitely smaller than y (and write $x \ll y$) if $nx < y$ for all $n \in \mathbb{N}$; we say x is infinitely larger than y (and write $x \gg y$) if $y \ll x$. If $x \ll 1$, we say x is infinitely small; if $x \gg 1$, we say x is infinitely large. Infinitely small numbers are also called infinitesimals or differentials. Infinitely large numbers are also called infinite. Non-negative numbers that are neither infinitely small nor infinitely large are also called finite.*

Definition 1.4. *(The Number d) Let d be the element of $\mathcal{R}$ given by $d[1] = 1$ and $d[t] = 0$ for $t \neq 1$.*

Remark 1.5. *Given $m \in \mathbb{Z}$, then d^m is the positive $\mathcal{R}$-number given by*

$$d^m = \begin{cases} \underbrace{dd\cdots d}_{m \text{ times}} & \text{if } m > 0 \\ 1 & \text{if } m = 0 \\ \frac{1}{d^{-m}} & \text{if } m < 0 \end{cases}.$$

Moreover, given a rational number $q = m/n$ (with $n \in \mathbb{N}$ and $m \in \mathbb{Z}$), then d^q is the positive nth root of d^m in $\mathcal{R}$ (that is, $(d^q)^n = d^m$) and it is given by

$$d^q[t] = \begin{cases} 1 & \text{if } t = q \\ 0 & \text{otherwise} \end{cases}.$$

It is easy to check that $d^q \ll 1$ if $q > 0$ and $d^q \gg 1$ if $q < 0$ in $\mathbb{Q}$. Moreover, for all $x \in \mathcal{R}$ (resp. $\mathcal{C}$), the elements of $supp(x)$ can be arranged in ascending order, say $supp(x) = \{q_1, q_2, \ldots\}$ with $q_j < q_{j+1}$ for all j; and x can be written as $x = \sum_{j=1}^{\infty} x[q_j]d^{q_j}$, where the series converges in the valuation topology.

Altogether, it follows that $\mathcal{R}$ (resp. $\mathcal{C}$) is a non-Archimedean field extension of $\mathbb{R}$ (resp. $\mathbb{C}$). For a detailed study of these fields, we refer the reader to the survey paper [9] and the references therein. In particular, it is shown that $\mathcal{R}$ and $\mathcal{C}$ are complete with respect to the natural (valuation) topology.

It follows therefore that the fields $\mathcal{R}$ and $\mathcal{C}$ are just special cases of the class of fields discussed in [5]. For a general overview of the algebraic properties of formal power series fields in general, we refer the reader to the comprehensive overview by Ribenboim [4], and for an overview of the related valuation theory to the books by Krull [2], Schikhof [5] and Alling [1]. A thorough and complete treatment of ordered structures can also be found in [3].

Besides being the smallest ordered non-Archimedean field extension of the real numbers that is both complete in the order topology and real closed, the Levi-Civita field $\mathcal{R}$ is of particular interest because of its practical usefulness. Since the supports of the elements of $\mathcal{R}$ are left-finite, it is possible to represent these numbers on a computer; and having infinitely small numbers in the field allows for many computational applications. One such application is the computation of derivatives of real functions representable on a computer [10], where both the accuracy of formula manipulators and the speed of classical numerical methods are achieved.

2. MEASURE THEORY AND INTEGRATION ON $\mathcal{R}$, $\mathcal{R}^2$ AND $\mathcal{R}^3$

Using the nice smoothness properties of power series (see [7] and the references therein), we developed a Lebesgue-like measure and integration theory on $\mathcal{R}$ in [8, 12] that uses the $\mathcal{R}$-analytic functions (functions given locally by power series- Definition 2.4) as the building blocks for measurable functions instead of the step functions used in the real case. This was possible in particular because the family $\mathcal{S}(a, b)$ of analytic functions on a given interval $I(a, b) \subset \mathcal{R}$ (where $I(a, b)$ denotes any one of the intervals $[a, b]$, $(a, b]$, $[a, b)$ or (a, b)) satisfies the following crucial properties.

1. $\mathcal{S}(a,b)$ is an algebra that contains the identity function;

2. for all $f \in \mathcal{S}(a,b)$, f is Lipschitz on $I(a,b)$ and there exists an anti-derivative F of f in $\mathcal{S}(a,b)$, which is unique up to a constant;

3. for all differentiable $f \in \mathcal{S}(a,b)$, if $f' = 0$ on (a,b) then f is constant on $I(a,b)$; moreover, if $f' \geq 0$ on (a,b) then f is nondecreasing on $I(a,b)$.

Notation 2.1. *Let $a < b$ in $\mathcal{R}$ be given. Then by $l(I(a,b))$ we will denote the length of the interval $I(a,b)$, that is*

$$l(I(a,b)) = \text{ length of } I(a,b) = b - a.$$

Definition 2.2. *Let $A \subset \mathcal{R}$ be given. Then we say that A is measurable if for every $\epsilon > 0$ in $\mathcal{R}$, there exist a sequence of mutually disjoint intervals (I_n) and a sequence of mutually disjoint intervals (J_n) such that $\bigcup_{n=1}^{\infty} I_n \subset A \subset \bigcup_{n=1}^{\infty} J_n$, $\sum_{n=1}^{\infty} l(I_n)$ and $\sum_{n=1}^{\infty} l(J_n)$ converge in $\mathcal{R}$, and $\sum_{n=1}^{\infty} l(J_n) - \sum_{n=1}^{\infty} l(I_n) \leq \epsilon$.*

Given a measurable set A, then for every $k \in \mathbb{N}$, we can select a sequence of mutually disjoint intervals $\left(I_n^k\right)$ and a sequence of mutually disjoint intervals $\left(J_n^k\right)$ such that $\sum_{n=1}^{\infty} l\left(I_n^k\right)$ and $\sum_{n=1}^{\infty} l\left(J_n^k\right)$ converge in $\mathcal{R}$ for all k,

$$\bigcup_{n=1}^{\infty} I_n^k \subset \bigcup_{n=1}^{\infty} I_n^{k+1} \subset A \subset \bigcup_{n=1}^{\infty} J_n^{k+1} \subset \bigcup_{n=1}^{\infty} J_n^k \text{ and } \sum_{n=1}^{\infty} l\left(J_n^k\right) - \sum_{n=1}^{\infty} l\left(I_n^k\right) \leq d^k$$

for all $k \in \mathbb{N}$. Since $\mathcal{R}$ is Cauchy-complete in the order topology, it follows that $\lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(I_n^k\right)$ and $\lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(J_n^k\right)$ both exist and they are equal. We call the common value of the limits the measure of A and we denote it by $m(A)$. Thus,

$$m(A) = \lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(I_n^k\right) = \lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(J_n^k\right).$$

Contrary to the real case,

$$\sup\left\{\sum_{n=1}^{\infty} l(I_n) : I_n\text{'s are mutually disjoint intervals and } \bigcup_{n=1}^{\infty} I_n \subset A\right\}$$

and

$$\inf\left\{\sum_{n=1}^{\infty} l(J_n) : J_n\text{'s are mutually disjoint intervals and } A \subset \bigcup_{n=1}^{\infty} J_n\right\}$$

need not exist for a given set $A \subset \mathcal{R}$. However, as shown in [12], if A is measurable then both the supremum and infimum exist and they are equal to $m(A)$. This shows that the definition of measurable sets in Definition 2.2 is a natural generalization of that of the Lebesgue measurable sets of real analysis that corrects for the lack of suprema and infima in non-Archimedean ordered fields.

It follows directly from the definition that $m(A) \geq 0$ for any measurable set $A \subset \mathcal{R}$ and that any interval $I(a,b)$ is measurable with measure $m(I(a,b)) = l(I(a,b)) = b - a$. It also follows that if A is a countable union of mutually disjoint intervals $(I_n(a_n,b_n))$ such that $\sum_{n=1}^{\infty}(b_n - a_n)$ converges then A is measurable with $m(A) = \sum_{n=1}^{\infty}(b_n - a_n)$. Moreover, if $B \subset A \subset \mathcal{R}$ and if A and B are measurable, then $m(B) \leq m(A)$.

In [12] we show that the measure defined on $\mathcal{R}$ above has similar properties to those of the Lebesgue measure on $\mathbb{R}$. For example, we show that any subset of a measurable set of measure 0 is itself measurable and has measure 0. We also show that any countable unions of measurable sets whose measures form a null sequence is measurable and the measure of the union is less than or equal to the sum of the measures of the original sets; moreover, the measure of the union is equal to the sum of the measures of the original sets if the latter are mutually disjoint. Furthermore, we show that any finite intersection of measurable sets is also measurable and that the sum of the measures of two measurable sets is equal to the sum of the measures of their union and intersection.

It is worth noting that the complement of a measurable set in a measurable set need not be measurable. For example, $[0,1]$ and $[0,1] \cap \mathbb{Q}$ are both measurable with measures 1 and 0, respectively. However, the complement of $[0,1] \cap \mathbb{Q}$ in $[0,1]$ is not measurable. On the other hand, if $B \subset A \subset \mathcal{R}$ and if A, B and $A \setminus B$ are all measurable, then $m(A) = m(B) + m(A \setminus B)$.

The example of $[0,1] \setminus [0,1] \cap \mathbb{Q}$ above shows that the axiom of choice is not needed here to construct a nonmeasurable set, as there are many simple examples of nonmeasurable sets. Indeed, any uncountable real subset of $\mathcal{R}$, like $[0,1] \cap \mathbb{R}$ for example, is not measurable.

Then we define in [12] a measurable function on a measurable set $A \subset \mathcal{R}$ using Definition 2.2 and analytic functions (Definition 2.4 below).

Definition 2.3. *A sequence $(a_n)_{n=1}^{\infty}$ in $\mathcal{R}$ (or $\mathcal{C}$) is said to be regular if the union of the supports of all members of the sequence is a left-finite subset of $\mathbb{Q}$.*

Definition 2.4. *Let $a < b$ in $\mathcal{R}$ be given and let $f : I(a,b) \to \mathcal{R}$. Then we say that f is analytic on $I(a,b)$ if for all $x \in I(a,b)$ there exists a positive $\delta \sim b-a$ in $\mathcal{R}$, and there exists a regular sequence $(a_n(x))_{n=1}^{\infty}$ in $\mathcal{R}$ such that, under weak convergence,*

$$f(y) = \sum_{n=0}^{\infty} a_n(x)(y-x)^n \text{ for all } y \in (x-\delta, x+\delta) \cap I(a,b).$$

Definition 2.5. *Let $a < b$ in $\mathcal{R}$ be given and let $f : I(a,b) \to \mathcal{R}$ be analytic. Then there is a rational number $i(f)$ called the index of f and defined by*

$$i(f) := \min\{\lambda(f(x)) | x \in I(a,b)\}.$$

It is shown in [13] that the above minimum must exist and that $\lambda(f(x)) = i(f)$ for almost every $x \in I(a,b) \cap d^{\lambda(b-a)}\mathbb{R}$. Moreover, if $x \in I(a,b) \cap d^{\lambda(b-a)}\mathbb{R}$ satisfies $\lambda(f(x)) = i(f)$, then $\lambda(f(y)) = i(f)$ for all y satisfying $|y-x| \ll d^{\lambda(b-a)}$.

Definition 2.6. *Let $A \subset \mathcal{R}$ be a measurable subset of $\mathcal{R}$ and let $f : A \to \mathcal{R}$ be bounded on A. Then we say that f is measurable on A if for all $\epsilon > 0$ in $\mathcal{R}$, there exists a sequence of mutually disjoint intervals (I_n) such that $I_n \subset A$ for all n, $\sum_{n=1}^{\infty} l(I_n)$ converges in $\mathcal{R}$, $m(A) - \sum_{n=1}^{\infty} l(I_n) \leq \epsilon$ and f is analytic on I_n for all n.*

In [12], we derive a simple characterization of measurable functions and we show that they form an algebra. Then we show that a measurable function is differentiable almost everywhere and that a function measurable on two measurable subsets of $\mathcal{R}$ is also measurable on their union and intersection.

We define the integral of an analytic function over an interval $I(a,b)$ and we use that to define the integral of a measurable function f over a measurable set A. Before we do that, we recall the following result whose proof can be found in [6].

Proposition 2.7. *Let $a < b$ in $\mathcal{R}$ and let $f : I(a,b) \to \mathcal{R}$ be analytic on $I(a,b)$. Then*

- *f is Lipschitz on $I(a,b)$;*

- $\lim_{x\to a^+} f(x)$ *and* $\lim_{x\to b^-} f(x)$ *exist;*
- *the function* $g:[a,b]\to\mathcal{R}$*, given by*

$$g(x)=\begin{cases} f(x) & \text{if } x\in I(a,b)\\ \lim_{\xi\to a^+} f(\xi) & \text{if } x=a\\ \lim_{\xi\to b^-} f(\xi) & \text{if } x=b,\end{cases}$$

extends f *to an analytic function on* $[a,b]$ *when* $I(a,b)\subsetneq[a,b]$.

Definition 2.8. *Let* $a<b$ *in* $\mathcal{R}$*, let* $f:I(a,b)\to\mathcal{R}$ *be analytic on* $I(a,b)$*, and let* F *be an analytic anti-derivative of* f *on* $I(a,b)$*. Then the integral of* f *over* $I(a,b)$ *is the* $\mathcal{R}$ *number*

$$\int_{I(a,b)} f=\lim_{x\to b^-}F(x)-\lim_{x\to a^+}F(x).$$

The limits in Definition 2.8 account for the case when the interval $I(a,b)$ does not include one or both of the end points; and these limits exist by Proposition 2.7 above.

Now let $A\subset\mathcal{R}$ be measurable, let $f:A\to\mathcal{R}$ be measurable and let M be a bound for $|f|$ on A. Then for every $k\in\mathbb{N}$, there exists a sequence of mutually disjoint intervals $\left(I_n^k\right)_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} I_n^k\subset A$, $\sum_{n=1}^{\infty} l\left(I_n^k\right)$ converges, $m(A)-\sum_{n=1}^{\infty} l\left(I_n^k\right)\le d^k$, and f is analytic on I_n^k for all $n\in\mathbb{N}$. Without loss of generality, we may assume that $I_n^k\subset I_n^{k+1}$ for all $n\in\mathbb{N}$ and for all $k\in\mathbb{N}$. Since $\lim_{n\to\infty} l\left(I_n^k\right)=0$, and since $\left|\int_{I_n^k} f\right|\le Ml\left(I_n^k\right)$ (proved in [12] for analytic functions), it follows that

$$\lim_{n\to\infty}\int_{I_n^k} f=0 \text{ for all } k\in\mathbb{N}.$$

Thus, $\sum_{n=1}^{\infty}\int_{I_n^k} f$ converges in $\mathcal{R}$ for all $k\in\mathbb{N}$ [11].

We show that the sequence $\left(\sum_{n=1}^{\infty}\int_{I_n^k} f\right)_{k=1}^{\infty}$ converges in $\mathcal{R}$; and we define the unique limit as the integral of f over A.

Definition 2.9. *Let* $A\subset\mathcal{R}$ *be measurable and let* $f:A\to\mathcal{R}$ *be measurable. Then the integral of* f *over* A*, denoted by* $\int_A f$*, is given by*

$$\int_A f=\lim_{\substack{\sum_{n=1}^{\infty} l(I_n)\to m(A)\\ \bigcup_{n=1}^{\infty} I_n\subset A\\ I_n'\text{s are mutually disjoint}\\ f \text{ is analytic on } I_n\ \forall n}}\sum_{n=1}^{\infty}\int_{I_n} f.$$

It turns out that the integral in Definition 2.9 satisfies similar properties to those of the Lebesgue integral on $\mathbb{R}$ [12]. In particular, we prove the linearity property of the integral and that if $|f|\le M$ on A then $\left|\int_A f\right|\le Mm(A)$, where $m(A)$ is the measure of A. We also show that the sum of the integrals of a measurable function over two measurable sets is equal to the sum of its integrals over the union and the intersection of the two sets.

In [8], which is a continuation of the work done in [12] and complements it, we show, among other results, that the uniform limit of a sequence of convergent power series on an interval $I(a,b)$ is again a power series that converges on $I(a,b)$. Then we use that to prove the uniform convergence theorem in $\mathcal{R}$.

Theorem 2.10. *Let $A \subset \mathcal{R}$ be measurable, let $f : A \to \mathcal{R}$, for each $k \in \mathbb{N}$ let $f_k : A \to \mathcal{R}$ be measurable on A, and let the sequence (f_k) converge uniformly to f on A. Then f is measurable on A, $\lim_{k\to\infty} \int_A f_k$ exists, and*

$$\lim_{k\to\infty} \int_A f_k = \int_A f.$$

In [14] we generalize the results of [8, 12] to two and three dimensions. In particular, we define a Lebesgue-like measure on $\mathcal{R}^2$ (resp. $\mathcal{R}^3$). Then we define measurable functions on measurable sets using analytic functions in two (resp. three) variables and show how to integrate those measurable functions using iterated integration. The resulting double (resp. triple) integral satisfies similar properties to those of the single integral in [8, 12] as well as those properties satisfied by the double and triple integrals of real calculus. In order to have basic regions, like disks for example, measurable, it turns out that the so-called simple regions defined below, rather than rectangles, are the best choice for the building blocks for measurable sets. We recall the following definitions from [14] which will be needed later in this paper.

Definition 2.11 (Simple Region). *Let $G \subset \mathcal{R}^2$. Then we say that G is a simple region if there exist $a \leqslant b$ in $\mathcal{R}$ and analytic functions $h_1, h_2 : I(a,b) \to \mathcal{R}$, with $h_1 \le h_2$ on $I(a,b)$ such that*

$$G = \{(x,y) \in \mathcal{R}^2 : y \in I(h_1(x), h_2(x)), x \in I(a,b)\}$$

or

$$G = \{(x,y) \in \mathcal{R}^2 : x \in I(h_1(y), h_2(y)), y \in I(a,b)\}.$$

Definition 2.12 (λ_x and λ_y of a simple region). *Let $A \subset \mathcal{R}^2$ be a simple region. If $A = \{(x,y) \in \mathcal{R}^2 : y \in I(h_1(x), h_2(x)), x \in I(a,b)\}$ we define $\lambda_x(A) = \lambda(b-a)$ and $\lambda_y(A) = i(h_2(x) - h_1(x))$ on $I(a,b)$ where $i(h_2(x) - h_1(x))$ is the index of the analytic function $h_2 - h_1$ on $I(a,b)$.*

On the other hand, if $A = \{(x,y) \in \mathcal{R}^2 : x \in I(h_1(y), h_2(y)), y \in I(a,b)\}$, we define $\lambda_y(A) = \lambda(b-a)$ and $\lambda_x(A) = i(h_2(y) - h_1(y))$ on $I(a,b)$.

If $\lambda_x(A) = \lambda_y(A) = 0$ then we say that A is finite.

Definition 2.13 (Analytic Functions on $\mathcal{R}^2$). *Let $A \subset \mathcal{R}^2$ be a simple region. Then we say that $f : A \to \mathcal{R}^2$ is an analytic function on A if, for every $(x_0, y_0) \in A$, there exist a simple region A_0 containing (x_0, y_0) that satisfies $\lambda_x(A_0) = \lambda_x(A)$ and $\lambda_y(A_0) = \lambda_y(A)$, and a regular sequence $(a_{ij})_{i,j=0}^{\infty}$ such that for every $s, t \in \mathcal{R}$, if $(x_0 + s, y_0 + t) \in A \cap A_0$ then*

$$f(x_0 + s, y_0 + t) = \sum_{i,j=0}^{\infty} a_{ij} s^i t^j = f(x_0, y_0) + \sum_{\substack{i,j=0 \\ i+j\neq 0}}^{\infty} a_{ij} s^i t^j,$$

where the power series converges in the weak topology.

Given a simple region $S \subset \mathcal{R}^2$ and an analytic function $f : S \to \mathcal{R}$, we define the index of f on S by

$$i(f) = \min\{\lambda(f(x,y)) | (x,y) \in S\},$$

which is shown to exist [14]. We note that $\lambda(f(x,y)) = i(f)$ for almost every $(x,y) \in S \cap (d^{\lambda_x(S)}\mathbb{R} \times d^{\lambda_y(S)}\mathbb{R})$ and for any such point $(x,y) \in S \cap (d^{\lambda_x(S)}\mathbb{R} \times d^{\lambda_y(S)}\mathbb{R})$, we have that $\lambda(f(x',y')) = i(f)$ for all $(x',y') \in S$ satisfying $|x' - x| \ll d^{\lambda_x(S)}$ and $|y' - y| \ll d^{\lambda_y(S)}$.

With the above definitions, we can proceed to define measurable sets, measurable functions, and integration just as we did in $\mathcal{R}$, replacing intervals by simple regions. We can then extend the measure theory and integration to $\mathcal{R}^3$, $\mathcal{R}^4$, etc. in an inductive way and obtain similar properties for the resulting integrals as those for the single integral defined above.

3. THE DELTA FUNCTION ON THE LEVI-CIVITA FIELD

In the following, capitalizing on the existence of infinitely small and infinitely large numbers and the newly developed integration theory on $\mathcal{R}$, $\mathcal{R}^2$ and $\mathcal{R}^3$, we will introduce a measurable function that has similar properties to the Dirac Delta function and reduces to it when restricted to $\mathbb{R}$.

Definition 3.1. *Let $\delta : \mathcal{R} \to \mathcal{R}$ be given by*

$$\delta(x) = \begin{cases} \frac{3}{4}d^{-3}(d^2 - x^2) & \text{if } |x| < d \\ 0 & \text{if } |x| \geqslant d \end{cases}.$$

Proposition 3.2. *Let $I \subset \mathcal{R}$ be an interval. If $(-d, d) \subset I$ then*

$$\int_{x \in I} \delta(x) = 1.$$

Moreover, if $(-d, d) \cap I = \emptyset$ then

$$\int_{x \in I} \delta(x) = 0.$$

Proof. Note that $\delta(x)$ is measurable on I [12]. If $(-d, d) \subset I$ then

$$\begin{aligned} \int\limits_{x \in I} \delta(x) &= \int\limits_{x \in (-d,d)} \delta(x) \\ &= \int\limits_{x \in (-d,d)} \frac{3}{4}d^{-3}(d^2 - x^2) \\ &= \frac{3}{4}d^{-3}\left(\left[d^2 x\right]_{-d}^{d} - \left[\frac{1}{3}x^3\right]_{-d}^{d}\right) \\ &= \frac{3}{4}d^{-3}\left(2d^3 - \frac{2}{3}d^3\right) = 1. \end{aligned}$$

If $(-d, d) \cap I = \emptyset$ then $\delta(x) = 0$ for all $x \in I$; and hence

$$\int_{x \in I} \delta(x) = \int_{x \in I} 0 = 0.$$

□

Proposition 3.3. *Let $I \subset \mathcal{R}$ be an interval containing $(-d, d)$. Then $\delta(x)$ has a measurable anti-derivative on I that is equal to the Heaviside function on $I \cap \mathbb{R}$.*

Proof. Let $H : I \to \mathcal{R}$ be given by

$$H(x) = \begin{cases} 0 & \text{if } x \leqslant -d \\ \frac{3}{4}d^{-3}(d^2 x - \frac{1}{3}x^3) + \frac{1}{2} & \text{if } -d < x < d \\ 1 & \text{if } x \geqslant d \end{cases}.$$

Then $H(x)$ is measurable and differentiable on I with $H'(x) = \delta(x)$ on I. Moreover,

$$H(x)|_{\mathbb{R}} = \begin{cases} 0 & \text{if } x < 0 \\ 1/2 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases},$$

which is the so-called Heaviside function. □

Proposition 3.4. *Let $\alpha \in \mathcal{R}\setminus\{0\}$ be given, and let $I \subset \mathcal{R}$ be any interval satisfying $\left(-\frac{d}{|\alpha|}, \frac{d}{|\alpha|}\right) \subset I$. Then*

$$\int_{x\in I} \delta(\alpha x) = \frac{1}{|\alpha|}.$$

Proof. Note that, by definition of the delta function, we have that

$$\delta(\alpha x) = \begin{cases} \frac{3}{4}d^{-3}\left(d^2 - (\alpha x)^2\right) & \text{if } |\alpha x| < d \\ 0 & \text{if } |\alpha x| \geqslant d \end{cases}$$
$$= \begin{cases} \frac{3}{4}d^{-3}\left(d^2 - (\alpha x)^2\right) & \text{if } |x| < \frac{d}{|\alpha|} \\ 0 & \text{if } |x| \geqslant \frac{d}{|\alpha|} \end{cases}.$$

It follows that

$$\int\limits_{x\in I} \delta(\alpha x) = \int\limits_{x\in\left(-\frac{d}{|\alpha|}, \frac{d}{|\alpha|}\right)} \frac{3}{4}d^{-3}(d^2 - (\alpha x)^2) = \left[\frac{3}{4}d^{-1}\left(x - d^{-2}\frac{\alpha^2 x^3}{3}\right)\right]\Bigg|_{-\frac{d}{|\alpha|}}^{\frac{d}{|\alpha|}} = \frac{1}{|\alpha|}.$$

□

Lemma 3.5. *Let $f : I(0,1) \to \mathcal{R}$ be analytic with $i(f) = 0$. Then for every $x \in I(0,1)$ and for every $n \in \mathbb{N}$, we have that $\lambda(f^{(n)}(x)) \geq 0$.*

Proof. Let $x \in I(0,1)$ and let $q < 0$ in $\mathbb{Q}$ be given. Since f is analytic on the finite interval $I(0,1)$, there exists $\delta > 0$ in $\mathbb{R}$ such that for all $y \in (x-\delta, x+\delta) \cap I(0,1)$, we have that

$$f(y) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(y-x)^n.$$

Since $i(f) = 0$, it follows that, for almost every $h \in (0,\delta) \cap \mathbb{R}$, we have $\lambda(f(x+h)) = 0$. In other words, for almost every $h \in (0,\delta) \cap \mathbb{R}$ we have that $f(x+h)[q] = 0$. But

$$f(x+h)[q] = \left(\sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}h^n\right)[q] = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)[q]}{n!}h^n[0] = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)[q]}{n!}h^n.$$

So for almost every $h \in (0,\delta) \cap \mathbb{R}$ we have

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x)[q]}{n!}h^n = 0.$$

Since the above is a real power series, this is possible only if $f^{(n)}(x)[q] = 0$ for all $n \in \mathbb{N}$. Therefore, for any $n \in \mathbb{N}$ and $x \in I(0,1)$, we have that $\lambda(f^{(n)}(x)) \geq 0$. □

Theorem 3.6. *Let $a < b$ in $\mathcal{R}$ be given and let $f : I(a,b) \to \mathcal{R}$ be analytic on $I(a,b)$ with $i(f) = 0$. Then for any $x \in I(a,b)$ and for any $n \in \mathbb{N}$, we have that*

$$\lambda(f^{(n)}(x)) \geq -n\lambda(b-a).$$

Proof. Define $F : I(0,1) \to \mathcal{R}$ by

$$F(x) = f((b-a)x + a).$$

Then F is analytic on $I(0,1)$ and $i(F) = i(f) = 0$; hence, by the previous lemma, for all $x \in I(0,1)$ and $n \in \mathbb{N}$, we have that

$$\lambda\left(F^{(n)}(x)\right) \geq 0.$$

Note that, for all $x \in I(0,1)$ and $n \in \mathbb{N}$, we have that

$$F^{(n)}(x) = (b-a)^n f^{(n)}((b-a)x + a).$$

It follows that

$$\begin{aligned} 0 \leq & \lambda(F^{(n)}(x)) = \lambda((b-a)^n f^{(n)}((b-a)x + a)) \\ & = n\lambda(b-a) + \lambda(f^{(n)}((b-a)x + a)); \end{aligned}$$

and hence $\lambda(f^{(n)}((b-a)x + a)) \geq -n\lambda(b-a)$. □

Proposition 3.7. *Let $a < b$ in $\mathcal{R}$ be such that $\lambda(b-a) < 1$ and let $f : I(a,b) \to \mathcal{R}$ be analytic on $I(a,b)$ with $i(f) = 0$. Then for any $x_0 \in [a+d, b-d]$, we have that*

$$\int_{x \in I(a,b)} f(x)\delta(x - x_0) =_0 f(x_0).$$

Proof. Fix $x_0 \in [a+d, b-d]$. Since f is a finite analytic function, there exists a $\eta > 0$ in $\mathcal{R}$ with $\lambda(\eta) = \lambda(b-a)$ such that, for any $x \in I(a,b)$ satisfying $|x - x_0| < \eta$, we have that $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$. Therefore,

$$\begin{aligned} \int_{x \in I(a,b)} f(x)\delta(x - x_0) &= \int_{x \in (x_0 - d, x_0 + d)} f(x)\delta(x - x_0) \\ &= \int_{x \in (x_0 - d, x_0 + d)} \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \delta(x - x_0) \\ &= \int_{x \in (x_0 - d, x_0 + d)} f(x_0)\delta(x - x_0) \\ &+ \int_{x \in (x_0 - d, x_0 + d)} \sum_{k=1}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \delta(x - x_0) \\ &= f(x_0) + \int_{x \in (x_0 - d, x_0 + d)} \sum_{k=1}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \delta(x - x_0). \end{aligned}$$

Now, for any $x \in (x_0 - d, x_0 + d)$, we have that $|x - x_0| < d$, and hence

$$\left| \int_{x \in (x_0 - d, x_0 + d)} \sum_{k=1}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \delta(x - x_0) \right| \leqslant \int_{x \in [x_0 - d, x_0 + d]} \sum_{k=1}^{\infty} \frac{|f^{(k)}(x_0)|}{k!} d^k \delta(x - x_0).$$

It follows that

$$\left| \int_{x \in (x_0 - d, x_0 + d)} \sum_{k=1}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \delta(x - x_0) \right| \leqslant \sum_{k=1}^{\infty} \frac{|f^{(k)}(x_0)|}{k!} d^k.$$

Thus

$$\lambda\left(\int_{x \in (x_0 - d, x_0 + d)} \sum_{k=1}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \delta(x - x_0) \right) \geqslant \lambda\left(\sum_{k=1}^{\infty} \frac{|f^{(k)}(x_0)|}{k!} d^k \right).$$

However, since $i(f)=0$ we can apply Theorem 3.6 to get that for all $k\in\mathbb{N}$, $\lambda(f^{(k)}(x_0))>-k$ and hence $\lambda\left(\sum_{k=1}^{\infty}\frac{|f^{(k)}(x_0)|}{k!}d^k\right)>0$. Thus,

$$\lambda\left(\int\limits_{x\in(x_0-d,x_0+d)}\sum_{k=1}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\delta(x-x_0)\right)>0.$$

Therefore,

$$\int\limits_{x\in(x_0-d,x_0+d)}\sum_{k=1}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\delta(x-x_0)=_0 0.$$

It follows that

$$\int\limits_{x\in I(a,b)}f(x)\delta(x-x_0)=_0 f(x_0).$$

□

Proposition 3.8. *Let $a<b<c$ in $\mathcal{R}$ be such $\lambda(b-a)<1$ and $\lambda(c-b)<1$; let $g:[a,b]\to\mathcal{R}$ and $h:[b,c]\to\mathcal{R}$ be analytic functions satisfying $g(b)=h(b)$ and $i(h)=i(g)=0$; and let $f:[a,c]\to\mathcal{R}$ be given by*

$$f(x)=\begin{cases}g(x) & \text{if } x\in[a,b)\\ h(x) & \text{if } x\in[b,c]\end{cases}.$$

Then for any $x_0\in[a+d,c-d]$, we have that

$$\int\limits_{x\in[a,c]}f(x)\delta(x-x_0)=_0 f(x_0).$$

Proof. Without loss of generality, we may assume that $b=0$. Fix $x_0\in[a+d,c-d]$. If $|x_0|\geqslant d$ then by Proposition 3.7 we are done; so without loss of generality we may assume that $|x_0|<d$. Thus, we have that

$$\int\limits_{x\in[a,c]}f(x)\delta(x-x_0)=\int\limits_{x\in[x_0-d,0]}g(x)\delta(x-x_0)+\int\limits_{x\in[0,x_0+d]}h(x)\delta(x-x_0).$$

Both g and h are analytic functions defined on $[a,0]$ and $[0,c]$, respectively; and hence they both can be expanded as power series centered at 0. Thus,

$$g(x)=\sum_{k=0}^{\infty}\alpha_k x^k$$

and

$$h(x)=\sum_{k=0}^{\infty}\beta_k x^k,$$

where

$$\alpha_k=\frac{g^{(k)}(0)}{k!}\text{ and }\beta_k=\frac{h^{(k)}(0)}{k!}\text{ for }k=0,1,2,\ldots.$$

Since $\lambda(b-a)=\lambda(-a)=\lambda(a)<1$ and $\lambda(c-b)=\lambda(c)<1$ both power series will have radii of convergence infinitely larger than d, and hence they will converge everywhere on $[x_0-d,0]$ and $[0,x_0+d]$, respectively. Thus,

$$\int_{x\in[x_0-d,0]} g(x)\delta(x-x_0)=\int_{x\in[x_0-d,0]}\sum_{k=0}^{\infty}\alpha_k x^k\delta(x-x_0)$$

and

$$\int_{x\in[0,x_0+d]} h(x)\delta(x-x_0)=\int_{x\in[0,x_0+d]}\sum_{k=0}^{\infty}\beta_k x^k\delta(x-x_0).$$

Therefore,

$$\begin{aligned}\int_{x\in[a,c]} f(x)\delta(x-x_0)&=\int_{x\in[x_0-d,0]}\sum_{k=0}^{\infty}\alpha_k x^k\delta(x-x_0)+\int_{x\in[0,x_0+d]}\sum_{k=0}^{\infty}\beta_k x^k\delta(x-x_0)\\&=\alpha_0\int_{x\in[x_0-d,0]}\delta(x-x_0)+\beta_0\int_{x\in[0,x_0+d]}\delta(x-x_0)\\&+\int_{x\in[x_0-d,0]}\sum_{k=1}^{\infty}\alpha_k x^k\delta(x-x_0)\\&+\int_{x\in[0,x_0+d]}\sum_{k=1}^{\infty}\beta_k x^k\delta(x-x_0).\end{aligned}$$

However, $\alpha_0=g(0)=f(0)=h(0)=\beta_0$, and hence

$$\alpha_0\int_{x\in[x_0-d,0]}\delta(x-x_0)+\beta_0\int_{x\in[0,x_0+d]}\delta(x-x_0)=f(0)\int_{x\in[x_0-d,x_0+d]}\delta(x-x_0)=f(0).$$

Thus,

$$\int_{x\in[a,c]} f(x)\delta(x-x_0)=f(0)+\int_{x\in[x_0-d,0]}\sum_{k=1}^{\infty}\alpha_k x^k\delta(x-x_0)+\int_{x\in[0,x_0+d]}\sum_{k=1}^{\infty}\beta_k x^k\delta(x-x_0).$$

But

$$\begin{aligned}&\lambda\left(\int_{x\in[x_0-d,0]}\sum_{k=1}^{\infty}\alpha_k x^k\delta(x-x_0)+\int_{x\in[0,x_0+d]}\sum_{k=1}^{\infty}\beta_k x^k\delta(x-x_0)\right)\\&\geqslant\lambda\left(\sum_{k=1}^{\infty}|\alpha_k|\,(2d)^k\int_{x\in[x_0-d,0]}\delta(x-x_0)+\sum_{k=1}^{\infty}|\beta_k|\,(2d)^k\int_{x\in[0,x_0+d]}\delta(x-x_0)\right)>0\end{aligned}$$

as in the proof of Proposition 3.7. Thus,

$$\int_{x\in[x_0-d,0]}\sum_{k=1}^{\infty}\alpha_k x^k\delta(x-x_0)+\int_{x\in[0,x_0+d]}\sum_{k=1}^{\infty}\beta_k x^k\delta(x-x_0)=_0 0,$$

and hence $\int_{x\in[a,c]} f(x)\delta(x-x_0) =_0 f(0)$. However, $f(0) =_0 f(x_0)$ [6]; therefore

$$\int_{x\in[a,c]} f(x)\delta(x-x_0) =_0 f(x_0).$$

□

Uniform differentiability on an interval of $\mathcal{R}$ is defined in the same way as in the real case.

Definition 3.9. *Let $a < b$ in $\mathcal{R}$ be given and let $f : I(a,b) \to \mathcal{R}$ be differentiable with derivative f' on $I(a,b)$. Then we say that f is uniformly differentiable on $I(a,b)$ if for every $\epsilon > 0$ in $\mathcal{R}$ there exists $\delta > 0$ in $\mathcal{R}$ such that for all $x, y \in I(a,b)$*

$$0 < |y-x| < \delta \Rightarrow \left|\frac{f(y)-f(x)}{y-x} - f'(x)\right| < \epsilon.$$

Lemma 3.10. *Let $f : I(0,1) \to \mathcal{R}$ be analytic with $i(f) = 0$. Then f is uniformly differentiable on $I(0,1)$.*

Proof. First we note that, by Theorem 3.6, we have that $\lambda(f^{(n)}(x)) \geq 0$ for all $n \in \mathbb{N}$ and for all $x \in I(0,1)$. Now let $\epsilon > 0$ in $\mathcal{R}$ be given and let

$$\delta = \min\left\{d^4\epsilon, d\right\}.$$

Then for $x, y \in I(0,1)$ satisfying $0 < |y-x| < \delta$, we have that

$$f(y) = f(x) + f'(x)(y-x) + \sum_{n=2}^{\infty} \frac{f^{(n)}(x)}{n!}(y-x)^n,$$

where the power series converges in the order topology since $\lambda(f^{(n)}(x)) \geq 0$ for all $n \in \mathbb{N}$ and since $0 < |y-x| < \delta \ll 1$ so that

$$\lim_{n\to\infty} \frac{f^{(n)}(x)}{n!}(y-x)^n = 0.$$

It follows that

$$\left|f(y) - f(x) - f'(x)(y-x)\right| = \left|\sum_{n=2}^{\infty} \frac{f^{(n)}(x)}{n!}(y-x)^n\right| < \sum_{n=2}^{\infty} \frac{d^{-1}}{n!} d^{n-2}(y-x)^2;$$

and hence

$$\begin{aligned}\left|\frac{f(y)-f(x)}{y-x} - f'(x)\right| &< \sum_{n=0}^{\infty} \frac{d^{n-3}}{n!}|y-x| < \left(\sum_{n=0}^{\infty} \frac{d^{n-3}}{n!}\right) d^4\epsilon \\ &= \left(\sum_{n=0}^{\infty} \frac{d^n}{n!}\right) d\epsilon = \frac{d}{1-d}\epsilon \\ &< \epsilon.\end{aligned}$$

□

Theorem 3.11. *Let $a < b$ in $\mathcal{R}$ be given and let $f : I(a,b) \to \mathcal{R}$ be analytic on $I(a,b)$. Then f is uniformly differentiable on $I(a,b)$.*

Proof. Let $F : I(0,1) \to \mathcal{R}$ be given by $F(x) = d^{-i(f)} f(a + (b-a)x)$. Then F is analytic on $I(0,1)$ with $i(F) = 0$; and hence, by Lemma 3.10 above, F is uniformly differentiable on $I(0,1)$. Now fix $\epsilon > 0$ in $\mathcal{R}$. Since F is uniformly differentiable on $I(0,1)$, there is a $\delta > 0$ in $\mathcal{R}$ such that if $x, y \in I(0,1)$ and $|y - x| < \frac{\delta}{b-a}$ then $|F(y) - F(x) - F'(x)(y-x)| < d^{-i(f)}\epsilon$. However,

$$\begin{aligned}&|F(y) - F(x) - F'(x)(y-x)|\\ &= |d^{-i(f)} f(a + (b-a)y) - d^{-i(f)} f(a - (b-a)x) - d^{-i(f)} f'(a + (b-a)x)(b-a)(y-x)|\\ &= d^{-i(f)} |f(a + (b-a)y) - f(a + (b-a)x) - f'(a + (b-a)x)(b-a)(y-x)|.\end{aligned}$$

Thus, if $x, y \in I(0,1)$ and $|y - x| < \frac{\delta}{b-a}$ then we have that

$$|f(a + (b-a)y) - f(a + (b-a)x) - f'(a + (b-a)x)(b-a)(y-x)| < \epsilon.$$

Now let $u, v \in I(a,b)$ be such that $|v - u| < \delta$; and let

$$x = \frac{u-a}{b-a} \text{ and } y = \frac{v-a}{b-a}.$$

Then $u = a + (b-a)x$, $v = a + (b-a)y$, $x, y \in I(0,1)$ and $|y - x| < \frac{\delta}{b-a}$. It follows that

$$\begin{aligned}&|f(v) - f(u) - f'(u)(v-u)|\\ &= |f(a + (b-a)y) - f(a + (b-a)x) - f'(a + (b-a)x)(b-a)(y-x)|\\ &< \epsilon.\end{aligned}$$

Thus, f is uniformly differentiable on $I(a,b)$. □

Notation 3.12. *In the following, and to avoid confusion with the number d, we will use D_x to denote the differential operator $\frac{d}{dx}$, moreover we will use D_x^n to denote $\frac{d^n}{dx^n}$.*

Proposition 3.13. *Let x_0, $a < b$, and $\epsilon > 0$ in $\mathcal{R}$ be given; and let $f : [x_0 - \epsilon, x_0 + \epsilon] \times [a,b] \to \mathcal{R}$ be a (double) power series. Then*

$$D_x \int\limits_{y \in [a,b]} f(x,y) = \int\limits_{y \in [a,b]} \frac{\partial}{\partial x} f(x,y).$$

Proof. Let $N \in \mathbb{N}$ be such that $d^N < \epsilon$. By Theorem 3.11 above we have that f is uniformly differentiable with respect to x and hence

$$\lim_{k\to\infty} \frac{f(x + d^{N+k}, y) - f(x,y)}{d^{N+k}} = \frac{\partial}{\partial x} f(x,y) \text{ (uniformly)}.$$

Moreover, by definition

$$D_x \int\limits_{y \in [a,b]} f(x,y) = \lim_{k\to\infty} \int\limits_{y \in [a,b]} \frac{f(x + d^{N+k}, y) - f(x,y)}{d^{N+k}}.$$

However, by Theorem 3.9 in [8] we have that

$$\begin{aligned}\lim_{k\to\infty} \int\limits_{y \in [a,b]} \frac{f(x + d^{N+k}, y) - f(x,y)}{d^{N+k}} &= \int\limits_{y \in [a,b]} \lim_{k\to\infty} \frac{f(x + d^{N+k}, y) - f(x,y)}{d^{N+k}}\\ &= \int\limits_{y \in [a,b]} \frac{\partial}{\partial x} f(x,y).\end{aligned}$$

This completes the proof of the proposition. □

Corollary 3.14. *Let x_0, $a < b$, and $\epsilon > 0$ in $\mathcal{R}$ be given; and let $f : [x_0 - \epsilon, x_0 + \epsilon] \times [a, b] \to \mathcal{R}$ be analytic on* : $[x_0 - \epsilon, x_0 + \epsilon] \times [a, b]$. *Then*

$$D_x \int\limits_{y \in [a,b]} f(x, y) = \int\limits_{y \in [a,b]} \frac{\partial}{\partial x} f(x, y).$$

Proof. This follows immediately from the fact that analytic functions are given locally by power series. □

Proposition 3.15 (Leibniz's Rule). *Fix $x_0 \in \mathcal{R}$ and let $\epsilon > 0$ in $\mathcal{R}$ be given. Let $\alpha, \beta : [x_0 - \epsilon, x_0 + \epsilon] \to \mathcal{R}$ be analytic functions with $\alpha(x) \le \beta(x)$ for all $x \in [x_0 - \epsilon, x_0 + \epsilon]$. Let S be the simple region given by*

$$S = \{(x, y) \in \mathcal{R}^2 : y \in [\alpha(x), \beta(x)], x \in [x_0 - \epsilon, x_0 + \epsilon]\}$$

and let $f : S \to \mathcal{R}$ be analytic. Then

$$D_x \int\limits_{y \in [\alpha(x), \beta(x)]} f(x, y) = f(x, \beta(x))\beta'(x) - f(x, \alpha(x))\alpha'(x) + \int\limits_{y \in [\alpha(x), \beta(x)]} \frac{\partial}{\partial x} f(x, y).$$

Proof. The proof is identical to that of the real case. □

Proposition 3.16. *Let x_0, $a < b$, and $\epsilon > 0$ in $\mathcal{R}$ be given and let $\mu : [x_0 - \epsilon, x_0 + \epsilon] \to [a, b]$ be a non-constant analytic function. Let $g : [x_0 - \epsilon, x_0 + \epsilon] \times [a, \mu(x)] \to \mathcal{R}$ and $h : [x_0 - \epsilon, x_0 + \epsilon] \times [\mu(x), b] \to \mathcal{R}$ be analytic and let $f : [x_0 - \epsilon, x_0 + \epsilon] \times [a, b]$ be given by*

$$f(x, y) = \begin{cases} g(x, y) & \text{if } y \le \mu(x) \\ h(x, y) & \text{if } y > \mu(x) \end{cases}.$$

Then

$$D_x \int\limits_{y \in [a,b]} f(x, y) = \int\limits_{y \in [a,b]} \frac{\partial}{\partial x} f(x, y)$$

if and only if $f(x, y)$ is continuous.

Proof. Observe that

$$D_x \int\limits_{y \in [a,b]} f(x, y) = D_x \int\limits_{y \in [a, \mu(x)]} g(x, y) + D_x \int\limits_{y \in [\mu(x), b]} h(x, y).$$

But by proposition (3.15) we have that

$$D_x \int\limits_{y \in [a, \mu(x)]} g(x, y) = g(x, \mu(x))\mu'(x) + \int\limits_{y \in [a, \mu(x)]} \frac{\partial}{\partial x} g(x, y)$$

and

$$D_x \int_{y \in [\mu(x), b]} h(x, y) = \int\limits_{y \in [\mu(x), b]} \frac{\partial}{\partial x} h(x, y) - h(x, \mu(x))\mu'(x).$$

So

$$D_x \int\limits_{y \in [a,b]} f(x, y) = g(x, \mu(x))\mu'(x) + \int\limits_{y \in [a, \mu(x)]} \frac{\partial}{\partial x} g(x, y)$$

$$+ \int\limits_{y\in[\mu(x),b]} \frac{\partial}{\partial x} h(x,y) - h(x,\mu(x))\mu'(x)$$

$$= \int_{y\in[a,b]} \frac{\partial}{\partial x} f(x,y) + [g(x,\mu(x)) - h(x,\mu(x))]\,\mu'(x).$$

Since μ is a non-constant analytic function we know that $\mu' \neq 0$; and it follows that the above expression equals $\int_{y\in[a,b]} \frac{\partial}{\partial x} f(x,y)$ if and only if $g(x,\mu(x)) = h(x,\mu(x))$ for all $x \in [x_0-\epsilon, x_0+\epsilon]$, that is if and only if f is continuous at $y=\mu(x)$ and hence everywhere (since g and h are analytic). □

4. EXAMPLES IN ONE DIMENSION

In this section, we present two simple examples in which we illustrate the applications of the delta function defined on $\mathcal{R}$ above.

Example 4.1. *[Solving Poisson's Equation in One Dimension] Suppose that we wish to find the solution to the real differential equation $\ddot{x}(t) = f(t)$ on the interval $[0,+\infty)$ and subject to the initial conditions $x(0)=0, \dot{x}(0)=0$. To begin, we observe that the piecewise analytic solution to $\frac{\partial^2}{\partial t^2} G(t,t') = \delta(t-t')$ is*

$$G(t,t') = \begin{cases} A_1(t-t') + B_1 & t' \le t-d \\ A_2(t-t') + B_2 + \frac{3}{8} d^{-3}(d^2(t-t')^2 - \frac{1}{6}(t-t')^4) & t-d < t' < t+d \\ A_3(t-t') + B_3 & t' \ge t+d \end{cases},$$

where A_1, A_2, A_3, B_1, B_2 and B_3 are constants to be determined.

To ensure that our solution satisfies the given initial conditions we must have that the real parts of $G(0,t')$ and $\frac{\partial G}{\partial t}(0,t')$ equal zero; and to accomplish that, it is enough to set $G(r,t')=0$ and $\frac{\partial G}{\partial t}(r,t')=0$ where $r \in \mathcal{R}$ is any number that is infinitely small in absolute value (we will use $r=-d$ since that turns out to be a convenient choice). In order to apply Proposition 3.16 we require that G be continuous (so that $D_t \int_{t'} G(t,t') = \int_{t'} \frac{\partial}{\partial t} G(t,t')$) and that $\frac{\partial G}{\partial t}(t,t')$ be continuous (so that $D_t \int_{t'} \frac{\partial}{\partial t} G(t,t') = \int_{t'} \frac{\partial^2}{\partial t^2} G(t,t')$). Using the initial conditions and continuity of $G(t,t')$ and its derivative at $t = t' \pm d$, we can solve for A_1, B_1, A_2, B_2, A_3, and B_3, to get

$$G(t,t') = \begin{cases} t-t'-d & t' \le t-d \\ \frac{1}{2}(t-t') - \frac{13}{16} d + \frac{3}{8} d^{-3}(d^2(t-t')^2 - \frac{1}{6}(t-t')^4) & t-d < t' < t+d \\ 0 & t' \ge t+d \end{cases}.$$

Note that when restricted to real points, the real part of $G(t,t')$ reduces to the classical Green's function for D_t^2. Applying Proposition 3.16, we obtain that

$$\begin{aligned} D_t^2 \int\limits_{t'\in[0,d^{-\frac{1}{2}}]} G(t,t')f(t') &= \int\limits_{t'\in[0,d^{-\frac{1}{2}}]} \frac{\partial^2}{\partial t^2} G(t,t')f(t') \\ &= \int\limits_{t'\in[0,d^{-\frac{1}{2}}]} \delta(t-t')f(t') \\ &=_0 f(t). \end{aligned}$$

It follows that $\left(\int\limits_{t'\in[0,d^{-\frac{1}{2}}]} G(t,t')f(t') \right)[0]$ is a (real) solution to the equation

$$\ddot{u}(t) = f(t)$$

with the initial conditions

$$\left(\int\limits_{t'\in[0,d^{-\frac{1}{2}}]} G(0,t')f(t')\right)[0] = \left(\int\limits_{t'\in[0,d^{-\frac{1}{2}}]} \frac{\partial G}{\partial t}(0,t')f(t')\right)[0] = 0;$$

and hence we must have that

$$x(t) = \left(\int\limits_{t'\in[0,d^{-\frac{1}{2}}]} G(t,t')f(t')\right)[0].$$

Now, if we set $f(t) = t$ then we see that

$$\int\limits_{t'\in[0,d^{-\frac{1}{2}}]} G(t,t')f(t') = \int\limits_{t'\in[0,t+d]} G(t,t')f(t')$$

$$= \int\limits_{t'\in[0,t-d]} (t-t'-d)t' + \int\limits_{t'\in[t-d,t+d]} \left(\frac{1}{2}(t-t') + \frac{3}{16}d + \frac{3}{8}d^{-3}\left(d^2(t-t')^2 - \frac{1}{6}(t-t')^4\right)\right)t'.$$

But

$$\int\limits_{t'\in[0,t-d]} (t-t-d')t = \frac{1}{6}(t-d)^3$$

and

$$\int\limits_{t'\in[t-d,t+d]} \left(\frac{1}{2}(t-t') + \frac{3}{16}d + \frac{3}{8}d^{-3}\left(d^2(t-t')^2 - \frac{1}{6}(t-t')^4\right)\right)t' = \frac{7}{5}td^2 + \frac{1}{3}d^3.$$

Thus, $\int\limits_{t'\in[0,d^{-\frac{1}{2}}]} G(t,t')f(t') = \frac{1}{6}(t-d)^3 + \frac{7}{5}td^2 + \frac{1}{3}d^3 =_0 \frac{1}{6}t^3$ *and hence the real solution is* $x(t) = \frac{1}{6}t^3$.

Example 4.2 (Damped Driven Harmonic Oscillator). *Consider now an underdamped, driven harmonic oscillator with mass m, viscous damping constant c, spring constant k, and driving force $f(t)$. Let $x(t)$ be the position of the oscillator at time t with $x(0) = 0$ and $\dot{x}(0) = 0$. The oscillator's equation of motion is*

$$\ddot{x}(t) + \frac{c}{m}\dot{x}(t) + \frac{k}{m}x(t) = \frac{f(t)}{m}. \tag{4.1}$$

With the following change of variables

$$\gamma = \frac{c}{2\sqrt{mk}} \text{ and } \omega_0 = \sqrt{\frac{k}{m}},$$

equation (4.1) takes the form

$$\ddot{x}(t) + 2\gamma\omega_0\dot{x}(t) + \omega_0^2 x(t) = \frac{f(t)}{m}.$$

Since the oscillator is underdamped we have that $\gamma^2\omega_0^2 - \omega_0^2 < 0$ which is equivalent to $\gamma < 1$. To solve the equation of motion we first find the Green's function for the differential operator $(D_t^2 + 2\gamma\omega_0 D_t + \omega_0^2)$; that is, we find a solution for the differential equation

$$\left(\frac{\partial^2}{\partial t^2} + 2\gamma\omega_0\frac{\partial}{\partial t} + \omega_0^2\right)G(t,t') = \delta(t-t').$$

First we observe that the analytic solution to the homogeneous partial differential equation

$$\left(\frac{\partial^2}{\partial t^2}+2\gamma\omega_0\frac{\partial}{\partial t}+\omega_0^2\right)G_{hom}(t,t')=0$$

is

$$G_{hom}(t,t')=e^{-\gamma\omega_0(t-t')}\left(A\sin(\omega(t-t'))+B\cos(\omega(t-t'))\right),$$

where $\omega=\sqrt{1-\gamma^2}\omega_0$ *and where* A *and* B *are arbitrary constants.*

One particular solution to the inhomogeneous partial differential equation

$$\left(\frac{\partial^2}{\partial t^2}+2\gamma\omega_0\frac{\partial}{\partial t}+\omega_0^2\right)G_{inhom}(t,t')=\frac{3}{4}d^{-3}(d^2-t^2)$$

is given by

$$G_{inhom}(t,t')=\frac{3}{\omega_0^2}d^{-3}\left(\frac{d^2-(t-t')^2}{4}+\frac{\gamma(t-t')}{\omega_0}+\frac{1-4\gamma^2}{2\omega_0^2}\right).$$

Since

$$\delta(t)=\begin{cases}0 & \text{if } t\leqslant -d\\ \frac{3}{4}d^{-3}(d^2-t^2) & \text{if } -d<t<d\ ,\\ 0 & \text{if } d\leqslant t\end{cases}$$

we must have

$$G(t,t')=\begin{cases}e^{-\gamma\omega_0(t-t')}(A_1\sin(\omega(t-t'))+B_1\cos(\omega(t-t'))) & \text{if } t'\leqslant t-d\\ e^{-\gamma\omega_0(t-t')}(A_2\sin(\omega(t-t'))+B_2\cos(\omega(t-t'))) \\ \quad +\frac{3}{\omega_0^2}(\frac{d^2-(t-t')^2}{4}+\frac{\gamma(t-t')}{\omega_0}+\frac{1-4\gamma^2}{2\omega_0^2}) & \text{if } t-d<t'<t+d\\ e^{-\gamma\omega_0(t-t')}(A_3\sin(\omega(t-t'))+B_3\cos(\omega(t-t'))) & \text{if } t'\geqslant t+d\end{cases}$$

where A_1,A_2,A_3,B_1,B_2,B_3 *are constants to be determined by the initial conditions.*

As in the previous example the real part of our Green's function must satisfy the same initial conditions as the desired solution. To this end we require

$$G(-d,t')=0 \text{ and } \frac{\partial G}{\partial t}(-d,t')=0.$$

Solving for the relevant constants yields

$$A_3=0 \text{ and } B_3=0.$$

Again, as in the previous example, requiring continuity of G *and* $\partial G/\partial t$ *gives us the remaining 4 constants:*

$$\begin{aligned}A_1&=\frac{3}{\omega_0^2}d^{-3}e^{-\gamma\omega_0 d}\left(\left(\frac{2\gamma^3}{\omega_0}-\frac{3\gamma}{2\omega_0}+\left(\gamma^2-\frac{1}{2}\right)d\right)\frac{\cos(\omega d)}{\omega}-\left(\frac{\gamma}{\omega_0}d-\frac{1-4\gamma^2}{2\omega_0^2}\right)\sin(\omega d)\right)\\&\quad+\frac{3}{\omega_0^2}d^{-3}e^{-\gamma\omega_0 d}\left(\left(\frac{3\gamma}{2\omega_0}-\frac{2\gamma^3}{\omega_0}+\left(\gamma^2-\frac{1}{2}\right)d\right)\frac{\cos(\omega d)}{\omega}+\left(\frac{\gamma}{\omega_0}d-\frac{1-4\gamma^2}{2\omega_0^2}\right)\sin(\omega d)\right)\\B_1&=\frac{3}{\omega_0^2}d^{-3}e^{-\gamma\omega_0 d}\left(\left(\frac{2\gamma^3}{\omega_0}-\frac{3\gamma}{2\omega_0}+\left(\gamma^2-\frac{1}{2}\right)d\right)\frac{\sin(\omega d)}{\omega}+\left(\frac{\gamma}{\omega_0}d-\frac{1-4\gamma^2}{2\omega_0^2}\right)\cos(\omega d)\right)\\&\quad-\frac{3}{\omega_0^2}d^{-3}e^{-\gamma\omega_0 d}\left(\left(\frac{3\gamma}{2\omega_0}-\frac{2\gamma^3}{\omega_0}+\left(\gamma^2-\frac{1}{2}\right)d\right)\frac{\sin(\omega d)}{\omega}-\left(\frac{\gamma}{\omega_0}d-\frac{1-4\gamma^2}{2\omega_0^2}\right)\cos(\omega d)\right)\\A_2&=\frac{3}{\omega_0^2}d^{-3}e^{-\gamma\omega_0 d}\left(\left(\frac{2\gamma^3}{\omega_0}-\frac{3\gamma}{2\omega_0}+\left(\gamma^2-\frac{1}{2}\right)d\right)\frac{\cos(\omega d)}{\omega}-\left(\frac{\gamma}{\omega_0}d-\frac{1-4\gamma^2}{2\omega_0^2}\right)\sin(\omega d)\right)\end{aligned}$$

$$B_2 = \frac{3}{\omega_0^2} d^{-3} e^{-\gamma\omega_0 d} \left(\left(\frac{2\gamma^3}{\omega_0} - \frac{3\gamma}{2\omega_0} + \left(\gamma^2 - \frac{1}{2}\right) d \right) \frac{\sin(\omega d)}{\omega} + \left(\frac{\gamma}{\omega_0} d - \frac{1-4\gamma^2}{2\omega_0^2} \right) \cos(\omega d) \right).$$

While at first glance these constants seem too cumbersome, we have that

$$A_1 =_0 \frac{1}{\omega} \text{ and } B_1 =_0 0;$$

and hence

$$G(t,t')|_{\mathbb{R}} =_0 \begin{cases} \frac{1}{\omega} e^{-\gamma\omega_0(t-t')} \sin(\omega(t-t')) & \text{if } t > t' \\ 0 & \text{if } t \leqslant t' \end{cases}$$

which is the classical Green's function for this problem.

Now, suppose that the driving force is given by

$$f(t) = m e^{-\gamma\omega_0 t}.$$

Then the equation of motion becomes

$$\ddot{x}(t) + 2\gamma\omega_0 \dot{x}(t) + \omega_0^2 x(t) = e^{-\gamma\omega_0 t}.$$

Thus, as in the previous example, we can obtain the real solution as the real part of

$$\int\limits_{t' \in [0, d^{-\frac{1}{2}}]} G(t,t') \frac{f(t')}{m}.$$

Therefore,

$$x(t) =_0 \int\limits_{t' \in [0, d^{-\frac{1}{2}}]} G(t,t') \frac{f(t')}{m}.$$

But $G(t,t') = 0$ *for* $t' > t + d$*, and hence*

$$\int\limits_{t' \in [0, d^{-\frac{1}{2}}]} G(t,t') \frac{f(t')}{m} = \int\limits_{t' \in [0, t+d]} G(t,t') e^{-\gamma\omega_0 t'}.$$

Thus,

$$\begin{aligned} x(t) &=_0 \int\limits_{t' \in [0, t+d]} G(t,t') e^{-\gamma\omega_0 t'} \\ &= e^{-\gamma\omega_0 t} \int\limits_{t' \in [t-d, t+d]} (A_2 \sin(\omega(t-t')) + B_2 \cos(\omega(t-t'))) \\ &+ e^{-\gamma\omega_0 t} \int\limits_{t' \in [t-d, t+d]} \left(\frac{3}{\omega_0^2} (\frac{d^2 - (t-t')^2}{4} + \frac{\gamma(t-t')}{\omega_0} + \frac{1-4\gamma^2}{2\omega_0^2} \right) \\ &+ e^{-\gamma\omega_0 t} \int\limits_{t' \in [0, t-d]} (A_1 \sin(\omega(t-t')) + B_1 \cos(\omega(t-t'))) \\ &= e^{-\gamma\omega_0 t} \left[\frac{2A_2 \sin(\omega d)}{\omega} + A_1 \frac{\sin(\omega t) - \sin(\omega d)}{\omega} + B_1 \frac{\cos(\omega t) - \cos(\omega d)}{\omega} \right. \\ &\left. - \frac{3}{\omega_0^2} d^{-3} \left(\frac{2}{4\gamma^3\omega_0^3} + \frac{1+4\gamma^2}{2\gamma\omega_0^3} \right) (e^{\gamma\omega_0 d} - e^{-\gamma\omega_0 d}) + \frac{3}{\omega_0^2} \left(\frac{d}{2\gamma^2\omega_0^2} + \frac{d}{\omega_0^2} \right) (e^{\gamma\omega_0 d} + e^{-\gamma\omega_0 d}) \right] \\ &=_0 e^{-\gamma\omega_0 t} \frac{\cos(\omega t) - 1}{\omega^2}, \end{aligned}$$

which agrees with the classical solution.

5. THE DELTA FUNCTION ON $\mathcal{R}^2$ AND $\mathcal{R}^3$

Using the integration theory developed in [14] it is possible to define integrable delta functions on $\mathcal{R}^2$ and $\mathcal{R}^3$.

Definition 5.1. *Let $\delta_2 : \mathcal{R}^2 \to \mathcal{R}$ be given by*

$$\delta_2(x, y) = \delta(x)\delta(y),$$

where δ is the delta function defined above on $\mathcal{R}$.

Proposition 5.2. *Let $S \subset \mathcal{R}^2$ be measurable. If $(-d, d) \times (-d, d) \subset S$ then*

$$\iint_S \delta_2(x, y) = 1.$$

If $(-d, d) \times (-d, d) \cap S = \emptyset$ then

$$\iint_S \delta_2(x, y) = 0.$$

Proof. If $(-d, d) \times (-d, d) \subset S$ then

$$\begin{aligned}\iint\limits_S \delta_2(x, y) &= \iint\limits_{(x,y)\in(-d,d)\times(-d,d)} \delta(x)\delta(y) \\ &= \int\limits_{x\in(-d,d)} \left(\delta(x) \int\limits_{y\in(-d,d)} \delta(y) \right) \\ &= \int\limits_{x\in(-d,d)} \delta(x) = 1.\end{aligned}$$

If $(-d, d) \times (-d, d) \cap S = \emptyset$, then $\delta_2(x, y) = 0$ everywhere on S; and hence

$$\iint\limits_S \delta_2(x, y) = \iint\limits_S 0 = 0.$$

□

Proposition 5.3. *Let $S \subset \mathcal{R}^2$ be a simple region with $\lambda_x(S) < 1$ and $\lambda_y(S) < 1$, let $f : S \to \mathcal{R}$ be an analytic function with index $i(f) = 0$ on S. Then, for any $(x_0, y_0) \in S$ that satisfies $(x_0 - a, x_0 + a) \times (y_0 - a, y_0 + a) \subset S$ for some positive $a \gg d$ in $\mathcal{R}$, we have that*

$$\iint\limits_{(x,y)\in S} f(x, y)\delta_2(x - x_0, y - y_0) =_0 f(x_0, y_0).$$

Proof. First we note that $\delta_2(x - x_0, y - y_0) = 0$ everywhere except on the simple region $(x_0 - d, x_0 + d) \times (y_0 - d, y_0 + d)$. Thus,

$$\iint\limits_{(x,y)\in S} f(x, y)\delta_2(x - x_0, y - y_0) = \int\limits_{x\in(x_0-d,x_0+d)} \int\limits_{y\in(y_0-d,y_0+d)} f(x, y)\delta_2(x - x_0, y - y_0).$$

Now, for a fixed $x \in (x_0 - d, x_0 + d)$, $h(y) := f(x, y)$ is an analytic function on $(y_0 - a, y_0 + a)$ which contains $(y_0 - d, y_0 + d)$; and hence, by Proposition 3.7, we have that

$$\int\limits_{y\in(y_0-d,y_0+d)} h(y)\delta(y - y_0) =_0 h(y_0) = f(x, y_0).$$

Furthermore, $g(x) := f(x, y_0)$ is analytic on $(x_0 - a, x_0 + a)$ which contains $(x_0 - d, x_0 + d)$; and hence

$$\int\limits_{x \in (x_0 - d, x_0 + d)} g(x)\delta(x - x_0) =_0 g(x_0) = f(x_0, y_0).$$

Thus,

$$\begin{aligned}\iint\limits_{(x,y)\in S} f(x,y)\delta_2(x - x_0, y - y_0) &= \int\limits_{x \in (x_0 - d, x_0 + d)} \left(\delta(x - x_0) \int\limits_{y \in (y_0 - d, y_0 + d)} \delta(y - y_0) f(x, y)\right) \\ &=_0 \int\limits_{x \in (x_0 - d, x_0 + d)} \delta(x - x_0) f(x, y_0) \\ &=_0 f(x_0, y_0).\end{aligned}$$

□

Definition 5.4. *Let $\delta_3 : \mathcal{R}^3 \to \mathcal{R}$ be given by*

$$\delta_3(x, y, z) = \delta(x)\delta(y)\delta(z).$$

It follows immediately from Definitions 5.1 and 5.4 that

$$\delta_3(x, y, z) = \delta_2(x, y)\delta(z) = \delta_2(x, z)\delta(y) = \delta(x)\delta_2(y, z).$$

Proposition 5.5. *Let $S \subset \mathcal{R}^3$ be measurable. If $(-d, d) \times (-d, d) \times (-d, d) \subset S$ then*

$$\iiint_S \delta_3(x, y, z) = 1.$$

If $(-d, d) \times (-d, d) \times (-d, d) \cap S = \emptyset$ then

$$\iiint_S \delta_3(x, y, z) = 0.$$

Proof. If $(-d, d) \times (-d, d) \times (-d, d) \subset S$ then

$$\begin{aligned}\iiint\limits_S \delta_3(x, y, z) &= \iiint\limits_{(x,y,z)\in(-d,d)\times(-d,d)\times(-d,d)} \delta_3(x, y, z) \\ &= \iint\limits_{(x,y)\in(-d,d)\times(-d,d)} \left(\delta_2(x, y) \int\limits_{z\in(-d,d)} \delta(z)\right) \\ &= \iint\limits_{(x,y)\in(-d,d)\times(-d,d)} \delta_2(x, y) = 1.\end{aligned}$$

If $(-d, d) \times (-d, d) \times (-d, d) \cap S = \emptyset$, then $\delta_3(x, y, z) = 0$ everywhere on S; and hence

$$\iiint\limits_S \delta_3(x, y, z) = \iiint\limits_S 0 = 0.$$

□

Proposition 5.6. *Let $S \subset \mathcal{R}^3$ be a simple region with $\lambda_x(S) < 1$, $\lambda_y(S) < 1$, $\lambda_z(S) < 1$, and let $f : S \to \mathcal{R}$ be an analytic function on S with $i(f) = 0$ on S. Then, for any $(x_0, y_0, z_0) \in S$ that satisfies*

$$(x_0 - a, x_0 + a) \times (y_0 - a, y_0 + a) \times (z_0 - a, z_0 + a) \subset S$$

for some positive $a \gg d$ in $\mathcal{R}$, we have that

$$\iiint_{(x,y,z)\in S} f(x, y, z)\delta_3(x - x_0, y - y_0, z - z_0) =_0 f(x_0, y_0, z_0).$$

Proof. First we note that $\delta_3(x - x_0, y - y_0, z - z_0) = 0$ everywhere except on the simple region $(x_0 - d, x_0 + d) \times (y_0 - d, y_0 + d) \times (z_0 - d, z_0 + d)$. Thus,

$$\begin{aligned}
&\iiint_{(x,y,z)\in S} f(x, y, z)\delta_3(x - x_0, y - y_0, z - z_0) \\
&= \iiint_{(x,y,z)\in(x_0-d,x_0+d)\times(y_0-d,y_0+d)\times(z_0-d,z_0+d)} f(x, y, z)\delta_3(x - x_0, y - y_0, z - z_0) \\
&= \iint_{(x,y)\in(x_0-d,x_0+d)\times(y_0-d,y_0+d)} \left(\delta_2(x - x_0, y - y_0) \int_{z\in(z_0-d,z_0+d)} f(x, y, z)\delta(z - z_0)\right).
\end{aligned}$$

Now, for a fixed $(x, y) \in (x_0 - d, x_0 + d) \times (y_0 - d, y_0 + d)$, $h(z) := f(x, y, z)$ is an analytic function on the interval $(z_0 - a, z_0 + a)$ which contains $(z_0 - d, z_0 + d)$; and hence, by Proposition 3.7, we have that

$$\int_{z\in(z_0-d,z_0+d)} h(z)\delta(z - z_0) =_0 h(z_0) = f(x, y, z_0).$$

Furthermore, $g(x, y) := f(x, y, z_0)$ is analytic on the simple region $S_{xy} := (x_0 - a, x_0 + a) \times (y_0 - a, y_0 + a)$ containing $(x_0 - d, x_0 + d) \times (y_0 - d, y_0 + d)$; and hence, by Proposition 5.3, we have that

$$\iint_{(x,y)\in(x_0-d,x_0+d)\times(y_0-d,y_0+d)} g(x, y)\delta_2(x - x_0, y - y_0) =_0 g(x_0, y_0) = f(x_0, y_0, z_0).$$

Thus,

$$\begin{aligned}
&\iiint_{(x,y,z)\in S} f(x, y, z)\delta_3(x - x_0, y - y_0, z - z_0) \\
&= \iint_{(x,y)\in(x_0-d,x_0+d)\times(y_0-d,y_0+d)} \left(\delta_2(x - x_0, y - y_0) \int_{z\in(z_0-d,z_0+d)} f(x, y, z)\delta(z - z_0)\right) \\
&=_0 \iint_{(x,y)\in(x_0-d,x_0+d)\times(y_0-d,y_0+d)} \delta_2(x - x_0, y - y_0)f(x, y, z_0) \\
&=_0 f(x_0, y_0, z_0).
\end{aligned}$$

□

As in the classical case it is also possible to define the delta function in spherical coordinates, in particular we have that

$$\delta_{sph}(\mathbf{r} - \mathbf{r}') = F(r, \phi, \theta)\delta_3(r - r', \phi - \phi', \theta - \theta'),$$

where of course $\mathbf{r}$ is the point (r, ϕ, θ), $\mathbf{r'}$ is the point (r', ϕ', θ'), and F is some as yet unknown function. Naturally we must have that

$$\iiint\limits_{\mathbf{r'}\in\mathcal{R}^3} \delta_{sph}(\mathbf{r}-\mathbf{r'}) = 1,$$

from which it follows that

$$\int\limits_{r'\in\mathcal{R}^+}\int\limits_{\phi'\in[0,2\pi]}\int\limits_{\theta'\in[0,\pi]} F(r',\phi',\theta')\delta_3(r-r',\phi-\phi',\theta-\theta')r'^2\sin\theta' = 1.$$

However, we already know that

$$\int\limits_{r'\in\mathcal{R}^+}\int\limits_{\phi'\in[0,2\pi]}\int\limits_{\theta'\in[0,\pi]} \delta_3(r-r',\phi-\phi',\theta-\theta') = 1$$

since $\delta_3(r-r',\phi-\phi',\theta-\theta')$ is normalized by definition, so we obtain

$$\int\limits_{r'\in\mathcal{R}^+}\int\limits_{\phi'\in[0,2\pi]}\int\limits_{\theta'\in[0,\pi]} \delta_3(r-r',\phi-\phi',\theta-\theta')$$
$$= \int\limits_{r'\in\mathcal{R}^+}\int\limits_{\phi'\in[0,2\pi]}\int\limits_{\theta'\in[0,\pi]} F(r',\phi',\theta')\delta_3(r-r',\phi-\phi',\theta-\theta')r'^2\sin\theta'$$

and hence

$$F(r',\phi',\theta') = \frac{1}{r'^2\sin\theta'}.$$

Definition 5.7. *We define $\delta_{sph}:\mathcal{R}^3\to\mathcal{R}$ by*

$$\delta_{sph}(\mathbf{r}-\mathbf{r'}) = \frac{\delta(r-r',\phi-\phi',\theta-\theta')}{r^2\sin\theta}.$$

Note that as in the classical case, if a problem has spherical symmetry then the delta function takes the form

$$\delta_{sph}(\mathbf{r}-\mathbf{r'}) = F(r')\delta(r-r')$$

and since

$$\int\limits_{\phi'\in[0,2\pi]}\int\limits_{\theta'\in[0,\pi]} r'^2\sin(\theta') = 4\pi r'^2$$

it follows that

$$F(r') = \frac{1}{4\pi r'^2}$$

and hence

$$\delta_{sph}(\mathbf{r}-\mathbf{r'}) = \frac{\delta(r-r')}{4\pi r'^2}.$$

Example 5.8 (Electric Field of a thick spherical shell)**.** *Suppose we wish to find the electric field of a thick spherical shell centred at the origin with inner radius R_1, outer radius R_2 and a uniform charge density. One way to accomplish this is to solve the differential equation implied by Gauss's law:*

$$\nabla\cdot\mathbf{E}(\mathbf{r}) = \rho(\mathbf{r})$$

where $\mathbf{E}(\mathbf{r})$ *is the electric field at the point* $\mathbf{r}$ *and* $\rho(\mathbf{r})$*, the charge density multiplied by a constant that depends on the system of units used, is given by*

$$\rho(\mathbf{r}) = \begin{cases} 0 & \text{if } r < R_1 \\ \rho_0 & \text{if } R_1 \leq r \leq R_2 \\ 0 & \text{if } r > R_2 \end{cases}.$$

As in previous examples we can solve this differential equation by finding the Green's function $\mathbf{G}(\mathbf{r}, \mathbf{r}') = G_r \mathbf{e_r} + G_\phi \mathbf{e}_\phi + G_\theta \mathbf{e}_\theta$ *corresponding to the operator* $\nabla\cdot$*, in particular* $\mathbf{G}(\mathbf{r}, \mathbf{r}')$ *must satisfy*

$$\nabla \cdot \mathbf{G}(\mathbf{r}, \mathbf{r}') = \delta_{sph}(\mathbf{r} - \mathbf{r}'). \tag{5.1}$$

However we have spherical symmetry about the origin; so we may infer that

$$G_\phi = G_\theta = 0$$

and

$$\delta_{sph}(\mathbf{r} - \mathbf{r}') = \frac{\delta(r - r')}{4\pi r^2}.$$

Thus, equation (5.1) reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 G_r(r, r')\right) = \frac{\delta(r - r')}{4\pi r^2}.$$

Solving this differential equation yields

$$G_r(r, r') = \begin{cases} \frac{c_1}{r^2} & \text{if } r' \leqslant r - d \\ \frac{1}{4\pi r^2}\frac{3}{4}d^{-3}\left(d^2(r - r') - \frac{1}{3}(r - r')^3\right) + \frac{c_2}{r^2} & \text{if } r - d < r' < r + d \\ \frac{c_3}{r^2} & \text{if } r' \geqslant r + d \end{cases}$$

where c_1*,* c_2*, and* c_3 *are constants of integration. We know that* $\mathbf{E}(\mathbf{r}) = 0$ *for* $r < R_1$ *since there is no charge inside the shell. To ensure our solution satisfies this initial condition we must have* $G_r(0, r') =_0 0$ *and as in previous examples we accomplish this by setting* $G_r(-d, r') = 0$*. Using this initial condition as well as the continuity of* G_r*, we are able to solve for the constants in* $G_r(r, r')$*, in fact we find that*

$$G_r(r, r') = \begin{cases} \frac{1}{4\pi r^2} & \text{if } r' \leqslant r - d \\ \frac{1}{4\pi r^2}\frac{3}{4}d^{-3}\left(d^2(r - r') - \frac{1}{3}(r - r')^3\right) + \frac{1}{8\pi r^2} & \text{if } r - d < r' < r + d \\ 0 & \text{if } r' \geqslant r + d \end{cases}.$$

Now that we know the Green's function of the operator $\nabla\cdot$ *and have made it satisfy the relevant boundary conditions we can solve for the (real) electric field of the spherical shell by recalling that*

$$\mathbf{E}(r) = E_r(r)\mathbf{e_r} =_0 \int\limits_{r' \in \mathcal{R}^+} \int\limits_{\phi' \in [0, 2\pi)} \int\limits_{\theta' \in [0, \pi]} G_r(r, r')\rho(r')r'^2 \sin\theta' \mathbf{e_r} = 4\pi \int\limits_{r' \in \mathcal{R}^+} G_r(r, r')\rho(r')r'^2 \mathbf{e_r}.$$

If $r < R_1$ *then we have that*

$$E_r(r) =_0 0.$$

If $R_1 \leq r \leq R_2$ *then we have that*

$$E_r(r) =_0 4\pi \left(\int\limits_{r' \in [R_1, r-d]} \frac{\rho_0 r'^2}{4\pi r^2} + \int\limits_{r' \in [r-d, r+d]} \left(\frac{1}{4\pi r^2}\frac{3}{4}d^{-3}\left(d^2(r - r') - \frac{1}{3}(r - r')^3\right) + \frac{1}{8\pi r^2} \right) \rho_0 r'^2 \right)$$

$$=_0 \frac{\rho_0}{3r^2}\left(r^3 - R_1^3\right).$$

Finally if $r > R_2$ then we get

$$E_r(r) =_0 4\pi \int\limits_{r' \in [R_1, R_2]} \frac{\rho_0 r'^2}{4\pi r^2} = \frac{\rho_0}{3r^2}\left(R_2^3 - R_1^3\right).$$

Hence the electric field of a uniformly charged thick spherical shell is given by

$$E_r(r) = \begin{cases} 0 & \text{if } r < R_1 \\ \frac{\rho_0}{3r^2}\left(r^3 - R_1^3\right) & \text{if } R_1 \leq r \leq R_2 \\ \frac{\rho_0}{3r^2}\left(R_2^3 - R_1^3\right) & \text{if } r > R_2 \end{cases}.$$

While the examples given in this section are admittedly simple they serve to illustrate how to use the newly defined delta functions. In the future we plan to engage in a more detailed study of the delta functions in two and three dimensions as well as in constructing more complex and challenging examples.

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