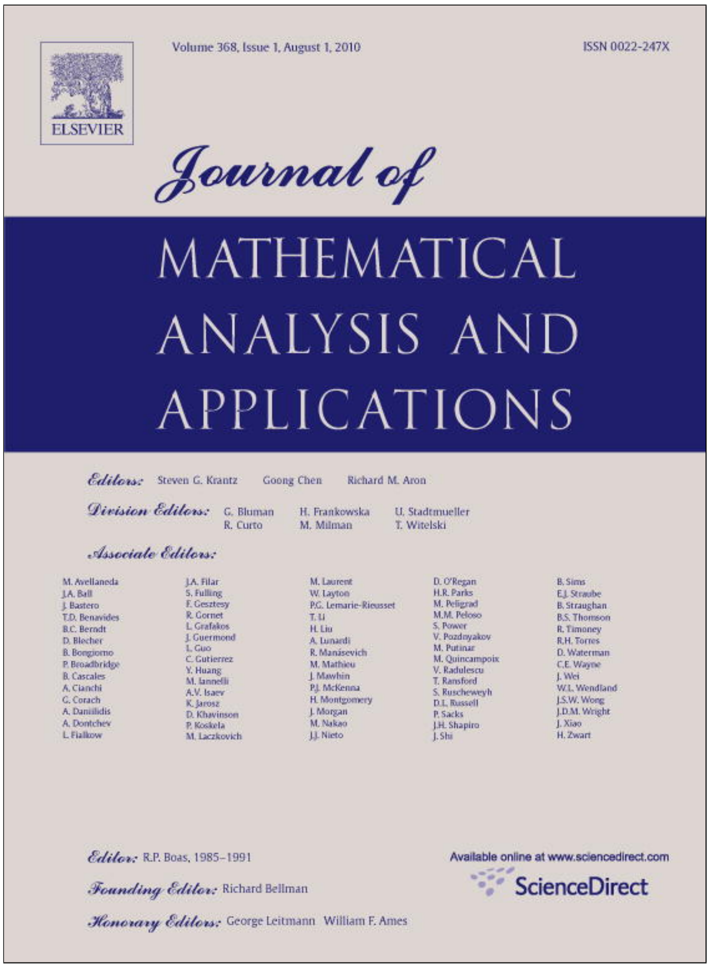
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On the topological structure of the Levi-Civita field

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ABSTRACT

Two topologies on the Levi-Civita field $\mathcal{R}$ will be studied: the valuation topology induced by the order on the field, and another weaker topology induced by a family of seminorms, which we will call weak topology. We show that each of the two topologies results from a metric on $\mathcal{R}$, that the valuation topology is not a vector topology while the weak topology is, and that $\mathcal{R}$ is complete in the valuation topology while it is not in the weak topology. Then the properties of both topologies will be studied in details; in particular, we give simple characterizations of open, closed, and compact sets in both topologies.

1. Introduction

The topological structure of the Levi-Civita field $\mathcal{R}$ [10,11] will be presented. We recall that the elements of $\mathcal{R}$ are functions from $\mathbb{Q}$ to $\mathbb{R}$ with left-finite support (denoted by supp). That is, below every rational number q, there are only finitely many points where the given function does not vanish. For the further discussion, it is convenient to introduce the following terminology.

Definition 1.1 *(λ, $\sim$, $\approx$, $=_r$).* We define $\lambda(x) = \min(\text{supp}(x))$ for $x \neq 0$ in $\mathcal{R}$ (which exists because of left-finiteness) and $\lambda(0) = +\infty$.

Given $x, y \in \mathcal{R} \setminus \{0\}$ and $r \in \mathbb{R}$, we say $x \sim y$ if $\lambda(x) = \lambda(y)$; $x \approx y$ if $\lambda(x) = \lambda(y)$ and $x[\lambda(x)] = y[\lambda(y)]$; and $x =_r y$ if $x[q] = y[q]$ for all $q \leqslant r$.

At this point, these definitions may feel somewhat arbitrary; but after having introduced an order on $\mathcal{R}$, we will see that λ describes orders of magnitude, the relation $\approx$ corresponds to agreement up to infinitely small relative error, while $\sim$ corresponds to agreement of order of magnitude.

The set $\mathcal{R}$ is endowed with formal power series multiplication (the exponents in the series forming left-finite sets of rational numbers) and with componentwise addition, which make it into a field in which we can isomorphically embed $\mathbb{R}$ as a subfield via the map $\Pi : \mathbb{R} \to \mathcal{R}$ defined by

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$$\Pi(x)[q] = \begin{cases} x & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases} \tag{1.1}$$

That $(\mathcal{R}, +, .)$ is a field is a classical result since it can be thought of as a generalized power series field where the coefficients are in $\mathbb{R}$ and the exponents form a left-finite (and hence well-ordered) subset of $\mathbb{Q}$.

Definition 1.2 *(Order in $\mathcal{R}$).* Let $x \neq y$ in $\mathcal{R}$ be given. Then we say $x > y$ if $(x - y)[\lambda(x - y)] > 0$; furthermore, we say $x < y$ if $y > x$.

With this definition of the order relation, $\mathcal{R}$ is an ordered field. Moreover, the embedding Π in Eq. (1.1) of $\mathbb{R}$ into $\mathcal{R}$ is compatible with the order. The order induces an absolute value on $\mathcal{R}$ in the natural way. We also note here that λ, as defined above, is a valuation; moreover, the relation $\sim$ is an equivalence relation, and the set of equivalence classes (the value group) is (isomorphic to) $\mathbb{Q}$.

Besides the usual order relations, some other notations are convenient.

Definition 1.3 *($\ll, \gg$).* Let $x, y \in \mathcal{R}$ be non-negative. We say x is infinitely smaller than y (and write $x \ll y$) if $nx < y$ for all $n \in \mathbb{N}$; we say x is infinitely larger than y (and write $x \gg y$) if $y \ll x$. If $x \ll 1$, we say x is infinitely small; if $x \gg 1$, we say x is infinitely large. Infinitely small numbers are also called infinitesimals or differentials. Infinitely large numbers are also called infinite. Non-negative numbers that are neither infinitely small nor infinitely large are also called finite.

Definition 1.4 *(The number d).* Let d be the element of $\mathcal{R}$ given by $d[1] = 1$ and $d[q] = 0$ for $q \neq 1$.

It is easy to check that $d^q \ll 1$ if and only if $q > 0$. Moreover, for all $x \in \mathcal{R}$, the elements of $\text{supp}(x)$ can be arranged in ascending order, say $\text{supp}(x) = \{q_1, q_2, \ldots\}$ with $q_j < q_{j+1}$ for all j; and x can be written as $x = \sum_{j=1}^{\infty} x[q_j] d^{q_j}$, where the series converges in the topology induced by the order [2].

Altogether, it follows that $\mathcal{R}$ is a non-Archimedean field extension of $\mathbb{R}$. For a detailed study of this field, we refer the reader to [2,18,3,19,25,20,4,21,26,22–24]. In particular, it is shown that $\mathcal{R}$ is complete with respect to the topology induced by the order. In the wider context of valuation theory, it is interesting to note that the topology induced by the order is the same as that introduced via the valuation λ, as shown in Remark 1.5 below.

Remark 1.5. The mapping $\Lambda : \mathcal{R} \times \mathcal{R} \to \mathbb{R}$, given by $\Lambda(x, y) = \exp(-\lambda(x - y))$, is an ultrametric distance (and hence a metric); the valuation topology it induces is equivalent to the order topology (we will use τ_v to denote either one of the two topologies in the rest of the paper). For if A is an open set in the order topology and $a \in A$, then there exists $r > 0$ in $\mathcal{R}$ such that, for all $x \in \mathcal{R}$, $|x - a| < r \Rightarrow x \in A$. Let $l = \exp(-\lambda(r))$, then apparently we also have that, for all $x \in \mathcal{R}$, $\Lambda(x, a) < l \Rightarrow x \in A$; and hence A is open with respect to the valuation topology. The other direction of the equivalence of the topologies follows analogously.

It follows therefore that the field $\mathcal{R}$ is just a special case of the class of fields discussed in [16]. For a general overview of the algebraic properties of formal power series fields in general, we refer the reader to the comprehensive overview by Ribenboim [13], and for an overview of the related valuation theory to the books by Krull [9], Schikhof [16] and Alling [1]. A thorough and complete treatment of ordered structures can also be found in [12].

Besides being the smallest ordered non-Archimedean field extension of the real numbers that is both complete in the order topology and real closed, the Levi-Civita field $\mathcal{R}$ is of particular interest because of its practical usefulness. Since the supports of the elements of $\mathcal{R}$ are left-finite, it is possible to represent these numbers on a computer [2]. Having infinitely small numbers, the errors in classical numerical methods can be made infinitely small and hence irrelevant in all practical applications. One such application is the computation of derivatives of real functions representable on a computer [18], where both the accuracy of formula manipulators and the speed of classical numerical methods are achieved.

In this paper, we study two topologies on $\mathcal{R}$: one induced naturally by the order, which we call the valuation topology, and another weaker topology induced by a family of seminorms, which we call weak topology [2]. We show that each of the two topologies results from a metric on $\mathcal{R}$, that the valuation topology is not a vector topology while the weak topology is, and that $\mathcal{R}$ is complete in the valuation topology while it is not in the weak topology. We study the properties of both topologies; in particular we look at open, closed, and compact sets in each topology.

2. Order (valuation) topology τ_v

We begin this section with the following definition.

Definition 2.1. Let $A \subset \mathcal{R}$. Then we say that A is compact in $(\mathcal{R}, \tau_v)$ if every open cover of A in $(\mathcal{R}, \tau_v)$ has a finite subcover.

Remark 2.2. Since τ_v is induced by a metric on $\mathcal{R}$, namely the valuation metric Λ mentioned in the introduction above, it follows by the Borel–Lebesgue theorem (see for example [7, Section 9.2]) that A is compact in $(\mathcal{R}, \tau_v)$ if and only if A is sequentially compact.

Theorem 2.3. *$(\mathcal{R}, \tau_v)$ is a totally disconnected topological space. It is Hausdorff and not locally compact. There are no countable bases. The topology induced to $\mathbb{R}$ is the discrete topology* [2].

Proof. Let $A \subset \mathcal{R}$ contain more than one point; and let $a \neq b$ in A be given. Without loss of generality, we may assume that $a < b$. Let

$$G_1 = \{x \in \mathcal{R}\colon |x - a| \ll b - a\} \quad \text{and} \quad G_2 = \mathcal{R} \setminus G_1.$$

Then G_1 and G_2 are disjoint and open in $(\mathcal{R}, \tau_v)$; $a \in G_1 \cap A$ and $b \in G_2 \cap A$; and $A \subset G_1 \cup G_2 = \mathcal{R}$. This shows that any subset of $(\mathcal{R}, \tau_v)$ containing more than one point is disconnected; and hence $(\mathcal{R}, \tau_v)$ is totally disconnected. It follows that $(\mathcal{R}, \tau_v)$ is Hausdorff. That $(\mathcal{R}, \tau_v)$ is Hausdorff also follows from the fact that it is a metric space [8, Problem 7(a), p. 66].

To prove that $(\mathcal{R}, \tau_v)$ is not locally compact, let $x \in \mathcal{R}$ be given and let U be a neighborhood of x. We show that the closure $\bar{U}$ of U is not compact. Let $\epsilon > 0$ in $\mathcal{R}$ be such that $(x - \epsilon, x + \epsilon) \subset U$ and consider the sets

$$M_{-1} = \{y \in \mathcal{R}\colon\ y < x \text{ or } y - x \gg d \cdot \epsilon\};$$
$$M_n = \big(x + (n-1)d \cdot \epsilon, x + (n+1)d \cdot \epsilon\big) \quad \text{for } n = 0, 1, 2, \ldots.$$

Then it is easy to check that M_n is open in $(\mathcal{R}, \tau_v)$ for all $n \geqslant -1$, and $\bigcup_{n=-1}^{\infty} M_n = \mathcal{R}$; in particular, $\bar{U} \subset \bigcup_{n=-1}^{\infty} M_n$. But it is impossible to select finitely many of the M_n's to cover $\bar{U}$ because each of the infinitely many elements $x + nd \cdot \epsilon$ of $\bar{U}$, $n = -1, 0, 1, 2, \ldots$, is contained only in the set M_n.

There cannot be any countable bases because the uncountably many open sets $M_X = (X - d, X + d)$, with $X \in \mathbb{R}$, are disjoint. The open sets induced on $\mathbb{R}$ by the sets M_X are just the singletons $\{X\}$. Thus, in the induced topology, all sets are open and the topology is therefore discrete. □

Remark 2.4. A detailed study of the properties in Theorem 2.3 reveals that they hold in an identical way in any ordered non-Archimedean field, and thus the above unusual properties are not specific to $\mathcal{R}$.

As an immediate consequence of the fact that $(\mathcal{R}, \tau_v)$ is not locally compact, we obtain the following result.

Corollary 2.5. *For all $a < b$ in $\mathcal{R}$, none of the intervals (a, b), $(a, b]$, $[a, b)$, or $[a, b]$ are compact in $(\mathcal{R}, \tau_v)$.*

Since τ_v is induced on $\mathcal{R}$ by the order, we define boundedness of a set in $(\mathcal{R}, \tau_v)$ as follows.

Definition 2.6. Let $A \subset \mathcal{R}$. Then we say that A is bounded in $(\mathcal{R}, \tau_v)$ if there exists $M > 0$ in $\mathcal{R}$ such that $|x| \leqslant M$ for all $x \in A$.

Proposition 2.7. *Let A be compact in $(\mathcal{R}, \tau_v)$. Then A is closed and bounded in $(\mathcal{R}, \tau_v)$. Moreover, A has an empty interior in $(\mathcal{R}, \tau_v)$; that is,*

$$\operatorname{int}_v(A) := \{a \in A\colon \exists r > 0 \text{ in } \mathcal{R} \ni (a - r, a + r) \subset A\} = \emptyset.$$

Proof. That A is closed in $(\mathcal{R}, \tau_v)$ follows from the fact that $(\mathcal{R}, \tau_v)$ is a Hausdorff topological space and A is compact in $(\mathcal{R}, \tau_v)$ (see [14, p. 36]).

Now we show that A is bounded in $(\mathcal{R}, \tau_v)$. For each $n \in \mathbb{N}$, let $G_n = (-d^{-n}, d^{-n})$. Then, for each $n \in \mathbb{N}$, G_n is open in $(\mathcal{R}, \tau_v)$. Moreover, $A \subset \bigcup_{n \in \mathbb{N}} G_n = \mathcal{R}$. Since A is compact in $(\mathcal{R}, \tau_v)$, we can choose a finite subcover; thus, there is $m \in \mathbb{N}$ and there exist $j_1 < j_2 < \cdots < j_m$ in $\mathbb{N}$ such that

$$A \subset \bigcup_{l=1}^{m} G_{j_l} = G_{j_m} = \big(-d^{-j_m}, d^{-j_m}\big).$$

It follows that $|x| < d^{-j_m}$ for all $x \in A$, and hence A is bounded in $(\mathcal{R}, \tau_v)$.

Finally, we show that $\operatorname{int}_v(A) = \emptyset$. Assume not. Then there exist $a < b$ in A such that $[a, b] \subset A$. Since $[a, b]$ is a closed subset of the compact set A, it follows that $[a, b]$ is compact in $(\mathcal{R}, \tau_v)$, which contradicts Corollary 2.5. □

Remark 2.8. Since A is compact in the metric space $(\mathcal{R}, \tau_v)$, A is bounded with respect to the metric. That A is bounded with respect to the absolute value (Definition 2.6) then follows from the fact that the two concepts of boundedness are equivalent as one can readily verify.

The following examples show that there are countably infinite closed and bounded sets that are not compact and there are uncountable sets that are compact in $(\mathcal{R}, \tau_v)$.

Example 2.9. Let $A = [0,1] \cap \mathbb{Q}$. Then clearly, A is countably infinite and bounded in $(\mathcal{R}, \tau_v)$. We show that A is closed in $(\mathcal{R}, \tau_v)$. Let $x \in \mathcal{R} \setminus A$ be given and let $G_0 = (x-d, x+d)$. If $G_0 \cap A \neq \emptyset$ then there exists $q \in A$ such that $G_0 \cap A = \{q\}$. Let $r = |q-x|$ and let $G = (x-r, x+r)$. Then G is open in $(\mathcal{R}, \tau_v)$ and $G \cap A = \emptyset$. Thus, $\mathcal{R} \setminus A$ is open, and hence A is closed in $(\mathcal{R}, \tau_v)$.

Next we show that A is not compact in $(\mathcal{R}, \tau_v)$. For each $q \in A$, let $G_q = (q-d, q+d)$. Then G_q is open in $(\mathcal{R}, \tau_v)$ for each q and $A \subset \bigcup_{q\in A} G_q$, but we can't select a finite subcover since each $t \in A$ is contained only in G_t.

Example 2.10. Let $C_{\mathcal{R}}$ denote the Cantor-like set constructed in the same way as the standard real Cantor set C; but instead of deleting the middle third, we delete from the middle an open interval $(1-2d)$ times the size of each of the closed subintervals of $[0,1]$ at each step of the construction (see [21]). Then $C_{\mathcal{R}}$ is compact in $(\mathcal{R}, \tau_v)$.

It turns out that if we view $\mathcal{R}$ as an infinite-dimensional vector space over $\mathbb{R}$ then τ_v is not a vector topology; that is, $(\mathcal{R}, \tau_v)$ is not a linear topological space.

Theorem 2.11. *τ_v is not a vector topology.*

Proof. Assume to the contrary that $(\mathcal{R}, \tau_v)$ is a vector topology. Then, by continuity of scalar multiplication, there exists an open set $O_{\mathbb{R}} \subset \mathbb{R}$ and there exists an open set $O_{\mathcal{R}} \subset \mathcal{R}$ such that $\alpha x \in (1-d, 1+d)$ for all $\alpha \in O_{\mathbb{R}}$ and for all $x \in O_{\mathcal{R}}$. Let $\alpha_0 \in O_{\mathbb{R}}$ and $x_0 \in O_{\mathcal{R}}$ be given. Since $O_{\mathbb{R}}$ is open, there exists $r > 0$ in $\mathbb{R}$ such that $(\alpha_0 - 2r, \alpha_0 + 2r) \subset O_{\mathbb{R}}$. Hence

$$\alpha_0 x_0 \in (1-d, 1+d) \quad \text{and} \quad (\alpha_0 + r)x_0 \in (1-d, 1+d).$$

Thus,

$$r|x_0| = \left|(\alpha_0 + r)x_0 - \alpha_0 x_0\right| < 2d,$$

which contradicts the fact that $r|x_0| \gg 2d$, since both r and $|x_0|$ are finite and d is infinitely small. □

Since any normed vector space, with the metric topology induced by its norm, is a linear topological space [6, Proposition III.1.3], we readily infer from Theorem 2.11 that there can be no norm on $\mathcal{R}$ that would induce the same topology as τ_v on $\mathcal{R}$.

3. Weak topology

In the following, we will think of $\mathcal{R}$ as an infinite-dimensional vector space over $\mathbb{R}$. We define a family of semi-norms on $\mathcal{R}$, which induces a topology weaker than the order (valuation) topology, called the weak topology [2].

Definition 3.1. Given $r \in \mathbb{R}$, we define a mapping $\|\cdot\|_r : \mathcal{R} \to \mathbb{R}$ as follows: $\|x\|_r = \max\{|x[q]|:\ q \in \mathbb{Q} \text{ and } q \leqslant r\}$.

The maximum in Definition 3.1 exists in $\mathbb{R}$ since, for any $r \in \mathbb{R}$, only finitely many of the $x[q]$'s considered do not vanish.

Definition 3.2. For $x \in \mathcal{R}$ and $r > 0$ in $\mathbb{R}$, we define the weak ball centered at x of radius r by

$$B_w(x, r) = \left\{y \in \mathcal{R}:\ \|y - x\|_{1/r} < r\right\}.$$

Lemma 3.3. *Let $0 < r_2 < r_1$ be given in $\mathbb{R}$, let $r = \min\{r_2, r_1 - r_2\}$, and let $x \in \mathcal{R}$ be given. Then for all $y \in B_w(x, r)$, we have that $B_w(y, r_2) \subset B_w(x, r_1)$. In particular, $B_w(x, r_2) \subset B_w(x, r_1)$.*

Proof. Let $y \in B_w(x, r)$ be given; we show that $B_w(y, r_2) \subset B_w(x, r_1)$. So let $z \in B_w(y, r_2)$ be given. Then $\|z - y\|_{1/r_2} < r_2$. It follows that

$$\begin{aligned}
\|z - x\|_{1/r_1} &\leqslant \|z - x\|_{1/r_2} \leqslant \|z - y\|_{1/r_2} + \|y - x\|_{1/r_2} \\
&< r_2 + \|y - x\|_{1/r_2} \leqslant r_2 + \|y - x\|_{1/r} \\
&< r_2 + r \leqslant r_2 + (r_1 - r_2) = r_1.
\end{aligned}$$

Thus $z \in B_w(x, r_1)$ for all $z \in B_w(y, r_2)$; and hence $B_w(y, r_2) \subset B_w(x, r_1)$.

Finally, since $x \in B_w(x, r)$, it follows that $B_w(x, r_2) \subset B_w(x, r_1)$. □

Proposition 3.4. *The family of subsets of $\mathcal{R}$*

$$\tau_w := \{O \subset \mathcal{R}\colon \forall x \in O,\ \exists r > 0 \text{ in } \mathbb{R} \text{ such that } B_w(x,r) \subset O\}$$

is a topology on $\mathcal{R}$.

Proof. Let $\{O_\alpha\}_{\alpha\in A}$ be a collection of elements of τ_w. We show that $\bigcup_{\alpha\in A} O_\alpha \in \tau_w$. So let $x \in \bigcup_{\alpha\in A} O_\alpha$ be given. Then there exists $\alpha_0 \in A$ such that $x \in O_{\alpha_0}$. Since $O_{\alpha_0} \in \tau_w$, there exists $r > 0$ in $\mathbb{R}$ such that $B_w(x,r) \subset O_{\alpha_0}$. Thus, $B_w(x,r) \subset \bigcup_{\alpha\in A} O_\alpha$.

Next we show that τ_w is closed under finite intersections: It suffices to show that if $O_1, O_2 \in \tau_w$ then $O_1 \cap O_2 \in \tau_w$. So let $O_1, O_2 \in \tau_w$ and let $x \in O_1 \cap O_2$ be given. Then there exist $r_1, r_2 > 0$ in $\mathbb{R}$ such that $B_w(x,r_1) \subset O_1$ and $B_w(x,r_2) \subset O_2$. Let $r = \min\{r_1, r_2\}$. Then, using Lemma 3.3, we obtain that $B_w(x,r) \subset B_w(x,r_1) \subset O_1$ and $B_w(x,r) \subset B_w(x,r_2) \subset O_2$. Thus, $B_w(x,r) \subset O_1 \cap O_2$.

That $\emptyset$ and $\mathcal{R}$ are both elements of τ_w is clear. It follows that τ_w is a topology on $\mathcal{R}$ and hence $(\mathcal{R}, \tau_w)$ is a topological space. □

It turns out that the weak topology is the most useful topology for considering convergence of sequences and series in general [19,24,23]. Moreover, it is of great importance for the implementation of the $\mathcal{R}$ calculus on computers [2,18].

Definition 3.5. Let $A \subset \mathcal{R}$. Then we say that A is open in $(\mathcal{R}, \tau_w)$ if $A \in \tau_w$. We say that A is closed in $(\mathcal{R}, \tau_w)$ if its complement $\mathcal{R} \setminus A \in \tau_w$.

Since, by Theorem 3.33 below, τ_w is induced by a metric on $\mathcal{R}$ we define compactness in $(\mathcal{R}, \tau_w)$ just as we did in $(\mathcal{R}, \tau_v)$ – see Definition 2.1 – and as in any other metric space. Moreover, the following result follows readily.

Proposition 3.6. *Let $A \subset \mathcal{R}$. Then A is closed in $(\mathcal{R}, \tau_w)$ if and only if whenever $(a_n)_{n\in\mathbb{N}}$ is a sequence of elements in A that converges in $(\mathcal{R}, \tau_w)$ to $a \in \mathcal{R}$, then $a \in A$.*

Proposition 3.7. *$(\mathcal{R}, \tau_w)$ is a Hausdorff topological space. The topology induced on $\mathbb{R}$ by the weak topology is the usual order topology on $\mathbb{R}$ [2].*

Proof. Let $x, y \in \mathcal{R}$ be given, let $r = \lambda(x - y)$, and let

$$\epsilon = \begin{cases} \min\{\frac{|(x-y)[r]|}{2}, \frac{1}{2|r|}\} & \text{if } r \neq 0, \\ \frac{|(x-y)[r]|}{2} & \text{if } r = 0. \end{cases}$$

Then $B_w(x,\epsilon)$ and $B_w(y,\epsilon)$ are disjoint and open in $(\mathcal{R}, \tau_w)$, and they contain x and y, respectively.

Considering elements of $\mathbb{R}$, their supports (when viewed as elements of $\mathcal{R}$) are all equal to $\{0\}$. Therefore, the open sets in $(\mathcal{R}, \tau_w)$ correspond to the open subsets of $\mathbb{R}$ in its order topology. □

Proposition 3.8. *Let $G \subset \mathcal{R}$ be open in $(\mathcal{R}, \tau_w)$. Then G is open in $(\mathcal{R}, \tau_v)$.*

Proof. Let $x \in G$ be given. Then there exists $r > 0$ in $\mathbb{R}$ such that $B_w(x,r) \subset G$. Let $n \in \mathbb{N}$ be such that $n > 1/r$. We show that $B(x, d^n) \subset G$.

Let $y \in B(x, d^n)$ be given. Then $|y - x| < d^n$. Hence $(y - x)[q] = 0$ for all $q < n$. In particular, $(y - x)[q] = 0$ for all $q \leqslant 1/r$; and hence $\|y - x\|_{1/r} = 0 < r$. Thus, $y \in B_w(x,r) \subset G$ for all $y \in B(x, d^n)$. It follows that $B(x, d^n) \subset G$, and hence G is open in $(\mathcal{R}, \tau_v)$. □

The following example shows that the converse of Proposition 3.8 is not true.

Example 3.9. The interval $(-1, 1) \subset \mathcal{R}$ is open in $(\mathcal{R}, \tau_v)$; but we show that it is not open in $(\mathcal{R}, \tau_w)$. Let $r > 0$ in $\mathbb{R}$ be given. Let $x = (r/2)d^{-1}$; then $x \notin (-1, 1)$, but $x \in B_w(0,r)$ since $\|x\|_{1/r} = r/2 < r$. It follows that $B_w(0,r) \not\subset (-1, 1)$ for all $r > 0$; and hence $(-1, 1)$ is not open in $(\mathcal{R}, \tau_w)$.

Remark 3.10. Similarly, we can show that none of the intervals (a, b), $(a, b]$, $[a, b)$, or $[a, b]$ are open in $(\mathcal{R}, \tau_w)$ for all $a < b$ in $\mathcal{R}$.

It follows from Proposition 3.8 and Example 3.9 that the weak topology is strictly weaker than the valuation topology ($\tau_w \subsetneq \tau_v$).

Corollary 3.11. *Let $A \subset \mathcal{R}$ be closed in $(\mathcal{R}, \tau_w)$. Then A is closed in $(\mathcal{R}, \tau_v)$.*

Corollary 3.12. *For all $a < b$ in $\mathcal{R}$, none of the intervals (a, b), $(a, b]$, $[a, b)$, or $[a, b]$ are closed in $(\mathcal{R}, \tau_w)$.*

Corollary 3.13. *Let $A \subset \mathcal{R}$ be compact in $(\mathcal{R}, \tau_v)$. Then A is compact in $(\mathcal{R}, \tau_w)$.*

Proof. Let $\{G_\alpha\}_{\alpha \in \mathcal{G}}$ be an open cover for A in $(\mathcal{R}, \tau_w)$. Then $\{G_\alpha\}_{\alpha \in \mathcal{G}}$ is an open cover for A in $(\mathcal{R}, \tau_v)$ by Proposition 3.8. Since A is compact in $(\mathcal{R}, \tau_v)$, then there exists $m \in \mathbb{N}$ and there exist $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathcal{G}$ such that $A \subset \bigcup_{j=1}^m G_{\alpha_j}$. Hence A is compact in $(\mathcal{R}, \tau_w)$. $\square$

One of the advantages of the weak topology τ_w over the valuation topology τ_v is that the former is a vector topology as the following theorem shows while the latter is not (Theorem 2.11).

Theorem 3.14. *$(\mathcal{R}, \tau_w)$ is a linear topological space*; *that is, τ_w is a vector topology.*

Proof. First we show that $+$ is continuous on $(\mathcal{R}, \tau_w) \times (\mathcal{R}, \tau_w)$. Let O be open in $(\mathcal{R}, \tau_w)$. We need to show that the inverse image A of O under $+$ is open in $(\mathcal{R}, \tau_w) \times (\mathcal{R}, \tau_w)$. So let $(x_1, x_2) \in A$ be given. Then $x_1 + x_2 \in O$. Since O is open in $(\mathcal{R}, \tau_w)$, there exists $r > 0$ in $\mathbb{R}$ such that $B_w(x_1 + x_2, r) \subset O$. Now let $y \in B_w(x_1, r/2)$ and $z \in B_w(x_2, r/2)$ be given. Then

$$\begin{aligned}\|y + z - x_1 - x_2\|_{1/r} &\leqslant \|y - x_1\|_{1/r} + \|z - x_2\|_{1/r} \\ &\leqslant \|y - x_1\|_{2/r} + \|z - x_2\|_{2/r} \\ &< \frac{r}{2} + \frac{r}{2} = r.\end{aligned}$$

Thus, $y + z \in B_w(x_1 + x_2, r) \subset O$; and hence $(y, z) \in A$. It follows that $B_w(x_1, r/2) \times B_w(x_2, r/2) \subset A$. Hence A is open in $(\mathcal{R}, \tau_w) \times (\mathcal{R}, \tau_w)$.

Next we show that scalar multiplication $\cdot : \mathbb{R} \times (\mathcal{R}, \tau_w) \to (\mathcal{R}, \tau_w)$ is continuous. Let O be open in $(\mathcal{R}, \tau_w)$ and let S denote the inverse image of O under $\cdot$. We show that S is open in $\mathbb{R} \times (\mathcal{R}, \tau_w)$. So let $(\alpha, x) \in S$ be given. Then $\alpha x \in O$. Hence there exists $r > 0$ in $\mathbb{R}$ such that $B_w(\alpha x, r) \subset O$.

First assume that $\alpha = 0$, then $\alpha x = 0$. As a first subcase, assume that $\|x\|_{1/r} = 0$. Then we claim that $(-1, 1) \times B_w(x, r) \subset S$: Let $\beta \in (-1, 1)$ and $y \in B_w(x, r)$ be given. Then

$$\begin{aligned}\|\beta y\|_{1/r} &= |\beta| \|y\|_{1/r} < \|y\|_{1/r} \\ &\leqslant \|y - x\|_{1/r} + \|x\|_{1/r} \\ &= \|y - x\|_{1/r} \\ &< r.\end{aligned}$$

Thus, $\beta y \in B_w(0, r) \subset O$ and hence $(\beta, y) \in S$. As a second subcase, assume that $\|x\|_{1/r} \neq 0$. Let

$$r_1 = \min\left\{\frac{1}{2}, \frac{r}{2\|x\|_{1/r}}\right\}.$$

Then $r_1 > 0$ and $r_1 \in \mathbb{R}$. We claim that $(-r_1, r_1) \times B_w(x, r) \subset S$: Let $\beta \in (-r_1, r_1)$ and $y \in B_w(x, r)$ be given. Then

$$\begin{aligned}\|\beta y\|_{1/r} &= \left\|\beta(y - x)\right\|_{1/r} + \|\beta x\|_{1/r} \\ &\leqslant |\beta| \|y - x\|_{1/r} + |\beta| \|x\|_{1/r} \\ &< r_1 r + r_1 \|x\|_{1/r} \\ &\leqslant \frac{1}{2} r + \frac{r}{2\|x\|_{1/r}} \|x\|_{1/r} = r.\end{aligned}$$

Thus, $\beta y \in B_w(0, r) \subset O$ and hence $(\beta, y) \in S$.

Now assume that $\alpha \neq 0$. Let

$$r_1 = \min\left\{\frac{r}{2}, \frac{r}{2|\alpha|}\right\}$$

and

$$\eta = \begin{cases} 1/2 & \text{if } \|x\|_{1/r} = 0, \\ \min\{\frac{1}{2}, \frac{r}{4\|x\|_{1/r}}\} & \text{if } \|x\|_{1/r} \neq 0. \end{cases}$$

We claim that $(\alpha-\eta,\alpha+\eta)\times B_w(x,r_1)\subset S$: Let $\beta\in(\alpha-\eta,\alpha+\eta)$ and $y\in B_w(x,r_1)$ be given. Then

$$\begin{aligned}\|\beta y-\alpha x\|_{1/r}&=\big\|(\beta-\alpha)(y-x)+(\beta-\alpha)x+\alpha(y-x)\big\|_{1/r}\\&\leqslant|\beta-\alpha|\|y-x\|_{1/r}+|\beta-\alpha|\|x\|_{1/r}+|\alpha|\|y-x\|_{1/r}.\end{aligned}$$

Since $r_1\leqslant r/2<r$, we have that

$$\|y-x\|_{1/r}\leqslant\|y-x\|_{1/r_1}<r_1\leqslant\frac{r}{2|\alpha|};\quad\text{so}\quad|\alpha|\|y-x\|_{1/r}<\frac{r}{2}.$$

Also

$$|\beta-\alpha|\|y-x\|_{1/r}<|\beta-\alpha|r_1<\eta\frac{r}{2}\leqslant\frac{r}{4};$$

and

$$|\beta-\alpha|\|x\|_{1/r}\leqslant\eta\|x\|_{1/r}\leqslant\frac{r}{4}.$$

Altogether, we get that

$$\|\beta y-\alpha x\|_{1/r}<\frac{r}{2}+\frac{r}{4}+\frac{r}{4}=r.$$

Thus, $\beta y\in B_w(\alpha x,r)\subset O$ and hence $(\beta,y)\in S$. □

Because of the continuity of addition, it is easy to see that the mapping of translation by a fixed $x_0\in\mathcal{R}$ (that is, the map $x\mapsto x+x_0$, $x\in\mathcal{R}$) is a homeomorphism of $\mathcal{R}$ onto itself. For this reason, the neighborhood structure at any point is the same as the neighborhood structure at 0; and it is sufficient to study the neighborhoods of 0 (henceforth referred to as the zero-neighborhoods). Before we start our discussion of the zero-neighborhoods, we recall the following definitions.

Definition 3.15. Let $A\subset\mathcal{R}$. Then:

(a) We say that A is circled if $\alpha x\in A$ for every $x\in A$ and every $\alpha\in\mathbb{R}$ satisfying $-1\leqslant\alpha\leqslant 1$.
(b) We say that A is absorbing if for every $x\in\mathcal{R}$ there exists $\delta>0$ in $\mathbb{R}$ such that $0\leqslant t<\delta\Rightarrow tx\in A$.

Lemma 3.16. *For all $r>0$ in $\mathbb{R}$, the ball $B_w(0,r)$ is circled and absorbing.*

Proof. Let $r>0$ in $\mathbb{R}$ be given. First we show that $B_w(0,r)$ is circled. So let $x\in B_w(0,r)$ and let $\alpha\in\mathbb{R}$ be such that $-1\leqslant\alpha\leqslant 1$. Then

$$\|\alpha x\|_{1/r}=|\alpha|\|x\|_{1/r}\leqslant\|x\|_{1/r}<r;$$

and hence $\alpha x\in B_w(0,r)$.

Now we show that $B_w(0,r)$ is absorbing. So let $x\in\mathcal{R}$ be given. We need to find $\delta>0$ in $\mathbb{R}$ such that $0\leqslant t<\delta\Rightarrow tx\in B_w(0,r)$. Let

$$\delta=\begin{cases}\frac{r}{2\|x\|_{1/r}} & \text{if } \|x\|_{1/r}\neq 0,\\ 1 & \text{if } \|x\|_{1/r}=0.\end{cases}$$

Then $\delta>0$. Moreover, for $0\leqslant t<\delta$ we have

$$\|tx\|_{1/r}=t\|x\|_{1/r}\leqslant\delta\|x\|_{1/r}<r;$$

and hence $tx\in B_w(0,r)$. □

Of the family of circled and absorbing open balls $\{B_w(0,r)\colon\ 0<r\in\mathbb{R}\}$, we can select a countable local base for the topology τ_w at 0.

Proposition 3.17. $\{B_w(0,q)\colon\ 0<q\in\mathbb{Q}\}$ *is a local base for τ_w at 0.*

Proof. We need to show that for each $O\in\tau_w$ that contains 0, there exists $q>0$ in $\mathbb{Q}$ such that $B_w(0,q)\subset O$. So let $O\in\tau_w$ be given such that $0\in O$. Then there exists $r>0$ in $\mathbb{R}$ such that $B_w(0,r)\subset O$. Let $q\in\mathbb{Q}$ be such that $0<q<r$. Then it follows from Lemma 3.3 that $B_w(0,q)\subset B_w(0,r)\subset O$. □

Corollary 3.18. *$\{B_w(0,q)\colon 0<q\in\mathbb{Q}\}$ is a countable base for the zero-neighborhoods. That is, for each zero-neighborhood N there exists $q>0$ in $\mathbb{Q}$ such that $B_w(0,q)\subset N$.*

Definition 3.19. Given $x\in\mathcal{R}$ and $r>0$ in $\mathbb{R}$, we define

$$\overline{B_w(x,r)}=\bigl\{y\in\mathcal{R}\colon \|y-x\|_{1/r}\leqslant r\bigr\}.$$

Remark 3.20. It follows from the above discussion of the open weak balls $B_w(0,r)$ that $\overline{B_w(0,r)}$ is a circled and absorbing zero-neighborhood for each $r>0$ in $\mathbb{R}$. Moreover, $\{\overline{B_w(0,q)}\colon 0<q\in\mathbb{Q}\}$ is a countable base for the zero-neighborhoods.

Recall that in a Banach space a set is called bounded if it is bounded in norm. However, the appropriate generalization of this is not so obvious for spaces with no norm. Even in metric spaces problems can arise. If we try to mimic the Banach space situation and say that a set is bounded in $(\mathcal{R},\tau_w)$ if and only if it is contained in some metric ball (using for example the metric of Theorem 3.32 which, by Theorem 3.33, induces the topology τ_w on $\mathcal{R}$), then we have a problem: $\mathcal{R}$ and hence any subset of $\mathcal{R}$ is bounded since all of $\mathcal{R}$ is contained in a ball of radius one! We define boundedness of a set in $(\mathcal{R},\tau_w)$ as in any other linear topological space (see for example [15, p. 8]).

Definition 3.21. Let $A,B\subset\mathcal{R}$. Then we say that B absorbs A (or that A is absorbed by B) if there exists $\rho>0$ in $\mathbb{R}$ such that $A\subset rB$ for all $r\geqslant\rho$. We say that A is bounded in $(\mathcal{R},\tau_w)$ if every zero-neighborhood absorbs A.

Proposition 3.22. *Let $A\subset\mathcal{R}$ be compact in $(\mathcal{R},\tau_w)$. Then A is closed and bounded in $(\mathcal{R},\tau_w)$.*

Proof. That A is closed in $(\mathcal{R},\tau_w)$ follows from the fact that $(\mathcal{R},\tau_w)$ is a Hausdorff topological space [14, p. 36].

Now we show that A is bounded in $(\mathcal{R},\tau_w)$. We need to show that every zero-neighborhood in $(\mathcal{R},\tau_w)$ absorbs A. So let U be a zero-neighborhood in $(\mathcal{R},\tau_w)$. Then there exists $r>0$ in $\mathbb{R}$ such that $B_w(0,r)\subset U$. Let $V=B_w(0,r/2)$; then $V+V\subset B_w(0,r)\subset U$, for if $x,y\in V$ then

$$\|x+y\|_{1/r}\leqslant\|x\|_{1/r}+\|y\|_{1/r}\leqslant\|x\|_{2/r}+\|y\|_{2/r}<\frac{r}{2}+\frac{r}{2}=r.$$

The family of sets $\{a+V\colon a\in A\}$ is an open cover of A in $(\mathcal{R},\tau_w)$. By compactness of A, we can select a finite subcover: Thus there exists $n\in\mathbb{N}$ and there exist $a_1,\dots,a_n\in A$ such that $A\subset\bigcup_{j=1}^{n}(a_j+V)$. Since $V=B_w(0,r/2)$ is absorbing in $(\mathcal{R},\tau_w)$, there exists $t>1$ in $\mathbb{R}$ such that $a_j\in tV$ for all $j\in\{1,\dots,n\}$. Thus for each $j=1,\dots,n$ we have that

$$a_j+V\subset a_j+tV\subset tV+tV=t(V+V)\subset tU;$$

and hence

$$A\subset\bigcup_{j=1}^{n}(a_j+V)\subset tU.$$

Thus, U absorbs A. □

Proposition 3.23. $(-1,1)$ *is not bounded in $(\mathcal{R},\tau_w)$.*

Proof. It suffices to show that $(-1,1)$ is not absorbed by $B_w(0,1)$. That is, it suffices to show that, for all $\alpha>0$ in $\mathbb{R}$, there exists $x\in(-1,1)$ such that $x\notin\alpha B_w(0,1)$. So let $\alpha>0$ in $\mathbb{R}$ be given. Let $x=2\alpha d$. Then $x\in(-1,1)$ but $x\notin\alpha B_w(0,1)$ since $\|x\|_1=2\alpha>\alpha$. □

Remark 3.24. Similarly, we can show that none of the intervals (a,b), $(a,b]$, $[a,b)$, or $[a,b]$ are bounded in $(\mathcal{R},\tau_w)$ for all $a<b$ in $\mathcal{R}$.

Corollary 3.25. *For all $a<b$ in $\mathcal{R}$, none of the intervals (a,b), $(a,b]$, $[a,b)$, or $[a,b]$ are compact in $(\mathcal{R},\tau_w)$.*

Proposition 3.26. *For all $r>0$ in $\mathbb{R}$, $\overline{B_w(0,r)}$ is closed but not bounded and hence not compact in $(\mathcal{R},\tau_w)$. Thus, $(\mathcal{R},\tau_w)$ is neither locally bounded nor locally compact.*

Proof. Let $y\notin\overline{B_w(0,r)}$. Then $\|y\|_{1/r}>r$. Let

$$t=\min\bigl\{\|y\|_{1/r}-r,r\bigr\}.$$

We show that $B_w(y,t) \cap \overline{B_w(0,r)} = \emptyset$: Let $z \in B_w(y,t)$ be given. Then $\|y-z\|_{1/t} = \|z-y\|_{1/t} < t$. It follows that

$$\begin{aligned}\|z\|_{1/r} &\geqslant \|y\|_{1/r} - \|y-z\|_{1/r}\\ &\geqslant \|y\|_{1/r} - \|y-z\|_{1/t}\\ &> \|y\|_{1/r} - t\\ &\geqslant \|y\|_{1/r} - \big(\|y\|_{1/r} - r\big) = r.\end{aligned}$$

This shows that $\mathcal{R} \setminus \overline{B_w(0,r)}$ is open in $(\mathcal{R}, \tau_w)$; and hence $\overline{B_w(0,r)}$ is closed in $(\mathcal{R}, \tau_w)$.

To show that $\overline{B_w(0,r)}$ is not bounded in $(\mathcal{R}, \tau_w)$, it suffices to show that there exists a zero-neighborhood in $(\mathcal{R}, \tau_w)$ which does not absorb $\overline{B_w(0,r)}$. Let $q \in \mathbb{Q}$ be such that

$$0 < q < \min\left\{\frac{r}{2}, \frac{1}{2r}\right\}.$$

We show that $\overline{B_w(0,r)}$ is not absorbed by $B_w(0,q)$. Let $\alpha > 0$ in $\mathbb{R}$ be given. Let $x = 2\alpha q d^{1/q}$. Then

$$\|x\|_{1/q} = 2\alpha q > \alpha q; \quad \text{and hence} \quad x \notin \alpha B_w(0,q).$$

However, since $q < r/2$, it follows that $1/q > 2/r > 1/r$; and hence

$$\|x\|_{1/r} = 0 < r, \quad \text{so} \quad x \in \overline{B_w(0,r)}. \qquad \square$$

Corollary 3.27. *For all $r > 0$ in $\mathbb{R}$, $B_w(0,r)$ is not bounded in $(\mathcal{R}, \tau_w)$.*

Remark 3.28. Since every p-normed space (with $0 < p \leqslant 1$) is locally bounded, we infer that there can be no p-norm (with $0 < p \leqslant 1$) that induces the topology τ_w on $\mathcal{R}$.

Using the results of Corollary 3.25 and Proposition 3.26 (or Corollary 3.27), we readily obtain the following result.

Corollary 3.29. *Let A be compact in $(\mathcal{R}, \tau_w)$. Then A has an empty interior in both $(\mathcal{R}, \tau_v)$ and $(\mathcal{R}, \tau_w)$; that is*

$$\begin{aligned}\operatorname{int}_v(A) &:= \big\{a \in A\colon \exists r > 0 \text{ in } \mathcal{R} \ni (a-r, a+r) \subset A\big\} = \emptyset, \quad \text{and}\\ \operatorname{int}_w(A) &:= \big\{a \in A\colon \exists r > 0 \text{ in } \mathbb{R} \ni B_w(a,r) \subset A\big\} = \emptyset.\end{aligned}$$

Proposition 3.30. *Let $B \subset \mathcal{R}$ be bounded in $(\mathcal{R}, \tau_w)$. Then there exists $M > 0$ in $\mathbb{R}$ such that $\|x\|_{1/M} \leqslant M$ for all $x \in B$; that is, $B \subset \overline{B_w(0,M)}$.*

Proof. Since B is bounded in $(\mathcal{R}, \tau_w)$, B is absorbed by every zero-neighborhood in $(\mathcal{R}, \tau_w)$. In particular, B is absorbed by $B_w(0,r)$ for some fixed $r > 0$ in $\mathbb{R}$. Thus, there exists $\alpha > 1$ in $\mathbb{R}$ such that $B \subset \alpha B_w(0,r)$. Hence $\|x\|_{1/r} < \alpha r$ for all $x \in B$. Let $M = \alpha r$. Then $M \in \mathbb{R}$ and $M > r > 0$. Thus, $0 < 1/M < 1/r$. Moreover, for all $x \in B$, we have that

$$\|x\|_{1/M} \leqslant \|x\|_{1/r} < \alpha r = M.$$

Hence $B \subset \overline{B_w(0,M)}$. $\square$

Remark 3.31. Proposition 3.26 and Corollary 3.27 show that the converse of Proposition 3.30 is not true.

Since $(\mathcal{R}, \tau_w)$ is a linear topological space with a countable local base, there is a translation invariant metric on $\mathcal{R}$ that induces the topology τ_w on $\mathcal{R}$ ([15, Theorem 1.24], see also [6, p. 105] and [5, p. 152]). In the remaining of the paper (Theorem 3.32 and Theorem 3.33), we present the details of the proof of the last statement.

Theorem 3.32. *The map $\Delta : \mathcal{R} \times \mathcal{R} \to \mathbb{R}$, given by*

$$\Delta(x,y) = \sum_{k=1}^{\infty} 2^{-k} \frac{\|x-y\|_k}{1 + \|x-y\|_k}, \tag{3.1}$$

is a translation invariant metric.

Proof. Δ is positive-definite: It is clear that $\Delta(x, y) \geqslant 0$ for all $x, y \in \mathcal{R}$. Moreover, for all $x, y \in \mathcal{R}$,

$$\begin{aligned}\Delta(x, y) = 0 \quad &\Leftrightarrow \quad \|x - y\|_k = 0 \quad \text{for all } k \in \mathbb{N}\\ &\Leftrightarrow \quad (x - y)[q] = 0 \quad \text{for all } q \leqslant k \text{ in } \mathbb{Q}, \text{ for all } k \in \mathbb{N}\\ &\Leftrightarrow \quad (x - y)[q] = 0 \quad \text{for all } q \in \mathbb{Q}\\ &\Leftrightarrow \quad x = y.\end{aligned}$$

Δ is symmetric: For all $x, y \in \mathcal{R}$, we have that

$$\Delta(x, y) = \sum_{k=1}^{\infty} 2^{-k} \frac{\|x - y\|_k}{1 + \|x - y\|_k} = \sum_{k=1}^{\infty} 2^{-k} \frac{\|y - x\|_k}{1 + \|y - x\|_k} = \Delta(y, x).$$

Δ satisfies the triangle inequality: Let $x, y, z \in \mathcal{R}$ be given. Then, for all $k \in \mathbb{N}$, we have that

$$\begin{aligned}\frac{\|x - y\|_k}{1 + \|x - y\|_k} &= 1 - \frac{1}{1 + \|x - y\|_k} \leqslant 1 - \frac{1}{1 + \|x - z\|_k + \|y - z\|_k}\\ &= \frac{\|x - z\|_k}{1 + \|x - z\|_k + \|y - z\|_k} + \frac{\|y - z\|_k}{1 + \|x - z\|_k + \|y - z\|_k}\\ &\leqslant \frac{\|x - z\|_k}{1 + \|x - z\|_k} + \frac{\|y - z\|_k}{1 + \|y - z\|_k}.\end{aligned}$$

Thus,

$$\begin{aligned}\Delta(x, y) &= \sum_{k=1}^{\infty} 2^{-k} \frac{\|x - y\|_k}{1 + \|x - y\|_k}\\ &\leqslant \sum_{k=1}^{\infty} 2^{-k} \frac{\|x - z\|_k}{1 + \|x - z\|_k} + \sum_{k=1}^{\infty} 2^{-k} \frac{\|y - z\|_k}{1 + \|y - z\|_k}\\ &= \Delta(x, z) + \Delta(y, z).\end{aligned}$$

Finally, for all $x, y, z \in \mathcal{R}$, we have that

$$\begin{aligned}\Delta(x + z, y + z) &= \sum_{k=1}^{\infty} 2^{-k} \frac{\|(x + z) - (y + z)\|_k}{1 + \|(x + z) - (y + z)\|_k}\\ &= \sum_{k=1}^{\infty} 2^{-k} \frac{\|x - y\|_k}{1 + \|x - y\|_k}\\ &= \Delta(x, y). \qquad \square\end{aligned}$$

It turns out that the metric Δ introduced above induces the same topology on $\mathcal{R}$ as the weak topology τ_w.

Theorem 3.33. *Let τ_Δ denote the topology induced by the metric Δ in Eq.* (3.1)*. Then $\tau_\Delta = \tau_w$.*

Proof. First we show that $\tau_\Delta \subseteq \tau_w$: Let $O \in \tau_\Delta$, and let $x \in O$ be given. Then there exists $r > 0$ in $\mathbb{R}$ such that

$$B_\Delta(x, r) := \{y \in \mathcal{R}\colon\ \Delta(x, y) < r\} \subset O.$$

Let $j \in \mathbb{N}$ be such that $j > 2/r$. Then

$$2^{-j} < \frac{1}{j} < \frac{r}{2}.$$

We show that $B_w(x, 1/j) \subset O$: Let $y \in B_w(x, 1/j)$ be given. Then $\|x - y\|_j < 1/j$. It follows that

$$\|x - y\|_k < \frac{1}{j} \leqslant \frac{1}{k} \quad \text{for } 1 \leqslant k \leqslant j.$$

Thus,

$$\begin{aligned}\Delta(x,y) &= \sum_{k=1}^{\infty} 2^{-k}\frac{\|x-y\|_k}{1+\|x-y\|_k}\\ &= \sum_{k=1}^{j} 2^{-k}\frac{\|x-y\|_k}{1+\|x-y\|_k} + \sum_{k=j+1}^{\infty} 2^{-k}\frac{\|x-y\|_k}{1+\|x-y\|_k}\\ &\leqslant \sum_{k=1}^{j} 2^{-k}\|x-y\|_k + \sum_{k=j+1}^{\infty} 2^{-k}\\ &< \frac{1}{j}\sum_{k=1}^{j} 2^{-k} + 2^{-j}\sum_{k=1}^{\infty} 2^{-k}\\ &< \frac{1}{j} + 2^{-j}\\ &< \frac{r}{2} + \frac{r}{2} = r.\end{aligned}$$

Hence $y \in B_\Delta(x,r) \subset O$. Thus, $B_w(x,1/j) \subset O$. This shows that $O \in \tau_w$.

Next we show that $\tau_w \subseteq \tau_\Delta$: Let $O \in \tau_w$; and let $x \in O$ be given. Since O is open in $(\mathcal{R},\tau_w)$, there exists $M \in \mathbb{R}$ such that $0 < M < 1$ and $B_w(x,M) \subset O$. Choose $j \in \mathbb{N}$ such that $j > 1/M$. We show that $B_\Delta(x, M2^{-(j+1)}) \subset O$: Let $y \in B_\Delta(x, M2^{-(j+1)})$ be given. Then

$$\Delta(x,y) = \sum_{k=1}^{\infty} 2^{-k}\frac{\|x-y\|_k}{1+\|x-y\|_k} < M2^{-(j+1)}.$$

Thus,

$$2^{-j}\frac{\|x-y\|_j}{1+\|x-y\|_j} < \frac{M}{2}2^{-j}; \quad \text{and hence} \quad \frac{\|x-y\|_j}{1+\|x-y\|_j} < \frac{M}{2}.$$

It follows that

$$\|x-y\|_j < \frac{M}{2-M} < M \quad \text{since } 0 < M < 1.$$

Therefore,

$$\|x-y\|_{1/M} \leqslant \|x-y\|_j < M;$$

and hence $y \in B_w(x,M) \subset O$. Thus, $B_\Delta(x, M2^{-(j+1)}) \subset O$. This shows that $O \in \tau_\Delta$. $\square$

Remark 3.34. Convergence of sequences and series in both $(\mathcal{R},\tau_v)$ and $(\mathcal{R},\tau_w)$ has been studied in details in [2,17,19,22, 24,23]. In particular, it is shown that $(\mathcal{R},\tau_v)$ is complete but $(\mathcal{R},\tau_w)$ is not. For example, the sequence $(a_n)_{n\in\mathbb{N}}$, where $a_n = \sum_{j=1}^{n} d^{-j}/j$ for each $n \in \mathbb{N}$, is Cauchy in $(\mathcal{R},\tau_w)$ but it does not converge in $(\mathcal{R},\tau_w)$.

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