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Summary on non-Archimedean valued fields

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Abstract. This article summarizes the main properties of ultrametric spaces, valued fields, ordered fields and fields with valuations of higher rank, highlighting their similarities and differences. The most used non-Archimedean valued fields are reviewed, like a completion in the case of the p-adic numbers fields and the Levi-Civita fields, or like an algebraic closure as is sometimes the case of the Puiseux series fields. Also a study of spherically complete valued fields is presented and their characterization as maximally complete fields is given where the Hahn fields and the Mal'cev-Neumann fields (their p-adic analogues) play an important role as "spherical completions". Finally several of the metric, topological, algebraic and order properties of the most used non-Archimedean valued fields are collected in a catalog where we can appreciate the analogy between fields that have the same characteristic as their residue class fields and fields that do not satisfy this property.

Introduction.

When a graduate student or any mathematician in general begin their journey in non-Archimedean Analysis, the freedom of choice for the base valued field presents different cases to consider, most of which do not occur when we work with the fields of real or complex numbers. The existence of discrete valuations, or the possible non-existence of vectors of norm 1 sometimes lead to the use of new procedures for proofs of results in Functional Analysis. The need and type of these procedures depend greatly on the properties of the base valued field. This article is prepared as a summary and will serve as a tool and source of references for the mathematician who wants to get familiar with the most commonly used non-Archimedean valued fields and/or needs to consider different valued fields in order to understand the potential extension of results that are valid for a particular valued field. Also this article helps one realize that any non-Archimedean valued field can be embedded in a Hahn field or a p-adic Mal'cev-Neumann field, thus the structure of a non-Archimedean valued field is not as diverse as one initially may think. The unique and remarkable features of this article are the catalog of valued fields (section 8) that collects all the fields and properties discussed in this paper, and the classification of valued fields diagram (section 9).

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1. Ultrametric Spaces.

A valued field is a mathematical entity with a topological and an algebraic structures that will be defined and studied in section 2. In this section we will introduce the notion of an ultrametric space which is a metric space used to study the metric and topological properties of a non-Archimedean valued field without worrying about its algebraic structure.

1.1. The Strong Triangle Inequality and its Consequences.

Definition 1.1. A ***metric*** on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying the following properties for all $x, y, z \in X$:

(1) $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$,
(3) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

The pair (X, d) is called a ***metric space***.

Notation 1.2. Let (X, d) be a metric space and let $a \in X$ and $r > 0$. The sets $B(a, r) := \{x \in X : d(x, a) < r\}$ and $B[a, r] := \{x \in X : d(x, a) \leq r\}$ are called the ***open and closed balls of center a and radius r***, respectively. The family of open balls forms a base of neighbourhoods for a uniquely determined Hausdorff topology on X. This topology is called the ***topology induced by d on X***. With respect to this topology the open balls are open sets and the closed balls are closed sets in X.

The ***diameter*** of a non-empty set $Y \subset X$ is $diam(Y) := \sup\{d(x, y) : x, y \in Y\}$ and the ***distance*** between two non-empty sets $Y, Z \subset X$ is $dist(Y, Z) := \inf\{d(y, z) : y \in Y, z \in Z\}$.

The set of values of a metric $d : X \times X \to \mathbb{R}$ is denoted and defined by $d(X \times X) := \{d(x, y) : x, y \in X\}$.

Definition 1.3. A metric $d : X \times X \to \mathbb{R}$ is called an ***ultrametric*** when it satisfies the so-called ***strong triangle inequality*** $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$. In that case, the pair (X, d) is called an ***ultrametric space***.

Examples 1.4.

(1) Let X be a set and $d : X \times X \to \mathbb{R}$ be the discrete metric, i.e. $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$ for all $x, y \in X$. Then (X, d) is an ultrametric space. In this case, for $x \in X$, we have that $B[x, 1] = B[x, r]$ for all $r \geq 1$. Also, for each y in $B(x, r)$, we have that $B(x, r) = B(y, r)$. As we will see, it is not rare to find a ball of an ultrametric space with infinitely many radii, and where each point of a ball is the center of such ball.
(2) Let p be a fixed prime. The p-adic metric on $\mathbb{Z}$ is defined by $d(n, m) = 0$ if $n = m$, and for $n \neq m$, $d(n, m) = p^{-r}$ where r is the largest non-negative integer such that p^r divides $m - n$. The pair $(\mathbb{Z}, d)$ is an ultrametric space.
(3) Let $\mathbb{R}[x]$ be the ring of all polynomials with real coefficients. For each nonzero polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ in $\mathbb{R}[x]$, put $\lambda(p) = \min\{i : a_i \neq 0\}$. Thus the map $d : \mathbb{R}[x] \times \mathbb{R}[x] \to \mathbb{R}$ defined by
$$d(p, q) := \begin{cases} e^{-\lambda(p-q)} & , \text{ if } p \neq q \\ 0 & , \text{ if } p = q \end{cases}$$
is an ultrametric on $\mathbb{R}[x]$.

(4) Let $\mathbb{N}$ be the set of positive integers and $d : \mathbb{N}\times\mathbb{N} \to \mathbb{R}$ be the map defined by

$$d(m,n) := \begin{cases} \max\{1+\frac{1}{m}, 1+\frac{1}{n}\} & , \text{ if } m \neq n \\ 0 & , \text{ if } m = n. \end{cases}$$

Then $(\mathbb{N}, d)$ is an ultrametric space.

(5) Any subset of a non-Archimedean valued field $(K, |\ |)$ with the map $(x, y) \mapsto |x-y|$ constitutes an ultrametric space (non-Archimedean valued fields will be introduced in the next section). Notice that with this example we have listed all ultrametric spaces, since W. Schikhof proved in [**39**] that any ultrametric space can isometrically be embedded into a non-Archimedean valued field.

Let (X, d) be a metric space. The following condition is called the ***Isosceles triangle principle***: If $d(x, z) \neq d(z, y)$ then $d(x, y) = \max\{d(x, z), d(z, y)\}$, i.e. every triangle with vertices in X is isosceles.

THEOREM 1.5 ([**32**, p. 3], [**49**, 2.A]). *Let (X, d) be a metric space. The metric d is an ultrametric if and only if it satisfies the Isosceles triangle principle.*

The following theorem collects the most remarkable results of an ultrametric space, all direct consequences of the strong triangle inequality.

THEOREM 1.6. *Let (X, d) be an ultrametric space. Then the following properties are satisfied:*

(1) *Each point of a ball is a center of the ball.*
(2) *Each ball in X is both closed and open ("**clopen**") in the topology induced by the ultrametric.*
(3) *Each ball has an empty boundary.*
(4) *Two balls are either disjoint, or one is contained in the other.*
(5) *Let $a \in Y \subset X$. Then $diam(Y) = \sup\{d(x, a) : x \in Y\}$.*
(6) *The radii of a ball B form the set $\{r \in \mathbb{R} : r_1 \leq r \leq r_2\}$, where $r_1 = diam(B)$, $r_2 = dist(B, X\backslash B)$ ($r_2 = \infty$ if $B = X$). It may happen that $r_1 < r_2$, so that a ball may have infinitely many radii.*
(7) *If two balls B_1, B_2 are disjoint, then $dist(B_1, B_2) = d(x, y)$ for all $x \in B_1, y \in B_2$.*
(8) *Let $U \neq \emptyset$ be an open subset of X. Given a sequence $(r_n)_n$ in $(0, \infty)$, strictly decreasing and convergent to 0, there exists a partition of U formed by balls of the form $B[a, r_n]$, with $a \in U$ and $n \in \mathbb{N}$.*
(9) *Let $\varepsilon \in \mathbb{R}^+$. For $x, y \in X$ the relation $d(x, y) < \varepsilon$ is an equivalence relation and induces a partition of X into open balls of radius ε. Analogously for $d(x, y) \leq \varepsilon$ and closed balls.*
(10) *Let $Y \subset X$, B a ball in X, $B \cap Y \neq \emptyset$. Then, $B \cap Y$ is a ball in Y.*
(11) *Let $(x_n)_n$ be a sequence in X converging to $x \in X$, then for each $a \in X \setminus \{x\}$, there exists $N \in \mathbb{N}$ such that $d(x_n, a) = d(x, a)$ for all $n \geq N$.*
(12) *There are not new values of an ultrametric after completion, i.e. if $(X^\wedge, d^\wedge)$ is the completion of (X, d), then $d(X \times X) = d^\wedge(X^\wedge \times X^\wedge)$.*
(13) *A sequence $(x_n)_n$ on X is Cauchy if and only if $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.*

PROOF. The property (8) can be found in [**41**, Theorem 18.6], while (5) in [**40**, 1.D]. The property (3) follows directly from (2) and the remaining proofs can be found in [**32**, pp. 3–4]. □

1.2. Compactness and Separability.

THEOREM 1.7 ([**41**, 19.2], [**40**, 1.E]). *Let (X, d) be a compact ultrametric space. Then the following statements are satisfied:*

(1) *$d(X \times X)$ is countable, it has a maximum, and it has 0 as the only possible accumulation point. Thus if $d(X \times X)$ is infinite, then $d(X \times X) = \{0\} \cup \{r_n : n \in \mathbb{N}\}$, where $(r_n)_n$ is a strictly decreasing sequence of $(0, \infty)$ such that $\lim_{n\to\infty} r_n = 0$.*
(2) *X is complete and separable.*
(3) *Every open covering of X has a finite refinement formed by disjoint balls.*
(4) *Every partition of X by clopen sets is finite.*
(5) *If $Y \subset X$ is clopen, $\emptyset \neq Y \neq X$, then dist$(Y, X \setminus Y)$ is positive.*

THEOREM 1.8 ([**32**, 1.1.3]). *If X is a locally compact ultrametric space then there exists a partition of X consisting of compact balls.*

THEOREM 1.9 ([**41**, 19.3]). *An ultrametric space is separable if and only if the set of all its balls is countable.*

1.3. Ultrametrizability.

DEFINITION 1.10. A topology on a set X is called ***ultrametrizable*** if it is the topology induced by an ultrametric. A topological space is called ultrametrizable if its topology is ultrametrizable.

If X is a topological space, then a subset of X is a ***clopen*** set if it is closed and open simultaneously.

A topological space is ***zero-dimensional*** if it has a base of clopen sets.

From 1.6 (2) we can deduce the following:

THEOREM 1.11. *Every ultrametrizable space is zero-dimensional, and hence totally disconnected.*

The converse of the previous theorem is false, since P. Roy [**38**] has constructed a zero-dimensional complete metric space that is not ultrametrizable. But we have a partial converse, which is a generalization of 1.9:

THEOREM 1.12 ([**12**, 2]). *A space is zero-dimensional, Hausdorff and has a countable base if and only if it is ultrametrizable and separable.*

The following result gathers different characterizations of ultrametrizable spaces.

THEOREM 1.13. *Let X be a topological space. The following statements are equivalent:*

(1) *X is ultrametrizable.*
(2) *X is homeomorphic to a subspace of a countable product of discrete spaces.*
(3) *X is homeomorphic to a subspace of a non-Archimedean valued field.*
(4) *X is metrizable and has large inductive dimension 0 (for every two closed, mutually disjoint subsets A and B of X, there exists a clopen $U \subset X$ with $A \subset U$ and $B \subset X \setminus U$. In the literature this is denoted by Ind $X = 0$).*
(5) *X is metrizable and has covering dimension 0 (every finite open covering of X has a refinement that is a partition formed by clopen sets. In the literature this is denoted by $\dim X = 0$).*

(6) *X is metrizable and its topology is non-Archimedean (it is Hausdorff and it has a base $\mathcal{B}$ such that if $A, B \in \mathcal{B}$ then either $A \subset B$ or $B \subset A$ or $A \cap B = \emptyset$).*

Proof. The equivalence (1)⇔(2) is proved in [**49**, 2.1] and the equivalence (1)⇔(3) is proved in [**39**], while the equivalence (1)⇔(4) is proved in [**12**, Theorem II]. The equivalence (4)⇔(5) is due to the equivalence $\operatorname{Ind} X = 0 \Leftrightarrow \dim X = 0$ (see [**31**, 2.3]). The implication (1)⇒(6) follows from 1.6(d) and the implication (6)⇒(4) follows from [**30**, p. 121] □

Theorem 1.14 ([**49**, 2.1]). *A countable product of ultrametrizable spaces is ultrametrizable.*

1.4. Spherical Completeness. Recall that a metric space is said to be Cauchy complete if every Cauchy sequence is convergent, or equivalently, if each nested sequence of closed balls whose radius approaches to 0, has a non-empty intersection. This motivates the following:

Definition 1.15. An ultrametric space is called ***spherically complete*** if each nested sequence of balls has a non-empty intersection.

Remark 1.16. The concept of spherical completeness plays a key role as a necessary and sufficient condition for the validity of the Hahn-Banach theorem in the non-Archimedean context (see [**49**, 4.10, 4.15]). At the end of this subsection we will discuss other important features of spherically complete spaces: a fixed point theorem and best approximations.

Theorem 1.17 ([**49**, 2.3]). *Let (X, d) be an ultrametric space. The following statements are equivalent:*

(1) *(X, d) is spherically complete.*
(2) *For any collection $(B_i)_{i\in I}$ of balls in X such that $B_i \cap B_j \neq \emptyset$ for any $i, j \in I$, we then have $\bigcap_{i\in I} B_i \neq \emptyset$.*
(3) *Every sequence of balls $B[a_1, \varepsilon_1] \supset B[a_2, \varepsilon_2] \supset \cdots$ for which $\varepsilon_1 > \varepsilon_2 > \cdots$ has nonempty intersection.*

It is clear that a spherically complete ultrametric space is Cauchy complete, but the converse is not always true. For instance, the space of the example 1.4(4) is a complete ultrametric space that is not spherically complete. However there is a partial converse:

Lemma 1.18 ([**43**, Lemma 1.7]). *Suppose that (X, d) is a Cauchy complete ultrametric space. If 0 is the only accumulation point of the set $d(X \times X)$, then X is spherically complete.*

Lemma 1.19. *If an ultrametric space is compact then it is spherically complete.*

Proof. It is enough to recall that if X is compact, then every collection of closed subsets of X, with the finite intersection property, has non-empty intersection. □

The following three results show methods to construct or identify ultrametric spaces that are not spherically complete. For that we need the following:

Definition 1.20. A metric on X is ***dense*** if for each ball B in X the set $d(B \times B)$ is dense in the interval $[0, diam(B)]$.

THEOREM 1.21 ([**41**, 20.4]). *If* (X, d) *is a complete ultrametric space and* d *is a dense metric, then there exists a subspace of* X *that is complete but not spherically complete.*

THEOREM 1.22 ([**41**, 20.5]). *If* (X, d) *is a separable ultrametric space and* d *is a dense metric, then* X *is not spherically complete.*

THEOREM 1.23 ([**43**, p. 5]). *Let* (X, d) *be a complete ultrametric space and let* $B_1 \supset B_2 \supset \cdots$ *be a decreasing sequence of balls in* X *such that* $diam(B_1) > diam(B_2) > \cdots$ *and* $\inf\{diam(B_n) : n \in \mathbb{N}\} > 0$. *Then the subspace* $X \setminus (\bigcap_n B_n)$ *is complete but not spherically complete.*

The concept of spherical completeness is geometrical rather than topological.

THEOREM 1.24 ([**49**, 2.F]). *Let* (X, d) *be a complete ultrametric space. Then the formula*

$$\sigma(x, y) := \inf\{2^{-n} : n \in \mathbb{Z}, d(x, y) \leq 2^{-n}\}$$

defines an ultrametric σ *such that* $d \leq \sigma \leq 2d$, *and* (X, σ) *is spherically complete.*

A fixed point theorem. One of the attributes of spherically complete ultrametric spaces is that they satisfy a stronger version of the fixed point theorem for complete metric spaces.

DEFINITION 1.25. Let (X, d) be a metric space. A function $f : X \to X$ is called a ***shrinking map*** when $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$, $x \neq y$. If there exists $k \in (0, 1)$ such that $d(f(x), f(y)) < kd(x, y)$ for all $x, y \in X$, $x \neq y$, then f is called a ***contraction***.

The fixed point theorem for complete metric spaces is the following:

THEOREM 1.26 ([**48**, 3.7.4]). *Every contraction of a complete metric space has a unique fixed point.*

Notice that the map $f : [1, \infty) \to [1, \infty)$, $f(x) = x + \dfrac{1}{x}$ is a shrinking map defined on a complete space but it is not a contraction and has no fixed point. In fact, $x = f(x)$ does not have a solution. Thus by 1.26, f cannot be a contraction. However:

$$|f(x) - f(y)| = |x - y|\big(1 - \frac{1}{xy}\big) < |x - y|.$$

For spherically complete ultrametric spaces we have the following:

THEOREM 1.27 ([**35**, 2.3]). *Every shrinking map of a spherically complete ultrametric space has a unique fixed point.*

Best Approximations.

DEFINITION 1.28. Let Y be a subset of an ultrametric space (X, d). Let $a \in X$ and $b \in Y$. Then b is a ***best approximation*** of a in Y if $d(a, b) = dist(a, Y)$.

Another attribute of a spherically complete ultrametric space is the existence of best approximations as it is stated in the following result.

THEOREM 1.29 ([**41**, 21.2]). *Let* $Y \neq \emptyset$ *be a spherically complete ultrametric space embedded in an ultrametric space* X. *Then each* $x \in X$ *has a best approximation in* Y, *i.e.* $\min\{d(y, x) : y \in Y\}$ *exists.*

In general, best approximations are not unique.

THEOREM 1.30 ([**41**, 21.1]). *Let $Y \neq \emptyset$ be a subset of an ultrametric space X. Suppose that Y has no isolated points. If an element $a \in X \setminus Y$ has a best approximation in Y then it has infinitely many.*

2. Valued fields.

2.1. Archimedean and non-Archimedean valuations.

DEFINITION 2.1. Let K be a field. A ***valuation*** on K is a map $|\ \ | : K \to \mathbb{R}$ satisfying the following axioms for all $x, y \in K$:

(1) $|x| \geq 0$, and $|x| = 0$ if and only if $x = 0$,
(2) $|xy| = |x||y|$,
(3) $|x + y| \leq |x| + |y|$.

The pair $(K, |\ \ |)$ is called a valued field.

It is not hard to see that $|1_K| = 1$, $|-x| = |x|$ and $|x^{-1}| = |x|^{-1}$ for $x \neq 0$. In the rest of the article we will denote the set $K \setminus \{0\}$ by K^*.

DEFINITION 2.2. A valuation $|\ \ |$ on K is called ***non-Archimedean*** if it satisfies the ***strong triangle inequality***: $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$. Otherwise it is called ***Archimedean***.

THEOREM 2.3 ([**49**, 1.1], [**41**, lemma 8.2]). *Let $(K, |\ \ |)$ be a valued field. The following conditions are equivalent:*

(1) *$|\ \ |$ is non-Archimedean.*
(2) *If $a, b \in K$ and $|a| < |b|$, then $|b - a| = |b|$ (Isosceles triangle principle).*
(3) *The set $\{|n1_K| : n \in \mathbb{N}\}$ is bounded.*
(4) *$|n1_K| \leq 1$ for every $n \in \mathbb{N}$.*
(5) *$|2 \cdot 1_K| \leq 1$.*

EXAMPLES 2.4. (1) The usual absolute values in $\mathbb{R}$ and $\mathbb{C}$ are valuations, and since the set $\{|n| : n \in \mathbb{N}\}$ is unbounded, they are Archimedean.

(2) Let K be a field. The map defined by $|x| = 1$ for $x \neq 0$ and $|0| = 0$ is called ***trivial valuation*** and is a non-Archimedean valuation.
(3) If K is a finite field, then the trivial valuation is the only valuation on K, for if there exists $x \in K^*$, $|x| \neq 1$, then the set $\{|x^n| : n \in \mathbb{Z}\}$ is infinite.
(4) If $char(K) \neq 0$ then any valuation on K is non-Archimedean. Indeed, in this case the prime subfield of K (the subfield of K generated by 1_K) is finite. Thus the set $\{|n1_k| : n \in \mathbb{N}\}$ is bounded.
(5) Let p be a prime number. The ***p-adic valuation*** $|\ \ |_p$ on $\mathbb{Q}$ is defined by

$$|0|_p = 0 \quad \text{and} \quad \left| p^v \frac{m}{n} \right| = \frac{1}{e^v}$$

where e is the base of the natural logarithms, $v \in \mathbb{Z}$, and $n, m \in \mathbb{Z} \setminus \{0\}$ are not divisible by p. Since the set $\{|n1|_p : n \in \mathbb{Z}\} = \{e^{-v} : v \in \mathbb{N}\} \cup \{0\}$ is bounded in $\mathbb{R}$, the p-adic valuation on $\mathbb{Q}$ is non-Archimedean.
(6) Let p be an irreducible polynomial in $K[x]$, where K is any field. The p-adic valuation $|\ \ |_p$ on the rational function field $K(x)$ (the quotient field of the ring of polynomials $K[x]$) is defined by $|0|_p = 0$ and

$$\left| p^v \frac{f}{g} \right| = \frac{1}{e^v}$$

where $v \in \mathbb{Z}$, and $f, g \in K[x] \setminus \{0\}$ are not divisible by p. This valuation is non-Archimedean since $\{|n1|_p : n \in \mathbb{Z}\} = \{0, 1\}$. For a description of all valuations on $K(x)$ see [**36**, 3.1.K] and [**2**].

A valuation $|\ |$ on a field K defines the metric $d(x, y) := |x-y|$ for $x, y \in K$. In particular, any valuation on K induces a metrizable topology on K. If the valuation $|\ |$ is non-Archimedean, then the induced metric d is an ultrametric.

DEFINITION 2.5. If two valuations induce the same topology on K, then they are called ***dependent***. If this is not the case then they are called ***independent***.

THEOREM 2.6 ([**14**, 1.1.2]). *Let $|\ |_1$ and $|\ |_2$ be two non-trivial valuations on a field K. The following conditions are equivalent:*

(1) *$|\ |_1$ and $|\ |_2$ are dependent.*
(2) *For all $x \in K$, $|x|_1 < 1$ implies $|x|_2 < 1$.*
(3) *There exists $\lambda > 0$, such that for all $x \in K$, $|x|_1 = (|x|_2)^\lambda$.*

NOTATION 2.7. The symbol $|\ |_0$ will denote the usual absolute value on any subfield of the field of the complex numbers $\mathbb{C}$.

THEOREM 2.8. *Let $(K, |\ |)$ be a valued field and let $|\ |_1 := (|\ |)^\lambda$ for $\lambda \in \mathbb{R}^+$. The following statements are satisfied:*

(1) *if $|\ |$ is non-Archimedean, then $|\ |_1$ and $|\ |$ are dependent valuations.*
(2) *if $|\ |$ is Archimedean, then $R := \{\rho \in \mathbb{R}^+ : |\ |^\rho$ is a valuation$\} = (0, r]$ for some $r \geq 1$. Moreover, if $\lambda \in R$, then $|\ |_1$ and $|\ |$ are dependent.*

PROOF. (1): the verification of the fact that $|\ |_1$ is a non-Archimedean valuation is straight forward. Since $|x| < 1 \Leftrightarrow |x|_1 < 1$, by 2.5 it follows that $|\ |_1$ and $|\ |$ are dependent valuations on K. (2): the proof of this statement can be found in [**36**, 1.2.D and 1.2.E]. $\square$

The next result is an approximation theorem for pairwise-independent valuations on a field.

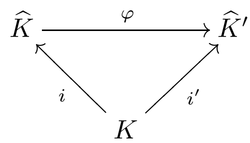
THEOREM 2.9 ([**14**, 1.1.3]). *Let K be a field and $|\ |_1, |\ |_2, \ldots, |\ |_n$ non-trivial pairwise-independent valuations on K. If $x_1, x_2, \ldots, x_n \in K$ and $\varepsilon \in \mathbb{R}^+$, then there exists $x \in K$ such that $|x - x_i|_i < \varepsilon$, for all $i \in \{1, \ldots, n\}$.*

The following theorem implies that in the examples 1.4 we have listed all the possible valuations on $\mathbb{Q}$ up to dependency.

THEOREM 2.10 ([**49**, 1.2](Ostrowski's Theorem)). *Let $|\ |$ be a non-trivial valuation on $\mathbb{Q}$. If $|\ |$ is Archimedean, then there exists $\lambda \in (0, 1]$, such that, $|\ | = (|\ |_0)^\lambda$. Otherwise, there are a prime p and $\lambda > 0$ such that $|\ | = (|\ |_p)^\lambda$.*

2.2. Completion of valued fields.

THEOREM 2.11 ([**14**, 1.1.4]). *Let $(K, |\ |)$ be a valued field. There exists a Cauchy complete valued field $(\widehat{K}, \widehat{|\ |})$, and an embedding $i : K \to \widehat{K}$, such that $|x| = \widehat{|i(x)|}$ for all $x \in K$, and the image $i(K)$ is dense in $\widehat{K}$. If $(\widehat{K}', \widehat{|\ |}', i')$ is another such trio, then there exists a unique isomorphism $\varphi : \widehat{K} \to \widehat{K}'$ satisfying $\widehat{|\varphi(x)|}' = \widehat{|x|}$ for all $x \in \widehat{K}$ and making the following diagram commutative:*



DEFINITION 2.12. A pair $(\widehat{K}, \widehat{|\ \ |})$ as in 2.11 is called a ***completion*** of the valued field $(K, |\ \ |)$.

REMARK 2.13. Let $(K, |\ \ |)$ be a valued field and $(\widehat{K}, \widehat{|\ \ |})$ its completion with embedding $i : K \to \widehat{K}$ such that $|x| = \widehat{|i(x)|}$ for all $x \in K$. Then

$$\{|n1_K| : n \in \mathbb{N}\} = \{\widehat{|i(n1_K)|} : n \in \mathbb{N}\} = \{\widehat{|n1_{\widehat{K}}|} : n \in \mathbb{N}\}.$$

Therefore the completion of an Archimedean valued field is an Archimedean valued field and the completion of a non-Archimedean valued field is a non-Archimedean valued field. In the latter case, we have that $|K| := \{|x| : x \in K\} = \{\widehat{|x|} : x \in \widehat{K}\} =: \widehat{|\widehat{K}|}$ by 1.6(12).

2.3. Completion of Archimedean valued fields.

THEOREM 2.14 ([**47**, 15.2.1](Gelfand-Tornheim's Theorem)). *Let $(K, \|\ \ \|)$ be a normed algebra over $(\mathbb{R}, (|\ \ |_0)^\lambda)$ for some $\lambda \in (0, 1]$. If K is a field, then either $K = \mathbb{R}\cdot 1_K$ or $K = (\mathbb{R}\cdot 1_K)(i)$, where, for the second possibility, $i \in K$ and $i^2 = -1$.*

As a consequence of the Gelfand-Tornheim's Theorem, we obtain a full description of all the Archimedean valued fields and their completions.

THEOREM 2.15 ([**47**, 15.2.2]). *If $(K, |\ \ |)$ is an Archimedean valued field, then there exist $\lambda \in (0, 1]$ and a field monomorphism $\sigma : K \to \mathbb{C}$, such that $|x| = (|\sigma(x)|_0)^\lambda$. Furthermore, if $(K, |\ \ |)$ is Cauchy complete, then either $\sigma(K) = \mathbb{R}$ or $\sigma(K) = \mathbb{C}$.*

2.4. Completion of non-Archimedean valued fields. In this subsection we will give a brief construction of our first non-trivial examples of Cauchy complete, non-Archimedean valued fields. Notice that in a field K with the trivial valuation, every Cauchy sequence is eventually constant, and thus K is complete. Regarding the fields already presented, we anticipate that $\mathbb{Q}$ is not Cauchy complete with respect to the usual absolute value $|\ \ |_0$ nor with respect to any p-adic valuation since $\mathbb{Q}$ is not a Baire space (for details see 2.5). Similarly, the field of rational functions $K(x)$ is not Cauchy complete with respect to the x-adic valuation (this is not trivial and it is proved in 2.5).

As we will see, in contrast with the Archimedean case, in the non-Archimedean case there are different families of Cauchy complete non-Archimedean valued fields.

Let $(K, |\cdot|)$ be a non-Archimedean valued field. Then $K^* = \{\lambda \in K : \lambda \neq 0\}$ is a multiplicative group and so is the ***value group*** of $(K, |\ \ |)$ defined by $|K^*| = \{|\lambda| : \lambda \in K^*\}$. The value group can either be dense in $(0, \infty)$ or discrete, where, for the second possibility, if the valuation is not trivial, then the group $|K^*|$ is cyclic with generator $\rho := \max\{r \in |K^*| : r < 1\}$ ([**47**, B.5.2]). Any element $\pi \in K$ for which $|\pi| = \rho$ is called a ***uniformizer*** for $|\ \ |$. Furthermore, $B[0, 1]$, the "closed" unit disk in K, is a ring and $B(0, 1)$, the "open" unit disk in K, is a maximal ideal of $B[0, 1]$ ([**49**, p. 4]).

Definition 2.16. Let $(K, |\ \ |)$ be a non-Archimedean valued field. The quotient field $B[0,1]/B(0,1)$ is called the ***residue class field*** of K. Moreover, if $|K^*|$ is discrete in $(0,\infty)$, then the valuation $|\ \ |$ is said to be ***discrete*** and if $|K^*|$ is dense in $(0,\infty)$, then the valuation is said to be ***dense***.

Examples 2.17. Fields with discrete valuation.

(1) Consider $(\mathbb{Q}, |\cdot|_p)$ for a prime number p. In this case,
$$|\mathbb{Q}^*| = \{e^{-n} : n \in \mathbb{Z}\} = \langle e^{-1}\rangle = \langle |p|_p\rangle$$
and therefore p is a uniformizer for $|\ \ |_p$.
$$B[0,1] = \left\{\frac{m}{n} : m, n \in \mathbb{Z},\ n \text{ is not divisible by } p\right\}$$
and
$$B(0,1) = \left\{\frac{pm}{n} : m, n \in \mathbb{Z},\ n \text{ is not divisible by } p\right\}.$$
Notice that $B[0,1] = (\mathbb{Z}\setminus(p))^{-1}\mathbb{Z}$, i.e. it is the localization of the ring $\mathbb{Z}$ at the prime ideal $(p) = p\mathbb{Z}$, and $B(0,1) = pB[0,1]$ is the maximal ideal of $B[0,1]$. Using the first isomorphism theorem for rings, we conclude that $B[0,1]/B(0,1) \simeq \mathbb{Z}/p\mathbb{Z}$. In other words, the residue class field of $(\mathbb{Q}, |\cdot|_p)$ is isomorphic to $\mathbb{F}_p$. For a simpler but larger proof see [**36**, p. 62].

(2) Consider $(K(x), |\ \ |_p)$ for an irreducible polynomial $p \in K[x]$, where K is a field. In this case, p is a uniformizer for $|\ \ |_p$,
$$B[0,1] = \left\{\frac{f}{g} : f, g \in K[x],\ g \text{ is not divisible by } p\right\}$$
and
$$B(0,1) = \left\{\frac{pf}{g} : f, g \in K[x],\ g \text{ is not divisible by } p\right\}$$
Analogously to the previous example, $B[0,1]$ is the localization of the ring $K[x]$ at the prime ideal (p), and $B(0,1) = pB[0,1]$ is the maximal ideal of $B[0,1]$. Thus the residue class field of $(K(x), |\ \ |_p)$ is isomorphic to $K[x]/(p)$. In particular, if $p(x) = x$, then the residue class field of $(K(x), |\ \ |_p)$ is isomorphic to K. For another proof see [**36**, p. 88].

Theorem 2.18 ([**14**, 1.3.4]). *Let $(K, |\ \ |)$ be a non-Archimedean valued field and $(\widehat{K}, \widehat{|\ \ |})$ its completion. If k and $\widehat{k}$ are their respective residue class fields, then $k \simeq \widehat{k}$ and $|K^*| = \widehat{|\widehat{K}^*|}$.*

The following result is going to bring us an explicit description of the completions of $(\mathbb{Q}, |\ \ |_p)$ and $(K(x), |\ \ |_p)$.

Theorem 2.19 ([**14**, 1.3.5]). *Let $|\ \ |$ be a discrete valuation on the field K, with uniformizer π and residue class field k. Then every element $x \in K^*$ can be written uniquely as a convergent series*
$$x = r_v\pi^v + r_{v+1}\pi^{v+1} + r_{v+2}\pi^{v+2} + \cdots = \lim_{n\to\infty}\sum_{i=v}^{n} r_i\pi^i$$
where $v = \log_{|\pi|}|x|$, $r_v \neq 0$, and the coefficients r_i are taken from a set $R \subset B[0,1]$ of representatives of the residue classes in k (i.e., the canonical map $B[0,1] \to k$ induces a bijection of R onto k).

REMARK 2.20. If π is a uniformizer then $|K^*| = \{|\pi|^n : n \in \mathbb{Z}\}$. Thus, for every $x \in K^*$, there is $v \in \mathbb{Z}$ such that $|x| = |\pi|^v$. Hence $\log_{|\pi|} |x| \in \mathbb{Z}$.

DEFINITION 2.21. Let p be a prime number. The completion of the field $(\mathbb{Q}, |\ \ |_p)$ is called the ***field of p-adic numbers*** and is denoted by $\mathbb{Q}_p$.

We checked in 2.17 that p is a uniformizer for $(\mathbb{Q}, |\ \ |_p)$ and that $\mathbb{F}_p$ is its residue class field. By 2.18, $\mathbb{Q}_p$ has the same residue class field and p as a uniformizer. According to 2.19, by taking $R = \{0, 1, \ldots, p-1\}$, this field has the following description

$$\mathbb{Q}_p = \left\{ \sum_{i=v}^{\infty} r_i p^i : v \in \mathbb{Z}, r_i \in R, \, r_v \neq 0 \right\} \cup \{0\},$$

where the valuation on $\mathbb{Q}_p$ is defined as $|0| = 0$ and $\left| \sum_{i=v}^{\infty} r_i p^i \right| = e^{-v}$ when $r_v \neq 0$.

The closed disk in $\mathbb{Q}_p$ of center 0 and radius 1, is the ***ring of p-adic integers***:

$$B[0,1] = \mathbb{Z}_p := \left\{ \sum_{i=0}^{\infty} r_i p^i : r_i = 0, 1, \ldots, p-1 \right\}.$$

Be aware of the fact that addition of two 'series' of the form $\sum_{i=v}^{\infty} r_i p^i$ is not coefficientwise, as the set $\{0, 1, \ldots, p-1\}$ is not closed under addition. As a simple example observe that choosing $p = 7$,

$$5p^i + 4p^i = 5p^i + 2p^i + 2p^i = 7p^i + 2p^i = p^{i+1} + 2p^i.$$

DEFINITION 2.22. Let K be a field and $p = x \in K[x]$. The completion of $(K(x), |\ \ |_p)$ is called the ***field of formal Laurent series*** and denoted by $K((x))$.

In 2.17, we saw that x is a uniformizer for $(K(x), |\ \ |_x)$ and that K is the residue class field of this valued field. By 2.18, $K((x))$ has x as a uniformizer and K as the residue class field as well. As a consequence of the previous theorem, this field has the following description

$$K((x)) = \left\{ \sum_{i=v}^{\infty} r_i x^i : v \in \mathbb{Z}, r_i \in K, \, r_v \neq 0 \right\} \cup \{0\},$$

where the valuation on $K((x))$ is defined as $|0| = 0$ and $\left| \sum_{i=v}^{\infty} r_i x^i \right| = e^{-v}$ when $r_v \neq 0$. The closed disk in $K((x))$ of center 0 and radius 1, is the ***ring of formal power series***:

$$B[0,1] = K[[x]] := \left\{ \sum_{i=0}^{\infty} r_i x^i : r_i \in K \right\}.$$

Notice that $K((x))$ is the quotient field of $K[[x]]$ ([**36**, 3.1.L]). Here the addition of two such series is defined coefficientwise and the multiplication of two nonzero series is defined as follows:

$$\left(\sum_{j=v_1}^{\infty} s_j x^j \right) \cdot \left(\sum_{k=v_2}^{\infty} t_k x^k \right) = \sum_{i=v}^{\infty} r_i x^i,$$

with $v = v_1 + v_2$, $r_i = \sum_{j+k=i} s_j t_k = \sum_{j=v_1}^{\infty} s_j t_{i-j} = \sum_{k=v_2}^{\infty} s_{i-k} t_k$, where $s_j = 0$ for $j < v_1$ and $t_k = 0$ for $k < v_2$.

Now, the following result is a direct consequence of 1.18.

THEOREM 2.23. *The valued fields* $\mathbb{Q}_p$ *and* $K((x))$ *are spherically complete.*

The following theorem shows how different a Cauchy complete non-Archimedean valued field is from their Archimedean analogs $\mathbb{R}$ and $\mathbb{C}$.

THEOREM 2.24 ([**4**, II.1.1]). *Let* $(K, |\ |)$ *be a Cauchy complete non-Archimedean valued field. If* $(x_n)_n$ *is a sequence of elements of* K*, then*

$$\sum_{n=1}^{\infty} x_n \text{ is convergent in } K \Leftrightarrow \lim_{n\to\infty} x_n = 0.$$

2.5. Incompleteness of $\mathbb{Q}$ and $K(x)$. In the following we will discuss the incompleteness of $\mathbb{Q}$ and $K(x)$ with respect to their non-Archimedean valuations. Since $\mathbb{Q}_p$ is the completion of $(\mathbb{Q}, |\ |_p)$, we can identify the elements of $\mathbb{Q}$ as elements of $\mathbb{Q}_p$. For example,

$$\frac{1}{1-p} = \sum_{i=0}^{\infty} p^i \qquad \text{and} \qquad -1 = \sum_{i=0}^{\infty} (p-1)p^i.$$

The algorithm to compute the coefficients of the p-adic expansion of an arbitrary rational number can be found in [**37**] in the proof of the following result.

THEOREM 2.25 ([**37**, 5.3]). *Let* $x = \sum_{i=v}^{\infty} r_i p^i \in \mathbb{Q}_p$ $(v \in \mathbb{Z},\ 0 \le r_i \le p-1)$. *Then* x *is a rational number if and only if the sequence* $(r_i)_i$ *of digits of* x *is eventually periodic, i.e. there exists* $n \in \mathbb{N}$ *such that the subsequence* $(r_i)_{i\ge n}$ *is periodic.*

We can use 2.25 to prove the incompleteness of $(\mathbb{Q}, |\ |_p)$. Consider $x = \sum_{i=0}^{\infty} p^{i^2} \in \mathbb{Q}_p$. Note that x is the limit of a convergent sequence of rational numbers, say $a_n = \sum_{i=0}^{n} p^{i^2}$. In fact, $|x - a_n|_p = e^{-(n+1)}$ for every $n \in \mathbb{N}$. Thus $(a_n)_n$ is a Cauchy sequence in $(\mathbb{Q}, |\ |_p)$ but 2.25 implies that $x \notin \mathbb{Q}$, i.e. $(a_n)_n$ is not convergent in $(\mathbb{Q}, |\ |_p)$. Another argument to prove the incompleteness of $(\mathbb{Q}, |\ |_p)$ is the following: if $(\mathbb{Q}, |\ |_p)$ is Cauchy complete, then it must be isomorphic to its completion. But $\mathbb{Q}$ is countable and $\mathbb{Q}_p$ is not, leading to a contradiction. Among the proofs of the incompleteness of $(\mathbb{Q}, |\ |_p)$ that don't invoke $\mathbb{Q}_p$, we highlight the following. By the Baire category theorem ([**28**, 48.2]), the space $(\mathbb{Q}, |\ |_p)$ is not Cauchy complete. In fact, for every $q \in \mathbb{Q}$, the set $\mathbb{Q} \setminus \{q\}$ is open and dense in $(\mathbb{Q}, |\ |_p)$, since $(q + p^{N+n})_n$ is a sequence of elements of $\mathbb{Q} \setminus \{q\}$ that converges to q, with $N = -\ln(|q|_p)$. Nevertheless $\bigcap_{q\in\mathbb{Q}} (\mathbb{Q} \setminus \{q\}) = \emptyset$, therefore, the space $(\mathbb{Q}, |\ |_p)$ is not Baire, hence it is not complete.

Let K be any field. Since $K((x))$ is the completion of $(K(x), |\ |_x)$, we can identify the elements of $K(x)$ as elements of $K((x))$. For example,

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i \qquad \text{and} \qquad \frac{1}{1-x-x^2} = 1 + \sum_{i=1}^{\infty} F_i x^i,$$

where $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ $(n > 1)$ are the Fibonacci numbers.

The algorithm to compute the coefficients of the x-adic expansion of an arbitrary rational function is analogous to the algorithm used to compute the coefficients of the p-adic expansion of rational numbers.

For the identification of rational functions in $K((x))$, the periodicity of the coefficients of the x-adic expansion is not adequate for such description and a weaker condition is needed: linear recurrence.

Definition 2.26. Let K be a field. A sequence $(r_n)_n$ of elements of K is ***linearly recurrent*** if there exist constants $c_1, c_2, \ldots, c_k \in K$ such that

$$r_{n+k} = c_1 r_n + c_2 r_{n+1} + \cdots + c_k r_{n+k-1} \tag{2.1}$$

for all $n \in \mathbb{N}$.

Remark 2.27. Linearly recurrent sequences are generalizations of periodic sequences. A periodic sequence with least period k is a linearly recurrent sequence $(r_n)_n$ with coefficients $c_1 = 1$ and $c_2 = c_3 = \cdots = c_k = 0$ satisfying the relation 2.1.

Theorem 2.28 ([**23**, V.5. Lemma 5], [**36**, 3.1.N]). *Let $f(x) = \sum_{i=0}^{\infty} r_i x^i \in K[[x]]$, where K is any field. For $m, s \geq 0$, let $A_{s,m}$ be the matrix $\{r_{s+i+j}\}_{0\leq i,j\leq m}$:*

$$\begin{pmatrix} r_s & r_{s+1} & r_{s+2} & \cdots & r_{s+m} \\ r_{s+1} & r_{s+2} & r_{s+3} & \cdots & r_{s+m+1} \\ r_{s+2} & r_{s+3} & r_{s+4} & \cdots & r_{s+m+2} \\ \vdots & \vdots & \vdots & & \vdots \\ r_{s+m} & r_{s+m+1} & r_{s+m+2} & \cdots & r_{s+2m} \end{pmatrix}$$

and let $N_{s,m} := \det(A_{s,m})$. Then $f(x) \in K(x)$ if and only if there exist integers $m \geq 0$ and S such that $N_{s,m} = 0$ whenever $s \geq S$, or equivalently, the sequence $(r_n)_n$ is eventually linearly recurrent, i.e. there exists $t \in \mathbb{N}$ such that the subsequence $(r_n)_{n\geq t}$ is linearly recurrent.

Now we are in position to prove the incompleteness of $(K(x), |\ |_x)$. Let $f(x) = \sum_{i=0}^{\infty} x^{i^2} \in K((x))$. Notice that $f(x)$ is the limit of a sequence of rational functions, say $a_n(x) = \sum_{i=0}^{n} x^{i^2} \in K[x]$. Thus $(a_n(x))_n$ is a Cauchy sequence in $K(x)$. We will show that $(a_n(x))_n$ is not convergent in $K(x)$ by showing that $f(x) \notin K(x)$.

Given m and S non-negative integers, choose a large enough $n \in \mathbb{N}$ such that $S < (n-1)^2 < n^2 - m < n^2 < n^2 + m < (n+1)^2$. By putting $\ell := n^2 - m$ we have

$$S < (n-1)^2 < \ell < \ell + m = n^2 < \ell + 2m < (n+1)^2.$$

Since all the members of an anti-diagonal of $A_{\ell,m}$ are equal, it follows that,

$$A_{\ell,m} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & ⋰ & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

and $N_{\ell,m} \neq 0$ with $\ell > S$. Then by 2.28, $f(x) \notin K(x)$.

2.6. Compact and Locally compact valued fields.

THEOREM 2.29 ([**36**, 1.7 Lemma 6]). *Every compact valued field is finite.*

THEOREM 2.30 ([**36**, 1.7 Theorem 5]). *Let $(K, |\ |)$ be a locally compact valued field. Then $(K, |\ |)$ is Cauchy complete.*

The next theorem classifies the locally compact Archimedean valued fields.

THEOREM 2.31 ([**36**, 1.7 Theorem 6]). *If $(K, |\ |)$ is a locally compact Archimedean valued field, then there exist $\lambda \in (0, 1]$ and a field monomorphism $\sigma : K \to \mathbb{C}$ such that $|x| = (|\sigma(x)|_0)^\lambda$ for all $x \in K$, satisfying either $\sigma(K) = \mathbb{R}$ or $\sigma(K) = \mathbb{C}$.*

PROOF. It follows from 2.30 and 2.15. □

The next two results describe every locally compact non-Archimedean valued field implicitly and explicitly respectively.

THEOREM 2.32 ([**41**, 12.2], [**49**, 1.B]). *A non-Archimedean valued field is locally compact if and only if it is Cauchy complete, its value group is discrete and its residue class field is finite.*

THEOREM 2.33 ([**49**, 3.U], [**7**, VI.9.3.1]). *Let $(K, |\ |)$ be a locally compact non-Archimedean valued field with residue class field k. The following statements are satisfied:*

1. *If $char(K) = 0$, then K is a finite algebraic extension of K_0, where K_0 is the closure of the prime subfield of K. Furthermore, there exists an isomorphism $T : K_0 \to \mathbb{Q}_p$ satisfying $|x| = (|T(x)|_p)^\tau$ for all $x \in K_0$, where $p = char(k)$ and $\tau > 0$ is such that $|p1_K| = p^{-\tau}$.*
2. *If $char(K) = p$, then $char(k) = p$ and K is isomorphic to $k((x))$. In particular, there exists a field monomorphism $\theta : k \to K$ such that the map $\sigma : k((x)) \to K$, $\sigma(\sum_{i=v}^{\infty} r_i x^i) = \sum_{i=v}^{\infty} \theta(r_i)\pi^i$ is an isometric isomorphism where π is a uniformizer for $|\ |$ and the valuation on $k((x))$ is defined by $\| \sum_{i=v}^{\infty} r_i x^i \| := |\pi|^v$ (for $r_v \neq 0$).*

COROLLARY 2.34. *If $(K, |\ |)$ is a locally compact non-Archimedean valued field, then it is spherically complete.*

PROOF. It follows immediately from 2.32 and 1.18. □

3. Ordered fields.

In this section we will present some examples of ordered fields and will discuss the concept of Archimedean extension of a field created by Hans Hahn in 1907 [**16**].

3.1. Formally real fields. By a ring, we will mean a commutative ring with unit $1 \neq 0$. Let A be a ring that is an ordered set such that its additive group $(A, +)$ is an ordered group (it has a total ordering which is compatible with the addition). The ring A is ordered if for all $x, y \in A$, $x > 0$ and $y > 0$ implies $xy > 0$. Note that an ordered ring is necessarily an integral domain. A field that is an ordered ring will be called an ***ordered field***.

DEFINITION 3.1. A field K is ***formally real*** if satisfies the following condition: given $a_1, \dots, a_n \in K$, such that, $\sum_{i=1}^{n} a_i^2 = 0$, then $a_1 = \cdots = a_n = 0$.

The following result characterizes the formally real fields as the fields that can be ordered.

THEOREM 3.2 ([**3**, 1.70(5) and 1.71(6)]). *Let K be a field. The following conditions are equivalent:*

(1) *K is formally real,*
(2) *-1 is not a sum of squares in K,*
(3) *There exists an order $\leq$ on K such that $(K, \leq)$ is an ordered field.*

EXAMPLES 3.3. (1) If $char(K) \neq 0$ then, $0 = \sum_{i=1}^{char(K)} 1^2$. Hence K is not formally real. Thus if K is formally real, then $char(K) = 0$.
(2) The field of complex numbers $\mathbb{C}$ cannot be an ordered field, since $-1 = i^2$ and therefore it is not formally real.
(3) If K is an ordered field then we can define an order in $K((x))$, which is compatible with the addition and multiplication. Thus $K((x))$ can be ordered, and therefore it is formally real. Such order is defined as follows: for every $z \in K((x))$ there are $r_i \in K$ such that $z = \sum_{i=v}^{\infty} r_i x^i$. We say that $z > 0$ if $z \neq 0$ and $r_v > 0$. Then $z_1 > z_2$ if $z_1 - z_2 > 0$.
(4) $\mathbb{Q}_p$ is not formally real because if $p = 2$, then -7 is a square and if $p > 2$ then $1 - p$ is a square ([**36**, p. 144]). Recall that in a formally real field the squares are non-negative elements. Since $\mathbb{Q} \subset \mathbb{Q}_p$, $char(\mathbb{Q}_p) = 0$.

3.2. General Hahn fields and the Embedding theorem. Let's review the concept of Archimedean extension of a field and the general Hahn fields.

DEFINITION 3.4. Let S be an ordered group. Two elements $x, y \in S^*$ are ***comparable*** if there exist $n, m \in \mathbb{N}$ such that $|x|_0 < n|y|_0$ and $|y|_0 < m|x|_0$, where

$$|a|_0 := \begin{cases} a & , a \geq 0 \\ -a & , a < 0 \, . \end{cases}$$

Let K be an ordered field. The relation of being comparable is an equivalence relation on K^* and to denote 'x and y are comparable' we write $x \sim y$. This relation defines a partition of K^* into equivalence classes, which are called the ***Archimedean classes of*** K. The equivalence class of $x \in K$ is denoted by $[x]$. Let's denote the class of all the Archimedean classes by G_K.

THEOREM 3.5. *Let K be an ordered field. The class G_K is an ordered abelian group under the order $\prec$ and addition $+$ defined as follows: for every $x, y \in K^*$*

(1) $[x] \prec [y] \iff \forall n \in \mathbb{N}, n|y|_0 < |x|_0 \iff y \not\sim x$ *and* $|y|_0 < |x|_0$.
(2) $[x] + [y] := [xy]$

In this group, the neutral element is $[1_K]$, and $-[x] = [x^{-1}]$.

DEFINITION 3.6. An ordered field K is ***Archimedean*** if $G_K = \{[1_K]\}$, i.e. when every two elements in K^* are comparable.

THEOREM 3.7. *An ordered field K is Archimedean if and only if it satisfies the Archimedean property, i.e. for every $x \in K$, there exist $n \in \mathbb{N}$ such that $|x|_0 < n1_K$.*

PROOF. If K is Archimedean, then every $x \in K^*$ is comparable with 1_K. Hence K satisfies the Archimedean property. If K is not Archimedean then there exist $x \in K^*$ such that $[x] \prec [1]$. Thus $n1_K < |x|_0$ for every $n \in \mathbb{N}$. □

Thus for every ordered field K, the group G_K measures the 'Archimedicity' or the 'non-Archimedicity' of K. The field $\mathbb{R}$ of real numbers (the only ordered, Dedekind complete field up to isomorphism) is characterized by the fact that each Archimedean ordered field can be embedded in $\mathbb{R}$ ([**17**, 3.5]). Hans Hahn in [**16**] (1907) generalized this property (see 3.13) and by doing so he ended up with ordered fields that extend all the ordered fields with a given "level of Archimedicity".

Definition 3.8. Let E/K be an extension of ordered fields, where the order on E restricted to K coincides with that of K. The field E is an ***Archimedean extension of*** K if every $x \in E$ is comparable to some $y \in K$. In that case, G_E and G_K are isomorphic ordered groups. An ordered field K is called ***Archimedean complete*** if it has no proper archimedean extension fields.

Definition 3.9. Let K be an ordered field. If G is an ordered abelian group isomorphic to G_K, then we say that ***K is of type G*** and G is called an ***Archimedean group of K***.

Theorem 3.10. *The field $\mathbb{R}$ is (up to isomorphism) the only Archimedean complete, ordered field of type $\{0\}$.*

Proof. An ordered field K is of type $\{0\}$ and Archimedean complete if and only if K is Hilbert complete (K is Archimedean and has no proper ordered Archimedean field extensions), which is in turn equivalent to being Archimedean and Dedekind complete ([**17**, 3.11]). □

Theorem 3.11 ([**3**, 6.20, 6.21, 7.32], [**11**, 2.15], [**16**]). *Let K be a field (not necessarily ordered) and G an ordered abelian group. The set*

$$K((G)) := \{f : G \to K : supp(f) \text{ is well-ordered}\},$$

where $supp(f) := \{x \in G : f(x) \neq 0\}$, is a field under the addition and multiplication defined as follows: for every $f, g \in K((G))$ and $x \in G$,

(1) $(f + g)(x) := f(x) + g(x)$,
(2) $fg(x) := \sum_{a+b=x} f(a)g(b)$

Fields of the form $K((G))$ are called ***general Hahn fields****.*

When K is an ordered field we can define an order on $K((G))$ generalizing the definition of the order in $K((x))$ (3.3).

Definition 3.12 (Ordered general Hahn fields). Let K be an ordered field and consider $\lambda : K((G))^* \to G$, $\lambda(f) = \min\{supp(f)\}$. For $f, g \in K((G))$ we define:

$$f < g \Leftrightarrow f \neq g \text{ and } (g - f)(\lambda(g - f)) > 0.$$

Then $(K((G)), \leq)$ is an ordered field.

Theorem 3.13 is crucial on ordered structures, while 3.14 generalizes 3.10.

Theorem 3.13 ([**10**], [**18**, 3.1], [**3**, 1.64], [**11**, 1.35], [**16**] (Hahn's Embedding Theorem)). *If K is an ordered field, then for every archimedean group G of K, there exists an order-preserving field monomorphism σ from K into $\mathbb{R}((G))$ such that $\mathbb{R}((G))$ is an Archimedean extension of $\sigma(K)$.*

Theorem 3.14 ([**10**, pp. 862–863], [**18**, 3.2], [**16**] (Hahn's Completeness Theorem)). *If G is an ordered abelian group, then the field $\mathbb{R}((G))$ is (up to isomorphism) the only Archimedean complete, ordered field of type G.*

3.3. Hahn Fields and Levi-Civita fields. In this subsection we will define a non-Archimedean valuation in some general Hahn fields and the family of the Levi-Civita fields will be introduced.

Definition 3.15. A ***Hahn field*** is a general Hahn field $K((G))$ for which G is a subgroup of $(\mathbb{R}, +)$ and K is any field.

The distinctive characteristic of a Hahn field is that we can define in a natural way a non-Archimedean valuation on them.

Theorem 3.16 ([**41**, A.9 pp. 288-292], [**42**, II.6 corollary, p. 51]). *Let G be a subgroup of $(\mathbb{R}, +)$ and K any field. If the map $|\ | : K((G)) \to \mathbb{R}$ is defined by*

$$|f| := \begin{cases} e^{-\min\{supp(f)\}} & , f \neq 0 \\ 0 & , f = 0, \end{cases}$$

then $(K((G)), |\ |)$ is a Cauchy complete non-Archimedean valued field with residue class field isomorphic to K and value group $|K((G))^*| = \{e^g \in \mathbb{R} : g \in G\}$. Moreover it is spherically complete.

When a field has a discrete valuation, then every nonzero element can be written in a unique way as a limit of a convergent power series (2.19). The following result shows that in some Hahn fields this also is possible when the valuation is dense.

Theorem 3.17. *Let K be a field and let $d : \mathbb{Q} \to K$ be the function defined by*

$$d(x) := \begin{cases} 1 & , x = 1 \\ 0 & , x \neq 1. \end{cases}$$

Then d is an element of the field $K((\mathbb{Q}))$, and for any $r \in \mathbb{Q}$, we have that

$$d^r(x) = \begin{cases} 1 & , x = r \\ 0 & , x \neq r. \end{cases}$$

The value group of $(K((\mathbb{Q})), |\ |)$ is $\{e^{-r} = |d^r| = |d|^r : r \in \mathbb{Q}\}$. Furthermore, every nonzero element f in $K((\mathbb{Q}))$ is the sum of a convergent generalized power series with respect to the valuation on $K((\mathbb{Q}))$, specifically:

$$f = \sum_{r \in \mathbb{Q}} f(r) d^r = \sum_{r \in supp(f)} f(r) d^r.$$

Additionally, every generalized power series of the form $\sum_{r \in \mathbb{Q}} a_r d^r$ for which $\{r \in \mathbb{Q} : a_r \neq 0\}$ forms a well ordered subset of $\mathbb{Q}$, is convergent in $K((\mathbb{Q}))$, and if two series of such form differ in at least one coefficient then their sum are different.

Theorem 3.18 ([**49**, 1.3]). *Let K be any field and let G be a subgroup of $(\mathbb{R}, +)$.*

$$L[G, K] := \{f : G \to K \mid supp(f) \cap (-\infty, n] \text{ is finite for every } n \in \mathbb{Z}\}$$

is a subfield of $K((G))$. When we restrict the valuation of $K((G))$ to $L[G, K]$, the latter becomes a Cauchy complete, non-Archimedean valued field with residue class field isomorphic to K and value group $|L[G, K]^*| = \{e^g : g \in G\}$. Fields of the form $L[G, K]$ are called ***Levi-Civita fields****.*

Theorem 3.19. *Let K be any field and G be a subgroup of $(\mathbb{R}, +)$. Then:*

1. *The fields $K((G))$ and $L[G, K]$ coincide if and only if G is discrete.*
2. *The field $L[G, K]$ is spherically complete if and only if G is discrete.*

(3) *If K is an ordered field, then $K((G))$ is an Archimedean extension of $L[G,K]$ with respect to the order defined in* 3.12. *If in addition K is Archimedean, then both $K((G))$ and $L[G,K]$ are of type G (see* 3.9*).*

PROOF. For (1), if G is not discrete, then it is dense in $\mathbb{R}$ by [**47**, B.5.2]. Thus there exists a sequence $(\alpha_n)_n$ of elements of G which is strictly increasing and converging to 0. Hence the function $f : G \to K$ defined by

$$f(g) = \begin{cases} 1_K & , g \in \{\alpha_n : n \in \mathbb{N}\} \\ 0 & , \text{else} \end{cases}$$

has a well-ordered support such that $supp(f) \cap (-\infty, 0]$ is infinite. Therefore $f \in K((G)) \setminus L[G,K]$. Conversely, if G is discrete then there exists $r > 0$ such that $G = r\mathbb{Z}$ ([**47**, B.5.2]). Let S be a well-ordered subset of G. Since $S \cap (-\infty, n]$ is finite for all $n \in \mathbb{Z}$, it follows that $K((G)) = L[G,K]$. For the proof of the statement (2) we will use terminology and a result from the section 6. By 3.18 and 3.16 it follows that $K((G))$ is an immediate extension field of $L[G,K]$. Therefore $L[G,K]$ is maximally complete if and only if $L[G,K] = K((G))$, since $K((G))$ is always maximally complete ([**42**, corollary, p. 51], [**24**, p. 193]). By (1), $L[G,K]$ is maximally complete if and only if G is discrete. Thus (2) follows from 6.12. □

EXAMPLE 3.20. If $G = (\mathbb{Z}, +)$ and K is any field, then $L[\mathbb{Z},K] = K((\mathbb{Z}))$ is isomorphic to $K((x))$ by the isomorphism $\varphi : K((x)) \to L[\mathbb{Z},K]$ defined by

$$\varphi\left(\sum_{i=m}^{\infty} a_i x^i\right) = \sum_{i=m}^{\infty} a_i d^i,$$

with $|\varphi(q)| = |q|_x$ for each $q \in K((x))$, where $|\ \ |_x$ is the x-adic valuation in $K((x))$.

EXAMPLE 3.21. The fields $\mathbb{F}_p((\mathbb{Z})) = L[\mathbb{Z},\mathbb{F}_p]$ and $\mathbb{Q}_p$ are Cauchy complete with respect to their valuations, both have the same value group $\{e^n : n \in \mathbb{Z}\}$, and their residue class fields are isomorphic to $\mathbb{F}_p$. However, these fields are not isomorphic since $L[\mathbb{Z},\mathbb{F}_p]$ has characteristic p while $\mathbb{Q}_p$ has characteristic 0.

3.4. Real-closed field extensions of $\mathbb{R}$. In this section we will study real closed field extensions of $\mathbb{R}$ with non-Archimedean valuations, in particular, we will see that under some conditions, the smallest among such (proper) extensions is a Levi-Civita field. Recall that a field K is ***algebraically closed*** if every polynomial in $K[x]$ has a root in K. If L/K is a field extension then $a \in L$ is ***algebraic*** over K if it is the root of a polynomial in $K[x]$. If every element of L is algebraic over K, then L is an ***algebraic extension*** of K. Also, K is ***real-closed*** if K is formally real and does not admit a proper algebraic extension that is formally real.

THEOREM 3.22 ([**26**, Chapter XI], [**3**, 1.71(21),1.71(22)], [**8**, 5.4.4], [**6**, Chapter 5, Section 4, Lemma 4.1]). *Let K be a field. The following conditions are equivalent:*

(1) *K is real-closed,*
(2) *x^2+1 is irreducible in K and $K(i)$ is algebraically closed ($i^2 = -1$),*
(3) *K is an ordered field, each positive element of K has a square root and every $p \in K[x]$ of odd degree has a root in K,*
(4) *any sentence in the first-order language of fields is true in K if and only if it is true in $\mathbb{R}$,*
(5) *K is an ordered field and the intermediate value theorem holds for all polynomials over K.*

Therefore, in order to develop a theory of Calculus over ordered fields for which the intermediate value theorem holds, then our base field has to be real-closed.

THEOREM 3.23 ([**3**, 6.23 (1)–(2)]). *Let K be a field and G be an ordered abelian group. Then $K((G))$ is real-closed if and only if K is real-closed and G is divisible.*

THEOREM 3.24 ([**46**, p. 218]). *The Levi-Civita field $L[\mathbb{Q},\mathbb{R}]$ is real-closed under the order defined in* 3.12.

DEFINITION 3.25. Let (x_n) be a sequence of elements in an ordered field K. Then (x_n) is ***Cauchy*** if for every 0-neighborhood U with respect to the order topology in K, there exists $N \in \mathbb{N}$ such that $x_m - x_n \in U$ for all $m, n \geq N$. The sequence (x_n) is ***convergent*** to $x \in K$ if for every 0-neighborhood U with respect to the order topology in K, there exists $N \in \mathbb{N}$ such that $x_n - x \in U$ for all $n \geq N$. An ordered field K is said to be ***Cauchy complete*** if every Cauchy sequence of K is convergent in the order topology.

THEOREM 3.26 ([**45**, 3.11]). *Let $K/\mathbb{R}$ be a field extension where K is a Cauchy complete ordered field such that: (1) the order on K extends the one in $\mathbb{R}$, (2) there exists $\delta \in K$ such that $0 < \delta < r$ for every $r \in \mathbb{R}^+$, and (δ^n) converges to 0 in the order topology, (3) for all $x \in K, x > 0$, for every $n \in \mathbb{N}$, there exists $y \in K, y > 0$ such that $x = y^n$. If d is the function defined in* 3.17, *then there exists an order-preserving field monomorphism $\sigma : L[\mathbb{Q},\mathbb{R}] \to K$ defined by*

$$\sigma(f) = \sigma\left(\sum_{q \in supp(f)} f(q) d^q\right) = \sum_{q \in supp(f)} f(q)\delta^q.$$

In particular, we have the following result:

COROLLARY 3.27. *Let $K/\mathbb{R}$ be a field extension where K is a Cauchy complete ordered field such that: (1) the order on K extends the one in $\mathbb{R}$, (2) there exists $\delta \in K$ such that $0 < \delta < r$ for every $r \in \mathbb{R}^+$, and (δ^n) converges to 0 in the order topology, (3) it is real-closed. If d is the function defined in* 3.17, *then there exists an order-preserving field monomorphism $\sigma : L[\mathbb{Q},\mathbb{R}] \to K$ defined by*

$$\sigma(f) = \sigma\left(\sum_{q \in supp(f)} f(q) d^q\right) = \sum_{q \in supp(f)} f(q)\delta^q.$$

REMARK 3.28. The embedding mentioned in 3.26 and 3.27 may exist even when the field K does not satisfy the condition (2). For example, the real closed field $\mathbb{R}((\mathbb{Q}[x]))$ extends $L[\mathbb{Q},\mathbb{R}]$, but it does not satisfy the condition (2). In fact, we assert the following:

Claim: If K is an ordered field such that G_K has an infinite subset of pairwise non-comparable elements that is cofinal in G_K (3.4), then for each $x \in K^*$, the sequence $(x^n)_n$ does not converge to 0 in the order topology.

For example, the field $\mathbb{R}((\mathbb{Z}[x]))$ satisfies the hypothesis of the claim but $\mathbb{R}((\mathbb{Z}^3))$ does not. ***Proof:*** By the Hahn's embedding theorem 3.13, the field K can be embedded in $\mathbb{R}((G_K))$ so we can consider the Hahn valuation on K (5.7, (3)). Let $x \in K^*$ and $\lambda(x) = \min\{supp(x)\} \in G_K$. Then $\lambda(x^n) = n\lambda(x) \in G_K$ for all $n \in \mathbb{N}$. By hypothesis we can choose $\varepsilon \in K$, $\varepsilon > 0$ such that $\lambda(\varepsilon) > n\lambda(x)$ for all $n \in \mathbb{N}$. If $|x|_0 = \max\{x, -x\}$, then $\lambda(x^n) = \lambda((|x|_0)^n) = \lambda((|x|_0)^n - \varepsilon)$ for all $n \in \mathbb{N}$. Thus $((|x|_0)^n - \varepsilon)(\lambda((|x|_0)^n - \varepsilon)) = ((|x|_0)^n - \varepsilon)(\lambda((|x|_0)^n)) = (|x|_0)^n(\lambda((|x|_0)^n)) > 0.$

Therefore $(|x|_0)^n > \varepsilon$ for all $n \in \mathbb{N}$. In other words, $x^n \notin (-\varepsilon, \varepsilon)$ for all $n \in \mathbb{N}$. Hence, $(x^n)_n$ does not converge to 0 in the order topology.

The proof of 3.26 ([**45**, 3.11]) can be adapted to prove the following:

COROLLARY 3.29. *Let $K/\mathbb{R}$ be a field extension where K is real-closed and Cauchy complete with respect to a non-trivial valuation $\widehat{|\ |}$ such that $\widehat{|x|} = 1$ for all $x \in \mathbb{R}^*$. Then, for every $\delta \in K^*$ such that $\widehat{|\delta|} < 1$, there exists a field monomorphism $\sigma : L[\mathbb{Q}, \mathbb{R}] \to K$ defined by*

$$\sigma(f) = \sigma\left(\sum_{q \in supp(f)} f(q) d^q\right) = \sum_{q \in supp(f)} f(q) \delta^q,$$

satisfying $\widehat{|\sigma(f)|} = |f|^\tau$ for all $f \in L[\mathbb{Q}, \mathbb{R}]$, where $\tau > 0$ is such that $\widehat{|\delta|} = |d|^\tau$.

4. Algebraic closure of valued fields and their completions.

In this section we will present non-Archimedean valued fields that are algebraic closure of other valued fields, like certain Puiseux series fields or the algebraic closure of $\mathbb{Q}_p$. Also we will review the completion of algebraic closures like the field of p-adic complex numbers $\mathbb{C}_p$, or the completion of a Puiseux series field.

Recall that an ***algebraic closure*** of a field K is an algebraically closed, algebraic extension of K. Each field has an algebraic closure and any two algebraic closures of a field K are isomorphic by means of an isomorphism leaving K pointwise fixed ([**27**, I.8.25], [**20**, 66], [**26**, V 2.5, 2.9]). The algebraic closure of K will be denoted by K^a.

4.1. General Results.

THEOREM 4.1 ([**41**, 14.1] (Krull's Existence Theorem)). *Let K be a subfield of a field L and let $|\ \ |$ be a non-Archimedean valuation on K. Then there exists a non-Archimedean valuation on L that extends $|\ \ |$.*

THEOREM 4.2 ([**41**, 14.2] (Krull's Uniqueness Theorem)). *Let L/K be an algebraic field extension. If $|\ \ |$ is a non-Archimedean valuation on K such that $(K, |\ \ |)$ is Cauchy complete, then there exists a unique valuation on L that extends $|\ \ |$. This extension is also non-Archimedean.*

As a consequence of Krull's Theorems we have the following:

COROLLARY 4.3 ([**15**, 6.3]). *Let $(K, |\ \ |)$ be a Cauchy complete, non-Archimedean valued field. The valuation $|\ \ |$ can be uniquely extended to a valuation on K^a.*

THEOREM 4.4 ([**41**, 16.2, 16.3], [**15**, 6.4, 6.5, 6.6, 6.8]). *Let $(K, |\ \ |)$ be a Cauchy complete, non-Archimedean valued field with residue class field k. The following statements are true:*

1. *the field k^a is the residue class field of K^a,*
2. *the field k^a is infinite,*
3. *$|(K^a)^*| = \{r \in (0, \infty) : r^n \in |K^*| \text{ for some } n \in \mathbb{N}\}$,*
4. *the only valuation on K^a that extends the one in K is dense.*

THEOREM 4.5 ([**41**, 17.1], [**15**, 6.10]). *Let $(K, |\ \ |)$ be non-Archimedean valued field and let $|\ \ |'$ be the valuation on K^a that extends $|\ \ |$. Then the completion of $(K^a, |\ \ |')$ is algebraically closed.*

4.2. Examples.

THEOREM 4.6. *Consider the field* $\mathbb{Q}_p^a$ *with the only valuation that extends* $|\ |_p$ *on* $\mathbb{Q}_p$*, also denoted by* $|\ |_p$*. The following statements are satisfied:*

(1) $\mathbb{Q}_p^a$ *is a proper extension of* $\mathbb{Q}_p$*,*
(2) *the residue class field of* $\mathbb{Q}_p^a$ *is* $\mathbb{F}_p^a$*,*
(3) $|(\mathbb{Q}_p^a)^*| = \{e^r : r \in \mathbb{Q}\}$*,*
(4) $(\mathbb{Q}_p^a, |\ |_p)$ *is not locally compact,*
(5) $(\mathbb{Q}_p^a, |\ |_p)$ *is not Cauchy complete,*
(6) $\mathbb{Q}_p^a$ *is an infinite dimensional vector space over* $\mathbb{Q}_p$*.*

PROOF. To prove (1) it is enough to show that $\mathbb{Q}_p$ is not algebraically closed. In fact, the polynomial $x^2 - p$ is irreducible on $\mathbb{Q}_p$. Otherwise there would be an x in $\mathbb{Q}_p$ such that $|x^2|_p = |p|_p = e^{-1}$. Hence $|x| = \sqrt{e^{-1}} \in |\mathbb{Q}_p|$ which is not possible. The statements (2) and (3) follow from 4.4. The statement (4) follows from (3) and 2.32. The statements (5) and (6) are proved in [**41**, 16.6 and 16.7] respectively. □

DEFINITION 4.7. The completion of the field $(\mathbb{Q}_p^a, |\ |_p)$ is called the ***field of*** p***-adic complex numbers*** and it is denoted by $\mathbb{C}_p$. The valuation on $\mathbb{C}_p$ which extends $|\ |_p$ will also be denoted by $|\ |_p$.

THEOREM 4.8. *The field of* p*-adic complex numbers satisfies:*

(1) *the residue class field of* $\mathbb{C}_p$ *is* $\mathbb{F}_p^a$*,*
(2) $|\mathbb{C}_p^*| = \{e^r : r \in \mathbb{Q}\}$*,*
(3) $(\mathbb{C}_p, |\ |_p)$ *is not locally compact,*
(4) $\mathbb{C}_p$ *is algebraically closed,*
(5) $\mathbb{C}_p$ *is an infinite dimensional vector space over* $\mathbb{Q}_p$*,*
(6) $\mathbb{C}_p$ *is separable,*
(7) $\mathbb{C}_p$ *and* $\mathbb{C}$ *are isomorphic as fields.*
(8) $\mathbb{C}_p$ *is not spherically complete.*

PROOF. The statements (1), (2) and (5) follow from 2.18 and 4.6, while (3) follows from (2) and 2.32. The statement (4) follows from 4.5 whereas (6) can be found in [**41**, 17.2], (7) can be found in [**49**, page 83] and (8) in [**41**, 20.6]. □

DEFINITION 4.9. Let K be a field. The set

$$\begin{aligned} K\langle\langle x\rangle\rangle &:= \bigcup_{n=1}^{\infty} K((x^{\frac{1}{n}})) \\ &= \left\{ \sum_{i=v}^{\infty} r_i x^{\frac{i}{n}} : v \in \mathbb{Z}, r_i \in K,\ r_v \neq 0,\ \text{for some } n \in \mathbb{N} \right\} \cup \{0\}, \end{aligned}$$

is a field when we adopt the convention $x^{\frac{\ell p}{\ell q}} = x^{\frac{p}{q}}$ for all $\ell \in \mathbb{Z}$. In fact, if $x \in K((x^{\frac{1}{n}}))$ and $y \in K((x^{\frac{1}{m}}))$, then both x and y are elements of $K((x^{\frac{1}{nm}}))$, and therefore $x+y$ and xy are well-defined in $K((x^{\frac{1}{nm}}))$ and hence in $K\langle\langle x\rangle\rangle$.

The mapping $\theta : K\langle\langle x\rangle\rangle \to \bigcup_{n=1}^{\infty} K\big(\big(\frac{1}{n}\mathbb{Z}\big)\big)$ defined by

$$\theta\left(\sum_{i=v}^{\infty} r_i x^{\frac{i}{n}}\right) = \sum_{i=v}^{\infty} r_i d^{\frac{i}{n}},$$

is an isomorphism such that $|\theta(f)| = |f|$ for all $f \in K\langle\langle x\rangle\rangle$. Therefore $K\langle\langle x\rangle\rangle$ can be considered as a subfield of $K((\mathbb{Q}))$. When we restrict the valuation of $K((\mathbb{Q}))$ (see below of 2.22) to $K\langle\langle x\rangle\rangle$, the latter becomes a non-Archimedean valued field with residue class field isomorphic to K and value group equal to $\{e^r : r \in \mathbb{Q}\}$. Fields of the form $K\langle\langle x\rangle\rangle$ are called ***Puiseux series fields***.

With this field we obtain the following chain of field extensions:

$$K \subsetneq K(x) \subsetneq K((x)) \subsetneq K\langle\langle x\rangle\rangle \subsetneq L[\mathbb{Q}, K] \subsetneq K((\mathbb{Q})).$$

THEOREM 4.10. *If K is any field, then $L[\mathbb{Q}, K]$ is the completion of $(K\langle\langle x\rangle\rangle, |\ |)$.*

PROOF. Consider the identification $K\langle\langle x\rangle\rangle = \bigcup_{n=1}^{\infty} K\big(\big(\frac{1}{n}\mathbb{Z}\big)\big)$. By 3.17, every element of the field $L[\mathbb{Q}, K]$ has the form $f = \sum_{i=v}^{\infty} r_i d^{\alpha_i}$ where $v \in \mathbb{Z}$, $r_i \in K$ and $(\alpha_i)_i$ is a strictly increasing sequence in $\mathbb{Q}$ such that $\{\alpha_i : i = v, v+1, \dots\} \cap (-\infty, n]$ is finite for every $n \in \mathbb{Z}$. Notice that the partial sum sequence $(\sum_{i=v}^{n} r_i d^{\alpha_i})_n$ is Cauchy in $\bigcup_{n=1}^{\infty} K\big(\big(\frac{1}{n}\mathbb{Z}\big)\big)$ with limit f. Hence the theorem holds by 2.11, since $\bigcup_{n=1}^{\infty} K\big(\big(\frac{1}{n}\mathbb{Z}\big)\big)$ is dense in $L[\mathbb{Q}, K]$ and the latter is Cauchy complete (3.18). □

The next result shows the interesting analogy of the trios $(\mathbb{Q}_p, \mathbb{Q}_p^a, \mathbb{C}_p)$ and $(K((x)), K\langle\langle x\rangle\rangle, L[\mathbb{Q}, K])$ when K satisfies the following conditions:

THEOREM 4.11. *If K is an algebraically closed field of characteristic 0, then:*

(1) *$K((x))^a = K\langle\langle x\rangle\rangle$.*
(2) *$L[\mathbb{Q}, K]$ is the completion of the field $(K\langle\langle x\rangle\rangle, |\ |)$.*
(3) *$L[\mathbb{Q}, K]$ is algebraically closed.*

PROOF. The first statement is proved in [**13**, 13.15], the second statement is satisfied for any field K (4.10), and the last statement follows from 4.5. □

Now we are able to determine when the Intermediate Value Theorem is valid for polynomials over $K\langle\langle x\rangle\rangle$ and $L[\mathbb{Q}, K]$.

COROLLARY 4.12. *The following statements are equivalent:*

(1) *K is real-closed.*
(2) *$K\langle\langle x\rangle\rangle$ is real-closed.*
(3) *$L[\mathbb{Q}, K]$ is real-closed.*

PROOF. (1) $\Longrightarrow$ (2) follows from [**5**, 2.6 Theorem 2.91] while (2) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (1) are proved in [**3**, 6.23 (1)]. Let's prove (1) $\Longrightarrow$ (3): if K is real-closed, then $K(i)$ is an algebraically closed filed of characteristic 0, where i is a root for $x^2 + 1 = 0$ (3.22). By 4.11 the field $K(i)\langle\langle x\rangle\rangle$ is algebraically closed and hence $L[\mathbb{Q}, K](i) = L[\mathbb{Q}, K(i)]$ is algebraically closed as well (4.5, 4.10). Finally, $L[\mathbb{Q}, K]$ is real-closed by 3.22. □

Now we will see that the analogy of the trios breaks down when the characteristic of K is positive.

THEOREM 4.13 ([**1**, p. 904], [**9**, pp. 64–65]). *If K is algebraically closed and $char(K) = p$, then the fields $K\langle\langle x\rangle\rangle$ and $L[\mathbb{Q}, K]$ are not algebraically closed, since the polynomial $p(z) = z^p - z - x^{-1} \in K\langle\langle x\rangle\rangle[z]$ has the following factorization in $K((\mathbb{Q}))$:*

$$p(z) = \prod_{i=0}^{p-1} \Big(z + i - \sum_{i=1}^{\infty} x^{\frac{-1}{p^i}} \Big).$$

In order to describe the algebraic closure of $K((x))$ when K is algebraically closed of positive characteristic, we need a fair amount of terminology on Regular Languages that will not be presented here but can be found with full details in [**22**]. Such a description reads as follows.

THEOREM 4.14 ([**22**, Theorem 10.6]). *Let K be an algebraically closed field of characteristic p and $x = \sum_i r_i x^i \in K((\mathbb{Q}))$. Then $x \in K((x))^a$ if and only if for all $n \in \mathbb{Z}$, y_n is p-quasi-automatic and span$\{y_n : n \in \mathbb{Z}\}$ is a finite dimensional vector space over K, where*

$$y_n := \sum_{i \in [0,1) \cap \mathbb{Q}} r_{n+i} x^i \ .$$

COROLLARY 4.15. *The following statements are equivalent:*

(1) *K is algebraically closed and has characteristic* 0.
(2) *$K\langle\langle x \rangle\rangle$ is algebraically closed.*
(3) *$L[\mathbb{Q}, K]$ is algebraically closed.*

PROOF. (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) follow from 4.11. The implications (2) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (1) follow form [**3**, 6.23 (0)] and 4.13. $\square$

5. General valuations.

Note that in 3.16, we have restricted our attention to those Hahn fields of Archimedean group embedded in $(\mathbb{R}, +)$ in order to define a valuation with values in $\mathbb{R}$. However it is possible to define a valuation with values in any ordered abelian group. If $| \ | : K \to \mathbb{R}$ is a valuation on a field K and we consider the function $v : K \to \mathbb{R} \cup \{\infty\}$ defined by $v(x) = -\ln |x|$, then we are shifting our attention to the additive structure of $\mathbb{R}$ rather than the multiplicative one of $(0, \infty)$. In particular, if $| \ |$ is a non-Archimedean valuation, then the strong triangle inequality now has the form: $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in K$. In this section we will use this approach to define a generalization of a non-Archimedean valuation.

DEFINITION 5.1. Let K be a field and let $(G, +)$ be an ordered abelian group. A map $v : K \to G \cup \{\infty\}$ is a ***general valuation (or Krull valuation)*** on K if it satisfies:

(1) v is onto,
(2) $v(x) = \infty$ if and only if $x = 0$,
(3) $v(xy) = v(x) + v(y)$,
(4) $v(x + y) \geq \min\{v(x), v(y)\}$,

where ∞ is a symbol that satisfies, for all $g \in G$, the following axioms:

$$g < \infty \text{ and } \infty = \infty + \infty = g + \infty = \infty + g.$$

The group $G = v(K^*)$ is called the ***value group*** of (K, v) and the quotient $\{x \in K : v(x) \geq 0\}/\{x \in K : v(x) > 0\}$ is the residue class field of (K, v). If G is order-isomorphic to a subgroup of $(\mathbb{R}, +)$, then we say that v has ***rank* 1**. Otherwise v is of ***higher rank***. The general valuation v is called ***discrete*** if G is cyclic. In that case, if G is not trivial then it is isomorphic to $(\mathbb{Z}, +)$ and hence it has rank 1.

THEOREM 5.2 ([**4**, Chapter III, section 3]). *An ordered group $(G, +)$ is order-isomorphic to a subgroup of $(\mathbb{R}, +)$ if and only if it is Archimedean, i.e. for every $a, b \in G$, $b > 0$, there exists $n \in \mathbb{N}$ such that $a < nb$. In particular, every Archimedean ordered group is abelian.*

DEFINITION 5.3. Let K be a field with general valuation $v : K \to G \cup \{\infty\}$. The sets of the form $U_v[a, g] := U[a, g] := \{x \in K : v(x - a) \geq g\}$ for $a \in K, g \in G$, constitute a base for a topology on K called the ***valuation topology*** induced by v. When G is not trivial, the sets of the form $U_v(a, g) := U(a, g) := \{x \in K : v(x - a) > g\}$ also constitute a base for the valuation topology induced by v. The field K with this topology becomes a Hausdorff topological field ([**3**, 7.64]).

DEFINITION 5.4. Let K be an ordered field and let G_K be the ordered group of the Archimedean classes of K (3.4, 3.5). The map $\mu : K \to G_K \cup \{\infty\}$

$$\mu(x) := \begin{cases} [x] & , x \neq 0 \\ \infty & , x = 0, \end{cases}$$

is a general valuation on K and it is called ***the order valuation*** of K ([**3**, 1.61]).

THEOREM 5.5. *Let K be an Archimedean ordered field and let μ be the order valuation on K. Then the valuation topology induced by μ is the discrete topology.*

PROOF. Since $G_k = \{[1]\}$, then μ is the trivial valuation on K. □

THEOREM 5.6 ([**3**, 7.63, 7.64]). *Let K be a non-Archimedean ordered field and let μ be the order valuation on K. Then the order topology on K coincides with the valuation topology induced by μ.*

EXAMPLES 5.7.

(1) If $| \ | : K \to \mathbb{R}$ is a non-Archimedean valuation, then the function $v : K \to \mathbb{R} \cup \{\infty\}$ defined by $v(x) = -\ln |x|$ is a general valuation (after a suitable restriction of the codomain). The topology induced by $| \ |$ coincides with the valuation topology induced by v, since $B(x, e^{-r}) = U(x, r)$ and $B[x, e^{-r}] = U[x, r]$ for every $x \in K$ and $r \in \mathbb{R}$.

(2) If $v : K \to G \cup \{\infty\}$ is a general valuation of rank 1 on K and $i : G \to (\mathbb{R}, +)$ is an order-preserving homomorphism, then the map $| \ | : K \to \mathbb{R}$ given by

$$|x| := \begin{cases} e^{-i(v(x))} & , x \neq 0 \\ 0 & , x = 0, \end{cases}$$

is a non-Archimedean valuation on K. The topology induced by $| \ |$ coincides with the valuation topology induced by v, since $B(x, |y|) = U(x, v(y))$ and $B[x, |y|] = U[x, v(y)]$ for every $x, y \in K$.

(3) Let K be a field and G be any ordered abelian group. The map $\lambda : K((G)) \to G \cup \{\infty\}$ defined by

$$\lambda(f) := \begin{cases} \min\{supp(f)\} & , f \neq 0 \\ \infty & , f = 0, \end{cases}$$

is a general valuation on the general Hahn field $K((G))$ called ***the Hahn valuation*** on $K((G))$ (compare with the valuation in 3.16).

THEOREM 5.8. *Let K be an Archimedean ordered field, G a nontrivial ordered abelian group, μ the order valuation on $K((G))$ and λ the Hahn valuation on $K((G))$. The following statements are true:*

(1) *$\mu(f) = [f] = \{g \in K((G)) : \lambda(g) = \lambda(f)\}$ for every $f \neq 0$,*
(2) *the valuation topologies induced by μ and λ coincide with the order topology on $K((G))$ (defined in 3.12),*

(3) *If G is a subgroup of $(\mathbb{R},+)$, then the valuation $|\ \ |$ of the Hahn field $K((G))$ defined in 3.16, induces the order topology on $K((G))$.*

PROOF. To prove the first statement it is enough to note that f and g are comparable if and only if $\lambda(f) = \lambda(g)$. The second statement follows from 5.6 and the equalities $U_\mu(f,\mu(g)) = U_\lambda(f,\lambda(g))$ and $U_\mu[f,\mu(g)] = U_\lambda[f,\lambda(g)]$ for all $f, g \in K((G))$. The last statement follows from part (2) and from example 5.7(1). □

LEMMA 5.9. *Let F/K be a field extension and let $v : F \to G \cup \{\infty\}$ be a general valuation on F. The restriction $v|_K$ of v to K is a general valuation on K. Moreover, if τ_v ($\tau_{v|_K}$) denotes the valuation topology on F (on K) induced by v (by $v|_K$ respectively), and $\tau_v \cap K$ denotes the subspace topology on K induced by (F, τ_v), then $\tau_{v|_K} = \tau_v \cap K$.*

PROOF. After a suitable restriction of the codomain the map $v|_K$ is a general valuation on K. The second statement follows from the following equalities valid for all $x, y \in K$: $U_{v|_K}(x, v(y)) = \{z \in K : v(x - z) > v(y)\} = U_v(x, v(y)) \cap K$. □

THEOREM 5.10. *Let K be a non-Archimeden ordered field. By 3.13 K can be embedded in $\mathbb{R}((G_K))$ where G_K is the group of Archimedean classes of K. Let:*

- *τ_K be the order topology on K,*
- *τ_μ be the valuation topology on K induced by the order valuation μ of K,*
- *τ_0 be the order topology on $\mathbb{R}((G_K))$,*
- *τ_η be the valuation topology on $\mathbb{R}((G_K))$ induced by the order valuation η of $\mathbb{R}((G_K))$,*
- *τ_λ be the valuation topology on $\mathbb{R}((G_K))$ induced by the Hahn valuation λ of $\mathbb{R}((G_K))$.*

Then $\quad \tau_K = \tau_\mu = \tau_{\lambda|_K} = \tau_{\eta|_K} = \tau_\lambda \cap K = \tau_0 \cap K = \tau_\eta \cap K.$

PROOF. The equality $\tau_K = \tau_\mu$ follows from 5.6 while the equalities $\tau_\lambda \cap K = \tau_0 \cap K = \tau_\eta \cap K$ follow from 5.8. The equalities $\tau_{\lambda|_K} = \tau_\lambda \cap K$ and $\tau_{\eta|_K} = \tau_\eta \cap K$ follow from 5.9. Finally we will show that $\tau_\mu = \tau_\eta \cap K$. For every $f \in K$, $\mu(f) = \{g \in K : g \text{ is comparable to } f\}$. Since K is embedded in $\mathbb{R}((G_K))$ we have that $K \subset \mathbb{R}((G_K))$. Hence $\mu(f) \subset \eta(f) = \{g \in \mathbb{R}((G_K)) : \lambda(g) = \lambda(f)\}$. It follows that $\mu(f) = \mu(g)$ if and only if $\eta(f) = \eta(g)$ for all $f, g \in K$. Additionally, $\mu(f) < \mu(g)$ if and only if $\eta(f) < \eta(g)$. Hence $U_\eta(f, \eta(g)) \cap K = U_\mu(f, \mu(g))$ for all $f, g \in K$. □

REMARK 5.11. Let $(A, <)$ be an ordered set with order topology τ_A and let B be a subset of A. The order in A induces an order in B which induces a topology $\tau_<$ on B. In general the subspace topology $\tau_A \cap B$ on B may be different than the topology $\tau_<$ ([**28**, p. 90]). Also, it is well-known that if B is a convex subset of A, then $\tau_A \cap B = \tau_<$ ([**28**, 16.4]). The surprising thing about the previous result is that for every non-Archimedean ordered field K its order topology τ_K (which is the topology induced by the order induced by the order on $\mathbb{R}((G_K))$) always coincides with the the subspace topology $\tau_0 \cap K$, even when K is not convex in $\mathbb{R}((G_K))$. For example, $L[\mathbb{Q}, \mathbb{R}]$ is not a convex subfield of $\mathbb{R}((\mathbb{Q}))$ and by 5.10, the order topology on $L[\mathbb{Q}, \mathbb{R}]$ coincides with the subspace topology inherited from $\mathbb{R}((\mathbb{Q}))$.

More details about general valuations can be found in [**36**, 13.1] and [**33**].

6. Maximal, Pseudo and Spherical completeness.

In this section we will present briefly the concepts of Maximal completeness, Pseudo-completeness and Spherical completeness together with their equivalences in fields with general valuations or with non-Archimedean valuations.

6.1. Pseudo-completeness. When we study the Cauchy completeness of a space, usually an equivalent condition using sequences is employed. For spherical completeness of non-Archimedean valued fields we also have an equivalent condition using sequences: the notion of pseudo-completeness.

Definition 6.1. Let K be a field, $|\ | : K \to \mathbb{R}$ be a non-Archimedean valuation and $v : K \to G \cup \{\infty\}$ be a general valuation where G is an abelian ordered group. A well-ordered set $\{x_\rho\}$ of elements of K, without a maximal element, is ***pseudo-Cauchy*** (with respect to v) if $v(x_\sigma - x_\rho) < v(x_\tau - x_\sigma)$ for all $\tau > \sigma > \rho$. A sequence $(x_n)_n$ of elements of K is ***pseudo-Cauchy*** (with respect to $|\ |$) if there exists $n_0 \in \mathbb{N}$, such that $|x_n - x_m| < |x_m - x_k|$ for all $n > m > k \geq n_0$.

Notice that a sequence $(x_n)_n$ is pseudo-Cauchy with respect to $|\ |$ if and only if it is eventually pseudo-Cauchy with respect to $v(x) = -\ln|x|$.

Lemma 6.2 ([**42**, II.4. Lemma 7], [**29**, 2.1. Lemma 4]). *Let K be a field, $|\ |$ be a non-Archimedean valuation on K and $v : K \to G \cup \{\infty\}$ be a general valuation.*

1. *If the well-ordered set $\{x_\rho\}_\rho$ is pseudo-Cauchy, then for all $\sigma > \rho$, $v(x_\sigma - x_\rho) = v(x_{\rho+1} - x_\rho) =: \gamma_\rho$. Hence the net $(\gamma_\rho)_\rho$ is strictly increasing in G.*
2. *If the sequence $(x_n)_n$ is pseudo-Cauchy, then for some $n_0 \in \mathbb{N}$ and all $k > n \geq n_0$, $|x_k - x_n| = |x_{n+1} - x_n| =: \mu_n$. Hence the sequence $(\mu_n)_n$ is eventually strictly decreasing in $(0, \infty)$.*

Definition 6.3 ([**29**, 2.1. Lemma 5]). Let K be a field, $|\ | : K \to \mathbb{R}$ be a non-Archimedean valuation and $v : K \to G \cup \{\infty\}$ be a general valuation. An element $a \in K$ is a ***pseudo-limit*** of the pseudo-Cauchy set $\{x_\rho\}_\rho$ if $v(a - x_\rho) = \gamma_\rho$ for all ρ, or equivalently, if $v(a - x_\rho) < v(a - x_\sigma)$ for all $\sigma > \rho$. An element $a \in K$ is a ***pseudo-limit*** of the pseudo-Cauchy sequence $(x_n)_n$ if $|a - x_n| = \mu_n$ for all $n \geq n_0$, for some $n_0 \in \mathbb{N}$, or equivalently, if $|a - x_n| > |a - x_m|$ for all $m > n \geq n_0$, for some $n_0 \in \mathbb{N}$.

Definition 6.4. Let K be a field, $|\ | : K \to \mathbb{R}$ be a non-Archimedean valuation and $v : K \to G \cup \{\infty\}$ be a general valuation. If every pseudo-Cauchy set of K has a pseudo-limit in K (w.r.t. v), then (K, v) is said to be ***pseudo-complete***. If every pseudo-Cauchy sequence of K has a pseudo-limit in K (w.r.t. $|\ |$), then $(K, |\ |)$ is said to be ***pseudo-complete***.

The following result shows the equivalence of pseudo-completeness and spherical completeness.

Theorem 6.5 ([**29**, 2.1. Theorem 2]). *Let $(K, |\ |)$ be a non-Archimedean valued field. Then $(K, |\ |)$ is spherically complete if and only if it is pseudo-complete.*

6.2. Spherical completeness. In view of the fact that any ultrametric space can be embedded in a non-Archimedean valued field ([**39**]), the following definition is a generalization of the definition given in 1.15. Let K be a field with general valuation v and value group G. A set of the form $U[a, g] := \{x \in K : v(x - a) \geq g\}$ for $a \in K, g \in G$ is called a ***ball*** in K. Recall that the collection of all balls in K constitute a basis for the valuation topology on K (5.3).

DEFINITION 6.6. A field K with general valuation v is ***spherically complete*** if every nested sequence of balls $U[a_1, g_1] \supset U[a_2, g_2] \supset \cdots$ for which $g_1 < g_2 < \cdots$ has nonempty intersection.

THEOREM 6.7. *A field K with general valuation v is spherically complete if and only if it is pseudo-complete.*

PROOF. The proof of [**29**, 2.1. Theorem 2] can be adapted to our case. □

6.3. Maximal completeness. Just as a metric space has a completion, a non-Archimedean valued field has a "spherical completion". In this subsection we will formalize this notion, and study its existence, uniqueness and structure.

DEFINITION 6.8. A field F with general valuation v' is an ***immediate extension*** of a field K with general valuation v if

(1) F/K is a field extension,
(2) v' is an extension of v,
(3) $v'(F^*) = v(K^*)$,
(4) the residue class field of (F, v') is equal to the residue class field of (K, v).

If (K, v) admits no proper immediate extension, then it is ***maximally complete***.

In particular, if the general valuations have rank 1, then the previous definition can be rephrased as follows.

DEFINITION 6.9. A non-Archimedean valued field $(F, |\ |')$ is an ***immediate extension*** of a non-Archimedean valued field $(K, |\ |)$ if

(1) F/K is a field extension,
(2) $|\ |'$ is an extension of $|\ |$,
(3) $|F^*|' = |K^*|$,
(4) the residue class fields of $(F, |\ |')$ and $(K, |\ |)$ are equal.

If $(K, |\ |)$ admits no proper immediate extension, then it is ***maximally complete***.

EXAMPLE 6.10. The completion of a non-Archimedean valued field $(K, |\ |)$ is an immediate extension of $(K, |\ |)$ by 2.18.

THEOREM 6.11 ([**21**, Theorem 4], [**42**, II.6. Theorem 8]). *A field K with general valuation v is maximally complete if and only if it is is pseudo-complete, or equivalently, spherically complete.*

In particular, for non-Archimedean valued fields we have the following:

THEOREM 6.12 ([**29**, 2.3. Theorem 2, 2.1. Theorem 5]). *Let $(K, |\ |)$ be a non-Archimedean valued field. Then $(K, |\ |)$ is maximally complete if and only if it is pseudo-complete if and only if it is spherically complete.*

The next result guarantees the existence of a maximally complete immediate extension of any valued field.

THEOREM 6.13 ([**42**, II.3. Theorem 5], [**29**, 2.4. Theorem 1], [**24**, Theorem 24]). *For each field K with general valuation v, there exists at least one maximally complete immediate extension. This also holds for non-Archimedean valued fields.*

The uniqueness of a maximally complete immediate extension of a valued field is not always possible. However there are some conditions for which we have uniqueness. First, let's clarify the type of uniqueness we are referring to.

Definition 6.14. Two immediate extensions (F_1, v_1) and (F_2, v_2) of a field K with general valuation v are ***analytically equivalent*** (over (K, v)) if there exists a field isomorphism $\sigma : F_1 \to F_2$ such that $v_1(x) = v_2(\sigma(x))$ for all $x \in F_1$ and $\sigma(y) = y$ for all $y \in K$. Similarly, two immediate extensions $(F_1, |\ \ |_1)$ and $(F_2, |\ \ |_2)$ of a non-Archimedean valued field $(K, |\ \ |)$ are ***analytically equivalent*** (over $(K, |\ \ |)$) if there exists a field isomorphism $\sigma : F_1 \to F_2$ such that $|x|_1 = |\sigma(x)|_2$ for all $x \in F_1$ and $\sigma(y) = y$ for all $y \in K$.

Definition 6.15. Let the field K have a general valuation with value group G and residue class field k. If $char(k) = p > 0$, then the ***hypothesis A*** is the following pair of conditions:

(1) Any equation of the form $x^{p^n} + a_1 x^{p^{n-1}} + \cdots + a_{n-1}x^p + a_n x + a_{n+1} = 0$, with coefficients in k, has a root in k.
(2) $G = pG$.

Analogously for non-Archimedean valued fields we have the following:

Definition 6.16. Let the field K have a non-Archimedean valuation $|\ \ |$ with value group $|K^*|$ and residue class field k. If $char(k) = p > 0$, then the ***hypothesis A*** is the following pair of conditions:

(1) Any equation of the form $x^{p^n} + a_1 x^{p^{n-1}} + \cdots + a_{n-1}x^p + a_n x + a_{n+1} = 0$, with coefficients in k, has a root in k.
(2) $|K^*| = (|K^*|)^p$.

Theorem 6.17 ([**21**, Theorem 5], [**42**, VII.5. Theorem 4]). *Let K be a field with a general or a non-Archimedean valuation and let k be its residue class field.*

(1) *If $char(k) = 0$, then $char(K) = 0$ and the maximally complete immediate extension of K is uniquely determined up to analytical equivalence.*
(2) *If $char(k) = p > 0$, and k and the value group satisfy the hypothesis A, then the maximally complete immediate extension of K is uniquely determined up to analytical equivalence over K.*

Example 6.18. If K is any field and G is any ordered abelian group, then the general Hahn field $K((G))$ with the Hahn valuation (5.7) is maximally complete (see [**42**, corollary, page 51], [**24**, page 193]). Hence any maximally complete immediate extension of the Levi-Civita field $L[G, K]$ is analytically equivalent to $K((G))$ when K and G satisfy the conditions of 6.17. Recall that the value group of $L[G, K]$ is G and its residue class field is isomorphic to K.

Note that if a valued field K has a divisible value group G and its residue class field k is algebraically closed, then G and k satisfy the hypothesis A. Consequently:

Corollary 6.19. *Let K be a field with a general valuation. If the value group of K is divisible and the residue class field of K is algebraically closed, then the maximally complete immediate extension of K is uniquely determined up to analytical equivalence over K.*

The uniqueness obtained in 6.19 is related to the uniqueness of the algebraic closure of a field in the the following sense:

Theorem 6.20 ([**34**, Corollary 4], [**3**, 6.23. (0)]). *Let F be a maximally complete field with a value group G and a residue class field K. The field F is algebraically closed if and only if G is divisible and K is algebraically closed.*

REMARK 6.21. In [**21**, pp. 318–320] Irving Kaplansky presents a field that does not satisfy the Hypothesis A and has two maximally complete immediate extensions that are not analytically equivalent and not even isomorphic as fields. For a deeper understanding of the Hypothesis A, and generalizations of the uniqueness theorems of a maximally complete field see [**25**] where the Hypothesis A is put into a Galois theoretic setting.

7. Structures of Maximally Complete fields.

In this section we will describe the structures of maximally complete fields, which can be split into two cases: when the characteristics of the valued field and of its residue class field are equal (same characteristic case), and when they are different (mixed characteristic case).

7.1. Equal characteristic case.

DEFINITION 7.1 ([**42**, I.6, p. 23]). Let K be a field and G be an ordered abelian group. Let $f : G \times G \to K^*$ be a function such that for all $\alpha, \beta, \gamma \in G$:

$$f(\alpha,\beta) = f(\beta,\alpha)$$
$$f(0,0) = f(\alpha,0) = f(0,\beta)$$
$$f(\alpha,\beta+\gamma)f(\beta,\gamma) = f(\alpha+\beta,\gamma)f(\alpha,\beta).$$

The set $\{f(\alpha,\beta) : \alpha,\beta \in G\}$ is called a ***factor set*** for G in K. The field $K((G, f_{\alpha,\beta}))$ is defined as the group $(K((G)),+)$ with the multiplication induced by the relations $t^\alpha t^\beta = f(\alpha,\beta)t^{\alpha+\beta}$ for $\alpha,\beta \in G$. The Hahn valuation $\lambda : K((G, f_{\alpha,\beta})) \to G$, $\lambda(x) := \min\{\mathrm{supp}(x)\}$ is a general valuation on $K((G, f_{\alpha,\beta}))$.

THEOREM 7.2 ([**21**, Theorem 6], [**42**, VII.6 Theorem 6]). *Let F be a maximally complete field with value group G, residue class field k and $char(F) = char(k)$.*

1. *If $char(k) = 0$, then F is analytically equivalent to $(k((G, f_{\alpha,\beta})), \lambda)$ for some factor set $\{f(\alpha,\beta) : \alpha,\beta \in G\}$.*
2. *If $char(k) = p > 0$, and G and k satisfy the hypothesis A, then F is analytically equivalent to $(k((G, f_{\alpha,\beta})), \lambda)$ for some factor set $\{f(\alpha,\beta) : \alpha,\beta \in G\}$.*

With an extra hypothesis this result can be simplified as follows:

THEOREM 7.3 ([**21**, Theorem 8], [**42**, VII.6 Corollary]). *Let F be a maximally complete field with value group G and residue class field k such that $char(F) = char(k)$. Suppose that every element of k has an n-th root for all $n \in \mathbb{N}$.*

1. *If $char(k) = 0$, then F is analytically equivalent to $(k((G)), \lambda)$.*
2. *If $char(k) = p > 0$, and G and k satisfy the hypothesis A, then F is analytically equivalent to $(k((G)), \lambda)$.*

COROLLARY 7.4. *Let K be a field with a general valuation, a value group G and a residue class field k such that $char(K) = char(k)$. Suppose that every element of k has an n-th root for all $n \in \mathbb{N}$.*

1. *If $char(k) = 0$, then $(k((G)), \lambda)$ is the only maximally complete immediate extension of K up to analytical equivalence.*
2. *If $char(k) = p > 0$, and G and k satisfy the hypothesis A, then $(k((G)), \lambda)$ is the only maximally complete immediate extension of K up to analytical equivalence.*

Proof. It follows immediately from 6.13, 6.17 and 7.3. □

Corollary 7.5 ([**21**, Corollary], [**42**, VII.6 Corollary], [**34**, Corollary 6]). *An arbitrary field K with general valuation v and residue class field k such that $char(K) = char(k)$, is analytically isomorphic to a subfield of $k^a((G))$ for some divisible ordered abelian group G. Furthermore, there exists a maximally complete immediate extension of K analytically isomorphic to a subfield of $k^a((G))$.*

The corollaries 7.4 and 7.5 are generalizations of the Hahn's Embedding Theorem (3.13) in the sense that K does not need to be an ordered field to be embedded in a general Hahn field. Nevertheless the value of the Hahn's Embedding Theorem lies in the order-preserving nature of the embedding when K is an ordered field.

Definition 7.6. An ordered abelian group G is ***discrete*** if its isolated subgroups $\{G_\rho\}$ are well-ordered with respect to inclusion, and if each $G_{\rho+1}/G_\rho$ is isomorphic to $(\mathbb{Z}, +)$.

Another strong result about the structure of a maximally complete field is the following which does not invoke the hypothesis A.

Theorem 7.7 ([**19**, Theorem]). *Let F be a maximally complete field with value group G and residue class field k such that $char(F) = char(k)$. If G is discrete, then F is analytically equivalent to $(k((G)), \lambda)$.*

Corollary 7.8. *Let K be a field with a general valuation, a discrete value group G, a residue class field k and $char(K) = char(k)$. Then $(k((G)), \lambda)$ is the only maximally complete immediate extension of K up to analytical equivalence.*

Proof. It follows from 6.13 and 7.7. □

7.2. Mixed characteristic case.

Theorem 7.9 ([**44**, II.5. Theorem 3]). *If R is a perfect field of characteristic $p > 0$, then there exists a unique ring $W(R)$ (up to isomorphism) of characteristic 0, Cauchy complete with respect to a discrete valuation v, $v(p) = 1 \in \mathbb{Z}$, and residue class field R. Also, the field of fractions $F(W(R))$ of $W(R)$ is the only field (up to isomorphism) of characteristic 0, Cauchy complete with respect to a discrete valuation v, $v(p) = 1 \in \mathbb{Z}$ and with residue class field R.*

In this context, $W(R) = \{x \in F(W(R)) : v(x) \geq 0\}$ is called the ring of Witt vectors with coefficients in R (see [**44**, II.5]). For example, if $R = \mathbb{F}_p$, then $W(R) = \mathbb{Z}_p$ and $F(W(R)) = \mathbb{Q}_p$.

In [**34**], Bjorn Poonen describes a maximally complete field of characteristic 0 with prescribed value group and perfect residue class field of positive characteristic. In other words, he describes an analog of a general Hahn field for the mixed characteristic case. The construction is as follows:

Definition 7.10. Let R be a perfect field of characteristic $p > 0$, let G be an ordered abelian group and let $W(R)$ and $F(W(R))$ be as in 7.9. Define $W(R)((G))$ as the subring of the general Hahn field $F(W(R))((G))$ formed by all the generalized power series $\sum_g \alpha_g t^g$ where $g \in G$, $\alpha_g \in W(R)$, and $\{g : \alpha_g \neq 0\}$ is well-ordered. A series $\sum_g \alpha_g t^g \in W(R)((G))$ is ***null*** if for all $g \in G$, $\sum_{n\in\mathbb{Z}} \alpha_{g+n} p^n = 0$ in $F(W(R))$. The set N of all null series is an ideal of $W(R)((G))$ that contains the polynomial $t - p$ ([**34**, Proposition 3]).

THEOREM 7.11 ([**34**, Corollary 3]). *Let R be a perfect field of characteristic $p > 0$ and G be an ordered abelian group. The quotient*

$$L := W(R)((G))/N$$

is a field of characteristic 0*, with a general valuation, a valuation group G and residue class field R. These fields are called* ***p-adic Mal'cev-Neumann fields****.*

THEOREM 7.12 ([**34**, Proposition 4]). *Let R be a perfect field of characteristic $p > 0$ and let G be an ordered abelian group. If $S \subset W(R)$ is a set of representatives of the classes of R, then each series $\sum_g \alpha_g t^g$ in $W(R)((G))$ is null equivalent to a unique series of the form $\sum_g \beta_g t^g$ for $\beta_g \in S$. In other words, we have the following expansion for the elements of L:*

$$L = \left\{ \sum_{g \in G} \beta_g p^g : \beta_g \in S, \{g : \beta_g \neq 0\} \text{ is well-ordered} \right\}.$$

REMARK 7.13. The field $\mathbb{Q}_p$ is an example of a p-adic Mal'cev-Neumann field:

$$\mathbb{Q}_p = \mathbb{Z}_p((\mathbb{Z}))/N = W(\mathbb{F}_p)((\mathbb{Z}))/N = \left\{ \sum_{n=m}^{\infty} \beta_n p^n : m \in \mathbb{Z}, \beta_n = 0, 1, \ldots, p-1 \right\}.$$

The next result strengthens the analogy between general Hahn fields and p-adic Mal'cev-Neumann fields.

THEOREM 7.14 ([**34**, Theorem 1]). *Let G be an ordered abelian group and K be a perfect field of characteristic $p > 0$. The field $L = W(R)((G))/N$ is maximally complete under the Hahn valuation.*

REMARK 7.15. By 6.19, 7.11 and 7.14 the field $W(\mathbb{F}_p^a)((\mathbb{Q}))/N$ is the only maximally complete immediate extension of $\mathbb{Q}_p^a$ and of $\mathbb{C}_p$ up to analytic equivalence.

By 7.12 and 7.14, we can formulate the following corollary of 7.9:

COROLLARY 7.16. *If R is a perfect field of characteristic $p > 0$, then the p-adic Mal'cev-Neumann field $W(R)((\mathbb{Z}))/N$ is the only field (up to isomorphism) of characteristic* 0*, with a discrete valuation v such that the residue class field is R, $v(p) = 1 \in \mathbb{Z}$, and $W(R)((\mathbb{Z}))/N$ is Cauchy complete with respect to v. Furthermore $F(W(R)) = W(R)((\mathbb{Z}))/N$ is maximally complete.*

The next result is analogous to 7.5.

COROLLARY 7.17 ([**34**, Corollary 5, Corollary 6]). *An arbitrary field K of characteristic* 0 *with a general valuation and a residue class field k such that $char(k) = p > 0$, is analytically isomorphic to a subfield of $W(k^a)((G))/N$ for some divisible ordered abelian group G. Furthermore, there exists a maximally complete immediate extension of K analytically isomorphic to a subfield of $W(k^a)((G))/N$.*

8. Catalog of fields.

In the next table we summarize the fields and their properties presented so far. Let G be an ordered abelian group, let K be any field and let p be a positive prime integer. Assume that each of the listed fields has the usual non-Archimedean valuation (or general valuation) defined in this article and it is equipped with the respective induced topology. Denote the cardinalities of K and G by c and g respectively. In case that K is a perfect field of characteristic p, denote the field

$W(K)((G))/N$ by $\mathbb{K}_p^{sph}$ (see 7.10 and 7.11). Notice that the first 5 fields of the table form a chain of field extensions in the mixed characteristic case, i.e. each of the fields has a characteristic different than the characteristic of its residue class field:

$$\mathbb{Q} \subset \mathbb{Q}_p \subset \mathbb{Q}_p^a \subset \mathbb{C}_p \subset \mathbb{K}_p^{sph}.$$

The last 5 fields of the table form a chain of field extensions in the equal characteristic case, i.e. each of the fields has a characteristic equal to the characteristic of its residue class field:

$$K(x) \subset K((x)) \subset K\langle\langle x\rangle\rangle \subset L[\mathbb{Q}, K] \subset K((\mathbb{Q})).$$

Note that these chains are similar in several ways, for example the construction of the fields in the i-th position from the fields in the $(i-1)$-th position is identical (under certain conditions when it is necessary). Other metric, topological or algebraic similarities are easily seen from the table. If the m-th property is satisfied by the n-th field of the table, then the symbol ✓ will appear in the entry (m, n). Otherwise the symbol ✗ will take place. When a number (n) appears instead, the property is satisfied under certain conditions specified below the table.

	$\mathbb{Q}$	$\mathbb{Q}_p$	$\mathbb{Q}_p^a$	$\mathbb{C}_p$	$\mathbb{K}_p^{sph}$	$K(x)$	$K((x))$	$K\langle\langle x\rangle\rangle$	$L[\mathbb{Q}, K]$	$K((G))$
Totally disconnected	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Separable	✓	✓	✓	✓	(1)	(2)	(2)	(2)	(2)	(2)
Cauchy complete	✗	✓	✗	✓	✓	✗	✓	✗	✓	✓
Spherically complete	✗	✓	✗	✗	✓	✗	✓	✗	✗	✓
Locally compact	✗	✓	✗	✗	(3)	✗	(4)	✗	✗	(4)
Algebraically Closed	✗	✗	✓	✓	(5)	✗	✗	(6)	(6)	(5)
Formally real	✓	✗	✗	✗	✗	(7)	(7)	(7)	(7)	(7)
Real-closed	✗	✗	✗	✗	✗	✗	(8)	(9)	(9)	(8)
Archimedean complete	✗	✗	✗	✗	✗	✗	(10)	✗	✗	(10)
Cardinality	$\aleph_0$	$\aleph_1$	$\aleph_1$	$\aleph_1$	c^g	$c^{\aleph_0}$	$c^{\aleph_0}$	$c^{\aleph_0}$	$c^{\aleph_0}$	c^g
Residue class field	$\mathbb{F}_p$	$\mathbb{F}_p$	$\mathbb{F}_p^a$	$\mathbb{F}_p^a$	K	K	K	K	K	K

(1) Under the assumption that G is a subgroup of $(\mathbb{R}, +)$, the p-adic Mal'cev-Neumann field $\mathbb{K}_p^{sph}$ is separable if and only if K and G are countable (1.9). In particular, if $K = \mathbb{F}_p^a$ and $G = \mathbb{Q}$, then $\mathbb{K}_p^{sph}$ is separable and it is a field extension of $\mathbb{C}_p$, $\mathbb{Q}_p^a$, $\mathbb{Q}_p$ and $\mathbb{Q}$. Thus these subfields are separable.

(2) Under the assumption that G is a subgroup of $(\mathbb{R}, +)$, the field $K((G))$ is separable if and only if K and G are countable (1.9). Since $K \subset K(x) \subset K((x)) \subset K\langle\langle x\rangle\rangle \subset L[\mathbb{Q}, K] \subset K((\mathbb{Q}))$, each of these fields is separable if and only if K is separable.

(3) Under the assumption that G is a subgroup of $(\mathbb{R}, +)$, the p-adic Mal'cev-Neumann field $\mathbb{K}_p^{sph}$ is locally compact if and only if K is finite and G is cyclic (2.32, 7.11).

(4) Under the assumption that G is a subgroup of $(\mathbb{R}, +)$, the field $K((G))$ is locally compact if and only if K is finite and G is cyclic (2.32, 3.16). In particular, $K((x))$ is locally compact if and only if K is finite.

(5) The field $K((G))$ is is algebraically closed if and only if K is algebraically closed and G is divisible (6.20, 6.18). In the case when K is a perfect field of characteristic p, the p-adic Mal'cev-Neumann field $\mathbb{K}_p^{sph}$ is algebraically closed if and only if K is algebraically closed and G is divisible (6.20, 7.14).

(6) Each of the fields $K\langle\langle x\rangle\rangle$ and $L[\mathbb{Q}, K]$ is algebraically closed if and only if K is algebraically closed of characteristic 0 (4.15).

(7) The field $K((G))$ is formally real if and only if K is formally real (3.12). In particular, since $K \subset K(x) \subset K((x)) \subset K\langle\langle x\rangle\rangle \subset L[\mathbb{Q}, K] \subset K((\mathbb{Q}))$, each of these fields is formally real if and only if K is formally real.

(8) The field $K((G))$ is real-closed if and only if K is real-closed and G is divisible ([**3**, 6.23 (1)–(2)]). In particular, $K((x))$ is real-closed for no field K.

(9) Each of the fields $K\langle\langle x\rangle\rangle$ and $L[\mathbb{Q}, K]$ is real-closed if and only if K is real-closed (4.12).

(10) $K((G))$ is Archimedean complete of type G if and only if K is isomorphic to $\mathbb{R}$ as ordered field (3.14).

9. Classification of fields.

The following diagram classifies the fields that admit a valuation or general valuation (non-trivial valuation when the field is infinite) in 6 non-overlapping classes. With a suitable choice of some of these classes it is possible to obtain partitions for: the class of ordered fields, the class of Archimedean valued fields, the class of non-Archimedean valued fields, and the class of fields with a general valuation. Below each of the 6 classes, there are some examples of their members.

Archimedean valued fields (first two classes) — non-Archimedean valued fields (fourth and fifth classes)

Non-formally real Archimedean valued fields	Formally real Archimedean valued fields	Formally real fields where $G_K \neq \{0\}$ is non-Archimedean	Formally real non-Archimedean valued fields	Non-formally real non-Archimedean valued fields	non-formally real fields with general valuation of higher rank
$\mathbb{C}$, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{2}, i)$	$\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\pi)$	$\mathbb{R}((\mathbb{Z}^3))$, $L[\mathbb{R}^2, \mathbb{Q}]$, $\mathbb{Q}((x, y))$, $\mathbb{R}((L[\mathbb{Z}, \mathbb{Q}]))$	$\mathbb{R}(x)$, $\mathbb{R}((x))$, $\mathbb{Q}(e)\langle\langle x\rangle\rangle$, $L[\mathbb{Q}, \mathbb{R}]$, $\mathbb{Q}((\mathbb{Q}))$	$\mathbb{Q}_p$, $\mathbb{F}_q$, $\mathbb{C}_p$, $\mathbb{C}(x)$, $\mathbb{F}_q((x))$, $\mathbb{Q}^a\langle\langle x\rangle\rangle$, $\mathbb{Q}_p((\mathbb{Q}))$, $L[\mathbb{Q}, \mathbb{C}]$	$\mathbb{Q}_p((\mathbb{Z}^5))$, $\mathbb{F}_q((x, y))$, $\mathbb{C}((\mathbb{R} \oplus \mathbb{Q}))$, $L[\mathbb{Z}^2, \mathbb{Q}_p]$, $\mathbb{K}_p^{sph}$ with $K = \mathbb{F}_p$ and $G = \mathbb{Q}^3$

Ordered fields (second, third and fourth classes)

Fields with general valuations (third to sixth classes)

In the following, the 6 classes are ordered from left to right.

1. By 2.15 the second class can be described as the collection of all the fields that are isomorphic to a subfield of $\mathbb{R}$ or equivalently as the collection of all the Archimedean ordered fields.
2. By 5.2 the third class can be described as the collection of all the non-Archimedean ordered fields K for which G_K cannot be embedded in $(\mathbb{R}, +)$ ($G_K \neq \{0\}$ is not an Archimedean ordered group).
3. By 5.2 the fourth class can be described as the collection of all the non-Archimedean ordered fields K for which G_K can be embedded in $(\mathbb{R}, +)$ ($G_K \neq \{0\}$ is an Archimedean ordered group).

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