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On the Solutions of Linear Ordinary Differential Equations and Bessel-type Special Functions on the Levi-Civita Field

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Abstract—Because of the disconnectedness of a non-Archimedean ordered field in the topology induced by the order, it is possible to have non-constant functions with zero derivatives everywhere. In fact the solution space of the differential equation $y' = 0$ is infinite dimensional. In this paper, we give sufficient conditions for a function on an open subset of the Levi-Civita field to have zero derivative everywhere and we use the nonconstant zero-derivative functions to obtain non-analytic solutions of systems of linear ordinary differential equations with analytic coefficients. Then we use the results to introduce Bessel-type special functions on the Levi-Civita field and to study some of their properties.

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1. INTRODUCTION

Solutions of linear ordinary differential equations and some Bessel-type special functions on the Levi-Civita field $\mathcal{R}$ [5, 6] are presented in this paper. [1] We recall that the elements of $\mathcal{R}$ are functions from $\mathbb{Q}$ to $\mathbb{R}$ with left-finite support (denoted by "supp"). That is, below every rational number q, there are only finitely many points where the given function does not vanish. For the further discussion, it is convenient to introduce the following terminology.

Definition 1.1 (λ, $\sim$, $\approx$, $=_r$). *For $x \neq 0$ in $\mathcal{R}$, we let $\lambda(x) = \min(supp(x))$, which exists because of the left-finiteness of $supp(x)$, and we let $\lambda(0) = +\infty$.*

Given $x, y \neq 0$ in $\mathcal{R}$, we say that $x \sim y$ if $\lambda(x) = \lambda(y)$, and $x \approx y$ if $\lambda(x) = \lambda(y)$ and $x[\lambda(x)] = y[\lambda(y)]$.

Given $x, y \in \mathcal{R}$ and $r \in \mathbb{R}$, we say that $x =_r y$ if $x[q] = y[q]$ for all $q \leq r$.

At this point, these definitions may look somewhat arbitrary, but after having introduced an order on $\mathcal{R}$, we will see that λ describes orders of magnitude, the relation $\approx$ corresponds to agreement up to infinitely small relative error, while $\sim$ corresponds to agreement of order of magnitude. The set $\mathcal{R}$ is endowed with formal power series multiplication and componentwise addition, which make it into a field (see [3]) in which we can isomorphically embed $\mathbb{R}$ as a subfield via the map $\Pi : \mathbb{R} \to \mathcal{R}$ defined by

$$\Pi(x)[q] = \begin{cases} x & \text{if } q = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

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Definition 1.2 (Order in $\mathcal{R}$). *Let $x, y \in \mathcal{R}$ be given. Then we say $x \geq y$ if $x = y$ or [$x \neq y$ and $(x-y)[\lambda(x-y)] > 0$].*

It is easy to check that the relation "$\geq$" is a total order and $(\mathcal{R}, +, \cdot, \geq)$ is an ordered field (which we denote simply by $\mathcal{R}$). Moreover, the embedding Π in equation (1.1) of $\mathbb{R}$ into $\mathcal{R}$ is compatible with the order. The order induces an absolute value on $\mathcal{R}$ in the natural way: $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. We also note that λ, as defined above, is a valuation. Moreover, the relation "$\sim$" is an equivalence relation, and the set of equivalence classes (the value group) is (isomorphic to) $\mathbb{Q}$. Besides the usual order relations, some other notations are also convenient.

Definition 1.3 ($\ll, \gg$). *Let $x, y \in \mathcal{R}$ be non-negative. We say that x is infinitely smaller than y (and write $x \ll y$) if $nx < y$ for all $n \in \mathbb{N}$; we say that x is infinitely larger than y (and write $x \gg y$) if $y \ll x$. If $x \ll 1$, then we say that x is infinitely small; if $x \gg 1$, then we say that x is infinitely large. Infinitely small numbers are also called infinitesimals or differentials. Infinitely large numbers are also called infinite. Non-negative numbers that are neither infinitely small nor infinitely large are called finite numbers.*

Definition 1.4 (The Number d). *Let d be the element of $\mathcal{R}$ given by $d[1] = 1$ and $d[q] = 0$ for $q \neq 1$.*

It is easy to check that $d^q \ll 1$ if $q > 0$ and $d^q \gg 1$ if $q < 0$. Moreover, for all $x \in \mathcal{R}$, the elements of $\operatorname{supp}(x)$ can be arranged in ascending order, say $\operatorname{supp}(x) = \{q_1, q_2, \ldots\}$ with $q_j < q_{j+1}$ for all j, and x can be written as $x = \sum_{j=1}^{\infty} x[q_j] d^{q_j}$, where the series converges in the topology induced by the absolute value (see [3]).

Altogether, it follows that $\mathcal{R}$ is a non-Archimedean field extension of $\mathbb{R}$. For a detailed study of this field, we refer the reader to [10, 20], and references therein. In particular, it is shown that $\mathcal{R}$ is complete with respect to the topology induced by the absolute value, that is, every Cauchy sequence of elements of $\mathcal{R}$ converges to an element of $\mathcal{R}$. In the wider context of valuation theory, it is interesting to note that the topology induced by the absolute value is the same as that introduced via the valuation λ, as it was shown in [19].

It follows therefore that the field $\mathcal{R}$ is just a special case of the class of fields discussed in [9]. For a general overview of the algebraic properties of formal power series fields in general, we refer the reader to the comprehensive overview by Ribenboim [8], and for an overview of the related valuation theory to the books by Krull [4], Schikhof [9] and Alling [1]. A thorough and complete treatment of ordered structures can also be found in [7].

Besides being the smallest ordered non-Archimedean field extension of the real numbers that is both complete in the order topology and real closed, the Levi-Civita field $\mathcal{R}$ is of particular interest because of its practical usefulness. Since the supports of the elements of $\mathcal{R}$ are left-finite, it is possible to represent these numbers on a computer (see [3]). Having infinitely small numbers, the errors in classical numerical methods can be made infinitely small, and hence irrelevant in all practical applications. One such application is the computation of derivatives of real functions representable on a computer, where both the accuracy of formula manipulators and the speed of classical numerical methods are achieved (see [16]).

In this paper we present some tools to construct a large class of solutions for equation $y' = 0$ on $\mathcal{R}$. Then as an application of that, we define and study the properties of Bessel-type special functions on (open subsets of) $\mathcal{R}$.

2. PRELIMINARY RESULTS

For an easier study of systems of linear ordinary differential equations on $\mathcal{R}$, it is beneficial to introduce matrix exponentials on $\mathcal{R}$. We define matrices on $\mathcal{R}$ and matrix operations: addition, multiplication, determinant, just as we do in the real case.

Definition 2.1. *Let $\mathcal{M}_n(\mathcal{R})$ denote the set of all $n \times n$ matrices with entries in $\mathcal{R}$. For $A \in \mathcal{M}_n(\mathcal{R})$, we define $|\cdot| : \mathcal{M}_n(\mathcal{R}) \to \mathcal{R}$ by*

$$|A| = \max_{1 \leq i,j \leq n} \{|a_{ij}|\},$$

where $|a_{ij}| = \max\{a_{ij}, -a_{ij}\}$.

In what follows we deal only with square matrices whose entries are at most finite in absolute value, and we denote this class of matrices by $\mathcal{M}_n^f(\mathcal{R})$.

Definition 2.2. *Let $A \in \mathcal{M}_n^f(\mathcal{R})$, and for each $k \in \mathbb{N} \cup \{0\}$ let $c_k \in \mathcal{R}$ be given. We say that the series $\sum_{k=0}^{\infty} c_k A^k$ is convergent in $\mathcal{M}_n^f(\mathcal{R})$ if the series $\sum_{k=0}^{\infty} c_k |A^k|$ is convergent in $\mathcal{R}$ with respect to the weak topology discussed in [3, 12, 17].*

Definition 2.3. *Let $A \in \mathcal{M}_n^f(\mathcal{R})$ be given. We define the exponential of A by the series $e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$, where $A^0 = I_n$ is the $n \times n$ identity matrix.*

In the next theorem we show that the series in Definition 2.3 converges in the sense of Definition 2.2.

Theorem 2.1. *For any $A \in \mathcal{M}_n^f(\mathcal{R})$, the series*

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

is always convergent, and hence e^A is well-defined.

Proof. Let $A = (a_{ij})$, and let $A^2 = (b_{ij})$. Then, by the way we perform matrix multiplication, for all $i, j \in \{1, \ldots, n\}$ we have

$$|b_{ij}| \leq n \left(\max_{1 \leq i,j \leq n} \{|a_{ij}|\} \right) \left(\max_{1 \leq i,j \leq n} \{|a_{ij}|\} \right) = n|A|^2.$$

It follows that

$$\left|A^2\right| = \max_{1 \leq i,j \leq n} \{|b_{ij}|\} \leq n|A|^2.$$

Using induction on k, it is then easy to show that

$$\left|A^k\right| \leq n^{k-1} |A|^k, \quad \forall k \in \mathbb{N}. \tag{2.1}$$

Since

$$\sum_{k=1}^{\infty} \frac{n^{k-1}}{k!} |A|^k = \frac{1}{n} \left(\sum_{k=1}^{\infty} \frac{1}{k!} (n|A|)^k \right)$$

converges in the weak topology of $\mathcal{R}$ to $\left(e^{n|A|} - 1\right)/n$ (because $|A|$ is at most finite), it follows from equation (2.1) and the properties of weak convergence of infinite series (see [12, 17]) that

$$\sum_{k=0}^{\infty} \frac{1}{k!} \left|A^k\right| = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left|A^k\right|$$

converges weakly in $\mathcal{R}$. Hence the series $\sum_{k=0}^{\infty} \frac{1}{k!} A^k$ converges in the sense of Definition 2.2. Theorem 2.1 is proved.

Taking into account that power series on $\mathcal{R}$ can be differentiated term by term within their domain of convergence (see [19]), we obtain the following result.

Theorem 2.2. *Let $A \in \mathcal{M}_n^f(\mathcal{R})$ and let $F : \mathcal{R} \to \mathcal{M}_n(\mathcal{R})$ be given by $F(t) = e^{tA}$. Then F is differentiable at each $t \in \mathcal{R}$, with derivative $F'(t) = Ae^{tA}$.*

Proof. Since $F(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$, we can obtain $F'(t)$ by differentiating the series term by term as a function of t:

$$F'(t) = \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} A^k = A \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = Ae^{tA} = AF(t).$$

Note that all the series in the last equation are well-defined.

3. THE MAIN RESULTS

3.1. Linear ordinary differential equations on $\mathcal{R}$

One of the main goals of this paper is to obtain solutions of linear ordinary differential equations in the case where the coefficients are analytic functions of the independent variable, including non-analytic solutions in addition to the analytic ones. The basic idea for the construction of non-analytic solutions of linear ordinary differential equations on $\mathcal{R}$ is based on the following theorem (proved in [11]), which shows that even the simplest differential equation $y' = 0$ over $\mathcal{R}$ has infinitely many linearly independent solutions on $[-1, 1] \subset \mathcal{R}$.

Theorem 3.1. *The solution space of the differential equation $y' = 0$ on $[-1, 1]$ is infinite dimensional.*

Proof. For each $n \in \mathbb{N}$, let $g_n : [-1, 1] \to \mathcal{R}$ be given by $g_n(x)[q] = x[q/(n+1)]$. We show that, for all $n \in \mathbb{N}$, g_n is differentiable on $[-1, 1]$ with $g_n'(x) = 0$ for all $x \in [-1, 1]$. So let $n \in \mathbb{N}$ be given. We first observe that $g_n(x+y) = g_n(x) + g_n(y)$ for all $x, y \in [-1, 1]$. Now let $x \in [-1, 1]$ and $\epsilon > 0$ in $\mathcal{R}$ be given. Let $\delta = \min\{\epsilon^2, d\}$, and let $y \in [-1, 1]$ be such that $0 < |y - x| < \delta$. Then, taking into account that $g(y - x) \sim (y - x)^{n+1}$, we have

$$\left|\frac{g_n(y) - g_n(x)}{y - x}\right| = \left|\frac{g_n(y - x)}{y - x}\right| \sim |y - x|^n.$$

Next, since $|y - x| < \min\{\epsilon^2, d\}$, we obtain that $|y - x|^n \ll \epsilon$. Hence

$$\left|\frac{g_n(y) - g_n(x)}{y - x}\right| < \epsilon \text{ for all } y \in [-1, 1] \text{ satisfying } 0 < |y - x| < \delta.$$

It follows that g_n is differentiable at x, with $g_n'(x) = 0$. This is true for all $x \in [-1, 1]$ and for all $n \in \mathbb{N}$. Hence g_n is a solution of the differential equation $y' = 0$ on $[-1, 1]$ for all $n \in \mathbb{N}$.

Next, we show that the set $S = \{g_n : n \in \mathbb{N}\}$ is linearly independent on $[-1, 1]$. So let $j \in \mathbb{N}$ and let $n_1 < n_2 < \cdots < n_j$ in $\mathbb{N}$ be given. It is enough to show that $g_{n_1}, g_{n_2}, \ldots, g_{n_j}$ are linearly independent on $[-1, 1]$. To this end, we suppose that $c_1 g_{n_1} + c_2 g_{n_2} + \cdots + c_j g_{n_j} = 0$ for some $c_1, c_2, \ldots, c_j$ in $\mathcal{R}$, and show that $c_1 = c_2 = \cdots = c_j = 0$. Indeed, since $c_1 g_{n_1} + c_2 g_{n_2} + \cdots + c_j g_{n_j} = 0$, we obtain that $c_1 g_{n_1}(d) + c_2 g_{n_2}(d) + \cdots + c_j g_{n_j}(d) = 0$. Hence $c_1 d^{n_1} + c_2 d^{n_2} + \cdots + c_j d^{n_j} = 0$, from which we infer that $c_1 = c_2 = \cdots = c_j = 0$.

Remark 3.1. *For each $n \in \mathbb{N}$, it is easy to check that the mapping g_n in the proof of Theorem 3.1 is an order preserving field automorphism of $\mathcal{R}$; this is a special property of non-Archimedean structures since it is well-known that the only field automorphism of $\mathbb{R}$ is the identity map ([13]).*

In Propositions 3.1 - 3.3 that follow, we give sufficient conditions for a nonconstant function to be a solution of the differential equation $y' = 0$ on an open subset of $\mathcal{R}$.

Proposition 3.1. *Let $M \subseteq \mathcal{R}$ be open and let $f : M \to \mathcal{R}$ be such that, for some fixed $p > 1$ in $\mathbb{Q}$ and for some positive $\eta \ll 1$ in $\mathcal{R}$, we have*

$$\forall x, y \in M, \quad \lambda(x - y) \geq \lambda(\eta) \Rightarrow |f(x) - f(y)| \sim |x - y|^p.$$

Then f is differentiable on M with derivative $f'(x) = 0$ for all $x \in M$, that is, f is a solution for the differential equation $y' = 0$ on M.

Proof. Let $x \in M$ and $\epsilon > 0$ in $\mathcal{R}$ be given. Since M is open, there exists $\delta_0 > 0$ in $\mathcal{R}$ such that $(x - \delta_0, x + \delta_0) \subset M$. Let

$$\delta = \min\{\delta_0, \epsilon^{\frac{2}{p-1}}, \eta\}.$$

Then for all $y \in \mathcal{R}$ satisfying $0 < |y - x| < \delta$, we have that $y \in M$ and $\lambda(y - x) \geq \lambda(\delta) \geq \lambda(\eta)$. Hence

$$\left|\frac{f(x) - f(y)}{x - y}\right| \sim |x - y|^{p-1} < \delta^{p-1} = \min\left\{\delta_0^{p-1}, \epsilon^2, \eta^{p-1}\right\} \leq \min\left\{\epsilon^2, \eta^{p-1}\right\} \ll \epsilon.$$

The last step is justified by the fact that if $\epsilon \ll 1$, then $\epsilon^2 \ll \epsilon$ and if ϵ is finite or infinitely large, then $\eta^{p-1} \ll \epsilon$ since $\eta \ll 1$ and $p-1>0$. So in both cases, we have $\min\{\epsilon^2, \eta^{p-1}\} \ll \epsilon$. Thus, f is differentiable at x for all $x \in M$ with $f'(x)=0$.

Proposition 3.2. *Let $M \subseteq \mathcal{R}$ be open and let $f : M \to \mathcal{R}$ be such that, for some fixed $p>1$ in $\mathbb{Q}$ and for some positive $\eta \ll 1$ and positive α in $\mathcal{R}$, we have*

$$\forall x, y \in M, \quad \lambda(x-y) \geq \lambda(\eta) \Rightarrow |f(x)-f(y)| \leq \alpha |x-y|^p.$$

Then f is a solution for the differential equation $y'=0$ on M.

Proof. Let $x \in M$ and $\epsilon > 0$ in $\mathcal{R}$ be given. Then there exists $\delta_0 > 0$ in $\mathcal{R}$ such that $(x-\delta_0, x+\delta_0) \subset M$. Let

$$\delta = \min\left\{\delta_0, \eta, \left(\epsilon^2/\alpha\right)^{1/(p-1)}, (\eta/\alpha)^{1/(p-1)}\right\}.$$

Then $\delta > 0$ and for $y \in \mathcal{R}$ satisfying $0 < |y-x| < \delta$, we have that $y \in M$ and $\lambda(y-x) \geq \lambda(\delta) \geq \lambda(\eta)$. Hence

$$\left|\frac{f(x)-f(y)}{x-y}\right| \leq \alpha|x-y|^{p-1} < \alpha\delta^{p-1} = \min\left\{\alpha\delta_0^{p-1}, \alpha\eta^{p-1}, \epsilon^2, \eta\right\} \leq \min\left\{\epsilon^2, \eta\right\} \ll \epsilon.$$

This shows that f is differentiable at x for all $x \in M$ with $f'(x)=0$.

Remark 3.2. *We note that Proposition 3.1 follows from Proposition 3.2 if we take α to be any infinitely large positive number.*

Definition 3.1. *Let $M \subseteq \mathcal{R}$ and $h : M \to \mathcal{R}$. We say that h is level preserving on M and write $h \in P(M)$ if $\forall x, y \in M$ satisfying $\lambda(x) = \lambda(y)$ and $x =_r y$ it follows that $\lambda(h(x)) = \lambda(h(y))$ and $h(x) =_q h(y)$, where $q \geq \lambda(h(x)) + r - \lambda(x)$.*

Example 3.1. *Let $f : \mathcal{R} \to \mathcal{R}$ be given by $f(x)[q] = x[q-1]$. Then it is easy to check that $f \in P(\mathcal{R})$.*

Proposition 3.3. *Let $M \subset \mathcal{R}$ be open and such that $\lambda(x) \geq 0$ for all $x \in M$. Let $h : M \to \mathcal{R}$ be a level preserving function on M, and let $\alpha \in \mathbb{Q}, \alpha > 1$ be given. Then the function $f : M \to \mathcal{R}$, given by*

$$f(x)[q] = \begin{cases} h(x)\left[\frac{q\lambda(h(x))}{\lambda(x)\alpha}\right], & \text{if } \lambda(x) > 0, \\ h(x)\left[q + \lambda(h(x))\right], & \text{if } \lambda(x) = 0 \end{cases} \tag{3.1}$$

is differentiable on M with derivative $f'(x) = 0$ for all $x \in M$.

Proof. Let $x \in \mathcal{R}$ and $\epsilon > 0$ in $\mathcal{R}$ be given. Since M is open, there exists $\eta > 0$ in $\mathcal{R}$ such that $\eta \ll 1$ and $(x-\eta, x+\eta) \subset M$. Let $\delta = \min\{\epsilon^{\frac{2}{\alpha-1}}, \eta\}$. Then $0 < \delta \ll 1$ and $(x-\delta, x+\delta) \subset M$. Now assuming $y \in M$ such that $0 < |y-x| < \delta$, we show that

$$\left|\frac{f(x)-f(y)}{x-y}\right| < \epsilon. \tag{3.2}$$

We note that, since $|y-x| \ll 1$, then either $\lambda(x) = 0 = \lambda(y)$ or $[\lambda(x) > 0$ and $\lambda(y) > 0]$.

First assume that $\lambda(x) > 0$ (and hence $\lambda(y) > 0$). We distinguish three cases.

Case 1: $\lambda(x) \neq \lambda(y)$. In this case, we have $\lambda(f(x)) \neq \lambda(f(y))$, and it follows that

$$\left|\frac{f(x)-f(y)}{x-y}\right| \sim (x-y)^{\alpha-1}.$$

Hence

$$\lambda\left(\frac{f(x)-f(y)}{x-y}\right) = (\alpha-1)\lambda(y-x) \geq (\alpha-1)\lambda(\delta)$$

$$= (\alpha - 1)\max\left\{\tfrac{2}{\alpha-1}\lambda(\epsilon), \lambda(\eta)\right\} = \max\left\{2\lambda(\epsilon), (\alpha-1)\lambda(\eta)\right\} > \lambda(\epsilon),$$

implying that

$$\left|\frac{f(x)-f(y)}{x-y}\right| \ll \epsilon. \tag{3.3}$$

Case 2: $x \sim y$ and $x[\lambda(x)] \neq y[\lambda(y)]$. In this case the arguments are similar to those of Case 1.
Case 3: $x =_r y$ for some $r \in \mathbb{Q}$ with $r \geq \lambda(x)$. Then obviously $\lambda(|x-y|) = r_+$, where r_+ is a rational number such that $r_+ > r$. It follows that $\lambda(|f(x)-f(y)|) = \alpha r_+$. Thus, we have

$$\lambda\left(\frac{f(x)-f(y)}{x-y}\right) = \lambda(f(x)-f(y)) - \lambda(x-y) = (\alpha-1)r_+ > (\alpha-1)\lambda(\delta),$$

from which, as in the Case 1, we obtain (3.3). Finally, if $\lambda(x) = 0 = \lambda(y)$ then the proof of the inequality (3.2) follows by the same arguments as above (when $\lambda(x) > 0$), except that we have to use the appropriate expression for f from equation (3.1).

In the following definition we introduce the class of all functions that are differentiable with derivative equal to 0 everywhere on an open subset of $\mathcal{R}$.

Definition 3.2. *Let $M \subseteq \mathcal{R}$ be open and let $f : M \to \mathcal{R}$. We define the class of functions $D_0^1(M)$ as follows:*

$$D_0^1(M) = \left\{f : M \to \mathcal{R} | f \text{ is differentiable on } M,\ f'(x) = 0\ \forall x \in M\right\}.$$

3.2. Systems of linear ordinary differential equations on $\mathcal{R}$

In this subsection we investigate the solutions of systems of linear ordinary differential equations on $\mathcal{R}$, using the functions of class D_0^1. The main goal of this subsection is to obtain solutions of systems of linear ODE's of the form: $Y'(t) = A(t)Y(t) + B(t)$, where $Y(t)$ is a vector of dimension $n > 0$, $n \in \mathbb{N}$, which contains the unknown functions, $Y'(t)$ contains the derivatives of the functions from $Y(t)$, $A(t) \in \mathcal{M}_n^f(\mathcal{R})$, for all $t \in \mathcal{R}$, which contains the coefficient functions of the system, and $B(t)$ is a vector of dimension n, which contains functions that ensure the inhomogeneity of the system. In order to realize this we study a few cases, going from the most special to the most general ones.

Theorem 3.2. *Consider the linear homogeneous system of ordinary differential equations with constant coefficients which are at most finite in absolute value $Y'(t) = AY(t)$. Then the solution is given by*

$$Y(t) = e^{At}C + e^{At}U_{na}(t),$$

where $C \in \mathcal{M}_{n,1}(\mathcal{R})$ is a vector containing constants, and $U_{na}(t) \in \mathcal{M}_{n,1}(D_0^1)$ is a vector which contains functions of class D_0^1.

Proof. We rewrite the system in the form: $Y'(t) - AY(t) = O_{n,1}$, which is equivalent to

$$e^{-At}Y'(t) - e^{-At}AY(t) = O_{n,1} \quad \text{or} \quad \left(e^{-At}Y(t)\right)' = O_{n,1}.$$

It follows that $e^{-At}Y(t) = C + U_{na}(t)$, where $C \in \mathcal{M}_{n,1}(\mathcal{R})$ and the elements of $U_{na}(t)$ are functions of class D_0^1, and hence $Y(t) = e^{At}C + e^{At}U_{na}(t)$.

Remark 3.3. *Theorem 3.2 shows that the solutions of linear homogeneous systems with constant coefficients over $\mathcal{R}$ are very similar to those of the real case, except that the solutions in the non-Archimedean case may also involve non-analytic functions with zero-derivatives.*

Since we know how to integrate $\mathcal{R}$-analytic functions [14] in the Lebesgue-like theory developed in [15, 18], we can study next those inhomogeneous systems, where the functions ensuring the inhomogeneity are $\mathcal{R}$-analytic.

Theorem 3.3. *Consider the inhomogeneous system of linear ordinary differential equations with constant coefficients: $Y'(t) = AY(t) + B(t)$ on the interval $[a, b] \subset \mathcal{R}$, where $|A|$, $|a|$ and $|b|$ are at most finite in absolute value, and $B(t)$ is a vector, which contains functions that are $\mathcal{R}$-analytic on $[a, b]$. Then the solution is given by the equation*

$$Y(t) = e^{At}C + e^{At}U_{na}(t) + e^{At}\int_{[a,t]} e^{-As}B(s),$$

where $C \in \mathcal{M}_{n,1}(\mathcal{R})$ and the elements of $U_{na}(t)$ are functions of class D_0^1.

Proof. We rewrite the system in the form: $Y'(t) - AY(t) = B(t)$, which is equivalent to $\left(e^{-At}Y(t)\right)' = e^{-At}B(t)$. It follows that

$$e^{-At}Y(t) = C + U_{na}(t) + \int_{[a,t]} e^{-As}B(s),$$

where $C \in \mathcal{M}_{n,1}(\mathcal{R})$, and $U_{na}(t) \in \mathcal{M}_{n,1}(D_0^1)$, and hence

$$Y(t) = e^{At}C + e^{At}U_{na}(t) + e^{At}\int_{[a,t]} e^{-As}B(s).$$

Note that we have used the fact that $e^{At}B(t)$ is a vector whose components are products of functions that are $\mathcal{R}$-analytic on $[a, b]$, and hence the components themselves are $\mathcal{R}$-analytic on $[a, b]$.

The proofs of the next two theorems (Theorems 3.4 and 3.5) are similar to those of Theorems 3.2 and 3.3 above, and therefore they are stated without proofs.

Theorem 3.4. *Consider the homogeneous system of linear ordinary differential equations with non-constant coefficients $Y'(t) = A(t)Y(t)$ on $[a, b]$, where $|a|$ and $|b|$ are at most finite, and where $A(t)$ is an $n \times n$ matrix whose elements are $\mathcal{R}$-analytic on $[a, b]$ and such that $|A(t)|$ is at most finite for all $t \in [a, b]$. Then the solution is given by*

$$Y(t) = e^{\int_{[a,t]} A(s)}C + e^{\int_{[a,t]} A(s)}U_{na}(t),$$

where $C \in \mathcal{M}_{n,1}(\mathcal{R})$ and $U_{na}(t) \in \mathcal{M}_{n,1}(D_0^1)$.

Theorem 3.5. *Consider the inhomogeneous system of linear ordinary differential equations with non-constant coefficients: $Y'(t) = A(t)Y(t) + B(t)$ on $[a, b]$, where $|a|$ and $|b|$ are at most finite, $A(t)$ is an $n \times n$ matrix whose elements are $\mathcal{R}$-analytic on $[a, b]$ and such that $|A(t)|$ is at most finite for all $t \in [a, b]$, and $B(t)$ is a vector whose components are functions that are $\mathcal{R}$-analytic on $[a, b]$. Then the solution is given by*

$$Y(t) = e^{\int_{[a,t]} A(s)}C + e^{\int_{[a,t]} A(s)}U_{na}(t) + e^{\int_{[a,t]} A(s)}\int_{[a,t]} e^{-\int_{[a,s]} A(r)}B(s),$$

where $C \in \mathcal{M}_{n,1}(\mathcal{R})$ and $U_{na}(t) \in \mathcal{M}_{n,1}(D_0^1)$.

3.3. Bessel-type special functions on $\mathcal{R}$

In this subsection we study Bessel-type special functions on $\mathcal{R}$, with the help of the solutions for systems of linear ordinary differential equations that we developed in Subsection 3.2. We introduce such functions with the following problem.

Problem 1. *Let's study the solutions of the differential equation*

$$t^2y'' + ty' + (t^2 - \nu^2)y = 0. \qquad (3.4)$$

on $[a, 1] \subset \mathcal{R}$, where $0 < a < 1$, a is finite, and $\nu \in \mathbb{Q} \setminus \mathbb{Z}$.

We call equation (3.4) the Bessel equation of order ν, and we study its solutions below. It is easy to check, similar to the real case, that the functions

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{t}{2}\right)^{2n}, \tag{3.5}$$

$$J_{-\nu}(t) = \left(\frac{t}{2}\right)^{-\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n-\nu+1)} \left(\frac{t}{2}\right)^{2n} \tag{3.6}$$

are two linearly independent analytic solutions of equation (3.4), for all $t \in [a, 1]$, where Γ is the Euler's Gamma function. Moreover, it follows from Corollary 3.10 in [17] that the power series in equations (3.5) and (3.6) converge weakly in $[a, 1]$.

The main objective of Problem 1 is to construct solutions of equation (3.4) that involve functions of the class D_0^1, and to study their properties. We set $w_1 = y$, $w_2 = y'$, and consider the following system of two linear ordinary differential equations:

$$\begin{cases} w_1' = w_2 \\ w_2' = \left(1 - \frac{\nu^2}{t^2}\right) w_1 + \frac{1}{t} w_2, \end{cases}$$

which we can write in matrix form as

$$W'(t) = A(t)W(t), \tag{3.7}$$

where

$$W(t) = \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}; \quad W'(t) = \begin{pmatrix} w_1'(t) \\ w_2'(t) \end{pmatrix}; \quad A(t) = \begin{pmatrix} 0 & 1 \\ 1 - \frac{\nu^2}{t^2} & \frac{1}{t} \end{pmatrix}.$$

Because the elements of $A(t)$ are $\mathcal{R}$-analytic functions on $[a, 1]$ and they are at most finite in absolute value for all $t \in [a, 1]$, we can use Theorem 3.4 to write the solutions of equation (3.7), which are also the solutions of equation (3.4). Let

$$D(t) = \int_{[a,t]} A(s) = \begin{pmatrix} 0 & t-a \\ t - a + \nu^2(\frac{1}{t} - \frac{1}{a}) & \ln \frac{t}{a} \end{pmatrix},$$

where the function ln is $\mathcal{R}$-analytic on $[a, 1]$.

Thus, the solution of equation (3.7), and hence of equation (3.4), on $[a, 1]$ has the form

$$W(t) = e^{D(t)}C + e^{D(t)}U_{na}(t),$$

where $C \in \mathcal{M}_{2,1}(\mathcal{R})$ and $U_{na}(t) \in \mathcal{M}_{2,1}(D_0^1)$.

Taking into account that the analytic part of the solution has the form $c_1 J_\nu(t) + c_2 J_{-\nu}(t)$, we conclude that in the first row of the matrix $e^{D(t)}$ we can take the entries to be $D_{11} = J_\nu(t)$ and $D_{12} = J_{-\nu}(t)$. With $C = \binom{c_1}{c_2}$, it follows that the first component of $W(t)$, which is $y = w_1(t)$, has the form:

$$y = c_1 \left[J_\nu(t) + J_\nu(t) g_\nu(t)\right] + c_2 \left[J_{-\nu}(t) + J_{-\nu}(t) g_{-\nu}(t)\right],$$

where $g_\nu, g_{-\nu} \in D_0^1([a, 1])$.

Now we are in a position to define Bessel functions of the first kind on $\mathcal{R}$.

Definition 3.3. *For $0 < a < 1$, $a \in \mathcal{R}$ finite, we define the functions $\mathcal{J}_\nu, \mathcal{J}_{-\nu} : [a, 1] \subset \mathcal{R} \to \mathcal{R}$ by*

$$\mathcal{J}_\nu(t) = J_\nu(t) + J_\nu(t) g_\nu(t), \text{ and}$$

$$\mathcal{J}_{-\nu}(t) = J_{-\nu}(t) + J_{-\nu}(t) g_{-\nu}(t),$$

where $\nu \in \mathbb{Q} \setminus \mathbb{Z}$ and $g_\nu, g_{-\nu} \in D_0^1([a, 1])$. We call the functions $\mathcal{J}_\nu, \mathcal{J}_{-\nu}$ Bessel functions of the first kind and of order ν, and $-\nu$, respectively.

Next, we study some properties of the Bessel functions $\mathcal{J}_\nu$ and $\mathcal{J}_{-\nu}$.

Theorem 3.6. *Under the notation of Problem 1 and Definition 3.3 above, the following two statements are true for all $t \in [a, 1]$:*

$$(a)\quad \frac{2\nu}{t}\mathcal{J}_\nu(t) = \Big(J_{\nu-1}(t) + J_{\nu+1}(t)\Big)\Big(1 + g_\nu(t)\Big),$$
$$(b)\quad 2\mathcal{J}_\nu'(t) = \Big(J_{\nu-1}(t) - J_{\nu+1}(t)\Big)\Big(1 + g_\nu(t)\Big).$$

Proof. Using equation (3.5), we can write

$$\left(\frac{J_\nu(t)}{t^\nu}\right)' = \frac{1}{2^\nu}\sum_{n=1}^{\infty}\frac{(-1)^n t^{2n-1}}{(n-1)!\Gamma(n+\nu+1)2^{2n-1}} = -\frac{J_{\nu+1}(t)}{t^\nu}.$$

Then, using the fact that $\mathcal{J}_\nu(t) = J_\nu(t) + J_\nu(t)g_\nu(t)$ and $g_\nu'(t) = 0, \forall t \in [a, 1]$, we obtain

$$\left(\frac{\mathcal{J}_\nu(t)}{t^\nu}\right)' = -\frac{J_{\nu+1}(t)}{t^\nu} - \frac{J_{\nu+1}(t)}{t^\nu}g_\nu(t),$$

implying that

$$\mathcal{J}_\nu'(t) = \frac{\nu}{t}\mathcal{J}_\nu(t) - J_{\nu+1}(t) - J_{\nu+1}(t)g_\nu(t) \tag{3.8}$$

Similarly, we can show that

$$\frac{1}{t}\Big(t^\nu \mathcal{J}_\nu(t)\Big)' = t^{\nu-1}J_{\nu-1}(t) + t^{\nu-1}J_{\nu-1}(t)g_\nu(t),$$

implying that

$$\mathcal{J}_\nu'(t) = J_{\nu-1}(t) - \frac{\nu}{t}\mathcal{J}_\nu(t) + J_{\nu-1}(t)g_\nu(t). \tag{3.9}$$

If we subtract equation (3.9) from equation (3.8), we get statement (a) of the theorem, and if we add the two equations, we get (b).

Remark 3.4. *Just as we did in Theorem 3.6, we can obtain other recursive relations for $\mathcal{J}_\nu$ and $\mathcal{J}_{-\nu}$ that would extend the classical recursive relations for J_ν and $J_{-\nu}$ from Real Calculus to the non-Archimedean calculus on $\mathcal{R}$.*

3.4. Generalized Bessel functions of the first kind on $\mathcal{R}$

We give some basic definitions based on those given by Á. Baricz (see [2]) in the classical case (real and complex). As before, let $0 < a < 1$, $a \in \mathcal{R}$ finite, $p \in \mathbb{Q} \setminus \mathbb{Z}$, and let $b, c \in \mathbb{R}$ in the rest of this paper.

Definition 3.4. *The differential equation*

$$t^2 w''(t) + btw(t) + [ct^2 - p^2 + (1-b)p]w(t) = 0 \tag{3.10}$$

will be referred to as the generalized Bessel equation of order p (see [2]), and any solution of it will be called a generalized Bessel function of order p.

The generalized Bessel functions permit the study of Bessel functions, spherical Bessel functions and modified Bessel functions together. That is why it is very important to extend this kind of functions to the field $\mathcal{R}$.

Remark 3.5. *As in the classical case (see [2]), it can easily be verified that the function*

$$w_p(t) = \sum_{n=0}^{\infty}\frac{(-c)^n}{n!\Gamma(p+n+\frac{b+1}{2})}\left(\frac{t}{2}\right)^{2n+p}$$

is a solution of equation (3.10). Moreover, when $c = b = 1$ we get the Bessel function of the first kind of order p discussed in the previous subsection.

Similar to Definition 3.3, we introduce generalized Bessel functions of the first kind on $\mathcal{R}$ as follows.

Definition 3.5. *Let* $0 < a < 1$, $a \in \mathbb{R}$ *finite,* $p \in \mathbb{Q} \setminus \mathbb{Z}$, *and let* $b, c \in \mathbb{R}$. *Then the function* $\mathcal{W}_p(t) = w_p(t) + w_p(t)g_p(t)$, *where* $g_p \in D_0^1([a,1])$, *and*

$$w_p(t) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!\Gamma(p+n+\frac{b+1}{2})} \left(\frac{t}{2}\right)^{2n+p}$$

is a solution of equation (3.10). We call $\mathcal{W}_p(t)$ *a generalized Bessel function of order* p *on* $[a,1]$.

The proof of the next result is similar to that of Theorem 3.6, as well as to the proof of the corresponding result in the classical case (see [2], Lemma 1.1), and therefore we state it without proof.

Theorem 3.7. *Under the notation in Definition 3.5, the following statements are true for all* $t \in [a,1]$

$$(a) \qquad \tfrac{2p+b-1}{t}\mathcal{W}_p(t) = \Big(w_{p-1}(t) + cw_{p+1}(t)\Big)\Big(1 + g_p(t)\Big),$$

$$(b) \quad (2p+b-1)\mathcal{W}_p'(t) = \Big(pw_{p-1}(t) - (p+b-1)cw_{p+1}(t)\Big)\Big(1 + g_p(t)\Big).$$

REFERENCES

1. N. L. Alling, *Foundations of Analysis over Surreal Number Fields* (North Holland, Amsterdam, 1987).
2. Á. Baricz, *Generalized Bessel Functions of the First Kind* (Springer, Berlin, 2010).
3. M. Berz, "Calculus and numerics on Levi-Civita fields", In M. Berz, C. Bischof, G. Corliss, and A. Griewank, editors, SIAM, Computational Differentiation: Techniques, Applications, and Tools, 19–35, 1996.
4. W. Krull, "Allgemeine Bewertungstheorie", J. Reine Angew. Math., **167**, 160–196, 1932.
5. T. Levi-Civita, "Sugli infiniti ed infinitesimi attuali quali elementi analitici", Atti Ist. Veneto di Sc., Lett. ed Art., **7a**, 4:1765, 1892.
6. T. Levi-Civita, "Sui numeri transfiniti", Rend. Acc. Lincei, **5a**, 7:91,113, 1898.
7. S. Priess-Crampe, *Angeordnete Strukturen: Gruppen, Körper, projektive Ebenen* (Springer, Berlin, 1983).
8. P. Ribenboim, "Fields: Algebraically Closed and Others", Manuscripta Mathematica, **75**, 115–150, 1992.
9. W. H. Schikhof, *Ultrametric Calculus: An Introduction to p-Adic Analysis* (Cambridge University Press, Camridge, 1985).
10. K. Shamseddine, "New Elements of Analysis on the Levi-Civita Field", PhD thesis, Michigan State University, report MSUCL-1147, 1999.
11. K. Shamseddine, "On the existence and uniqueness of solutions of ordinary differential equations on the Levi-Civita field", Int. J. Differ. Equ. Appl., **4**, 375–386, 2002.
12. K. Shamseddine, "On the topological structure of the Levi-Civita field", J. Math. Anal. Appl., **368**, 281–292, 2010.
13. K. Shamseddine, "Nontrivial order preserving automorphisms of non-Archimedean fields", Contemp. Math., **547**, 217–225, 2011.
14. K. Shamseddine, "A brief survey of the study of power series and analytic functions on the Levi-Civita fields", Contemp. Math., **596**, 269–280, 2013.
15. K. Shamseddine, "New results on integration on the Levi-Civita field", Indag. Math. (N.S.), **24** (1), 199–211, 2013.
16. K. Shamseddine and M. Berz, "Exception handling in derivative computation with non-Archimedean calculus", In M. Berz, C. Bischof, G. Corliss, and A. Griewank, editors, SIAM, Computational Differentiation: Techniques, Applications, and Tools, 37–51, 1996. .
17. K. Shamseddine and M. Berz, "Convergence on the Levi-Civita field and study of power series", In Proc. Sixth International Conference on p-adic Functional Analysis, New York, Marcel Dekker, 283–299, 2000.
18. K. Shamseddine and M. Berz, "Measure theory and integration on the Levi-Civita field", Contemp. Math., **319**, 369–387, 2003.
19. K. Shamseddine and M. Berz, "Analytical properties of power series on Levi-Civita fields", Ann. Math. Blaise Pascal, **12** (2), 309–329, 2005.
20. K. Shamseddine and M. Berz, "Analysis on the Levi-Civita field, a brief overview", Contemp. Math., **508**, 215–237, 2010.