Sections 6.1-6.3, 7.1 and 7.2 (Jackson's book)

6.1 - 6.3: Maxwell's Equations for time dependent fields and sources; vector and Scalar potentials; Gauge transformations

So far, the laws of electricity and magnetism, in differential form, are

$$\vec{\nabla}\cdot\vec{D} = \rho \quad \text{(Coulomb's Law)} \quad (1)$$

$$\vec{\nabla}\times\vec{H} = \vec{J} \quad \left(\text{Ampère's Law, magnetostatics } \vec{\nabla}\cdot\vec{J}=0\right) \quad (2)$$

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = \vec{0} \quad \text{(Faraday's Law)} \quad (3)$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \text{(No free magnetic poles)} \quad (4)$$

(1), (2) and (4) we obtained from steady-state experimental observations → no guarantee they hold true also for time-dependent fields and sources

~ 1865 Maxwell noticed that (2) is not consistent with the continuity equation: $\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$ (which is a a statement about conservation of charge since currents are charges in motion!)

Using (1), the continuity equation can be rewritten as

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D}) = 0, \text{ or}$$

$$\vec{\nabla}\cdot\left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right) = 0 \rightarrow \text{Maxwell replaced (2) with}$$

Displacement current

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \quad \text{Maxwell-Ampère's Law}$$

Displacement current

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$: Maxwell-Ampère's law

which was verified experimentally !

$\Downarrow$

Maxwell's equations

$$\vec{\nabla} \cdot \vec{D} = \rho$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

which form the basis of all classical electromagnetic phenomena, including the fact that light is an electromagnetic wave and that electromagnetic waves of all frequencies (wavelengths) could be produced.

6.2 vector and Scalar potentials :

Since $\vec{\nabla} \cdot \vec{B} = 0$, $\vec{B} = \vec{\nabla} \times \vec{A}$ where $\vec{A}$ is a vector potential. Then

$$(3) \rightarrow \vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) = 0, \text{ or}$$

$$\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = \vec{0} \longrightarrow$$

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\phi$$ for some scalar potential ϕ.

$$\rightarrow \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

$$\rightarrow \quad \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

$\rightarrow$ solve (3) and (4)

The dynamic behavior of $\vec{A}$ and ϕ is contained in (1) and (2):

$$\vec{\nabla}\cdot\vec{D} = \rho$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

In vaccum, these become:

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \\ \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \end{cases}$$

$$\rightarrow \begin{cases} \nabla^2\phi + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\rho/\varepsilon_0 \\ \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J} + \frac{1}{c^2}\left(-\vec{\nabla}\frac{\partial\phi}{\partial t} - \frac{\partial^2\vec{A}}{\partial t^2}\right) \end{cases}$$

$$\rightarrow \begin{cases} \nabla^2\phi + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\rho/\varepsilon_0 \\ \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = -\mu_0\vec{J} \end{cases} \quad (*)$$

If we make the transformations

$$\left.\begin{aligned} \vec{A} &\longrightarrow \vec{A}' = \vec{A} + \vec{\nabla}\psi \\ \phi &\longrightarrow \phi' = \phi - \frac{\partial\psi}{\partial t} \end{aligned}\right\} (**)$$

both $\vec{B}$ and $\vec{E}$ remain unchanged: $\vec{E}' = \vec{E}$ and $\vec{B}' = \vec{B}$:

$\vec{B}' = \vec{B}$:

$$\vec{B}' = \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} = \vec{B}$$

$$\vec{E}' = -\vec{\nabla}\phi' - \frac{\partial \vec{A}'}{\partial t}$$

$$= -\vec{\nabla}\phi + \cancel{\frac{\partial}{\partial t}(\vec{\nabla}\psi)} - \frac{\partial \vec{A}}{\partial t} - \cancel{\frac{\partial}{\partial t}(\vec{\nabla}\psi)} = \vec{E} \checkmark$$

We can use the freedom implied by (**) to make

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \qquad \text{(Lorentz Gauge).}$$

Then $(*) \longrightarrow$

$$\begin{cases} \nabla^2 \phi - \dfrac{1}{c^2}\dfrac{\partial^2 \phi}{\partial t^2} = -\rho/\varepsilon_0 \\ \nabla^2 \vec{A} - \dfrac{1}{c^2}\dfrac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} \end{cases}$$

Recall: Coulomb (or transverse) Gauge : $\vec{\nabla}\cdot\vec{A} = 0$

Then, $(*) \longrightarrow$

$$\begin{cases} \nabla^2 \phi = -\rho/\varepsilon_0 \\ \nabla^2 \vec{A} - \dfrac{1}{c^2}\dfrac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \dfrac{1}{c^2}\vec{\nabla}\left(\dfrac{\partial \phi}{\partial t}\right) \quad (***) \end{cases}$$

$$\rightarrow \phi(\vec{x}, t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}', t)}{|\vec{x} - \vec{x}'|}\, d^3x'$$

(Instantaneous Coulomb potential; hence the name: Coulomb Gauge.)

Once we have $\phi(\vec{x}, t)$, we can get $\vec{\nabla}\left(\frac{\partial \phi}{\partial t}\right)$ and

[illegible]

Once we have $\phi(\vec{x},t)$, we can get $\vec{\nabla}\left(\frac{\partial\phi}{\partial t}\right)$ and then solve the equation for $\vec{A}$.

Since $\vec{\nabla}\times\left[\vec{\nabla}\left(\frac{\partial\phi}{\partial t}\right)\right]=\vec{0}$, we will show that $\frac{1}{c^2}\vec{\nabla}\left(\frac{\partial\phi}{\partial t}\right)=\mu_0\vec{J}_\ell$, where $\vec{J}_\ell$ is the longitudinal (irrotational) component of $\vec{J}$: $\vec{J}=\vec{J}_\ell+\vec{J}_t$

$$\vec{\nabla}\times\vec{J}_\ell=\vec{0} \quad \text{and} \quad \vec{\nabla}\cdot\vec{J}_t=0$$

In fact, let

$$\vec{J}_\ell(\vec{x},t)=-\frac{1}{4\pi}\vec{\nabla}\int\frac{\vec{\nabla}'\cdot\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3x', \text{ and}$$

$$\vec{J}_t=\frac{1}{4\pi}\vec{\nabla}\times\left[\vec{\nabla}\times\int\frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3x'\right]. \quad \text{Then}$$

$$\vec{\nabla}\times\vec{J}_\ell=\vec{0} \quad ✓$$

$$\vec{\nabla}\cdot\vec{J}_t=0 \quad ✓$$

$$\mu_0\vec{J}_\ell=+\frac{\mu_0}{4\pi}\vec{\nabla}\int\frac{\partial\rho(\vec{x}',t)/\partial t}{|\vec{x}-\vec{x}'|}d^3x'$$

$$=+\frac{\mu_0}{4\pi}\vec{\nabla}\frac{\partial}{\partial t}\int\frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3x'$$

$$=\frac{\mu_0}{\cancel{4\pi}}\vec{\nabla}\frac{\partial}{\partial t}\left[\cancel{4\pi}\varepsilon_0\phi(\vec{x},t)\right]$$

$$=\frac{1}{c^2}\vec{\nabla}\left(\frac{\partial\phi}{\partial t}\right) \quad ✓$$

Finally,

$$\vec{J}(\vec{x}) = \frac{1}{\ldots}\left[\vec{\nabla}\left(\vec{\nabla}\cdot\int\frac{\vec{J}(\vec{x}',t)}{\ldots}d^3x'\right)\right.$$

Finally,

$$\vec{J}_t(\vec{x},t) = \frac{1}{4\pi}\left[\vec{\nabla}\left(\vec{\nabla}\cdot\int \frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}\,d^3x'\right) - \nabla^2 \int \frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}\,d^3x'\right]$$

$$= \vec{J}(\vec{x},t) + \frac{1}{4\pi}\vec{\nabla}\left[\int \vec{J}(\vec{x}',t)\cdot\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)d^3x'\right]$$

$$= \vec{J}(\vec{x},t) - \frac{1}{4\pi}\vec{\nabla}\left[\int \vec{J}(\vec{x}',t)\cdot\vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)d^3x'\right]$$

$$= \vec{J}(\vec{x},t) - \frac{1}{4\pi}\vec{\nabla}\left[\int \vec{\nabla}'\cdot\left(\frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}\right)d^3x'\right] + \frac{1}{4\pi}\vec{\nabla}\left[\int \frac{\vec{\nabla}'\cdot\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}\,d^3x'\right]$$

$\rightarrow \vec{J}_t(\vec{x},t) = \vec{J}(\vec{x},t) - \vec{J}_\ell(\vec{x},t)$, and hence

$$\vec{J}(\vec{x},t) = \vec{J}_\ell(\vec{x},t) + \vec{J}_t(\vec{x},t) \quad \checkmark$$

Thus, $(***) \longrightarrow$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\left(\cancel{\vec{J}_\ell} + \vec{J}_t\right) + \cancel{\frac{1}{c^2}\nabla^2\left(\frac{\partial\phi}{\partial t}\right)}$$

$$\rightarrow \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}_t$$ (hence the name "transverse" Gauge!)

The Coulomb gauge is often used when $\rho = 0$ and $\vec{J} = \vec{0}$. Then $\phi = 0$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \vec{0}$$ (homogeneous wave equation)

$\rightarrow \vec{E} = -\underline{\partial\vec{A}}$ and $\vec{B} = \vec{\nabla}\times\vec{A}$.

$\rightarrow \quad \vec{E} = -\dfrac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \vec{\nabla}\times\vec{A}$.

7.1 - 7.2 plane waves in a nonconducting Medium: Linear and Circular polarization

Maxwell's Equations in an infinite medium with no charge and current sources:

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0}$$

$$\vec{\nabla}\cdot\vec{D} = 0 \qquad \vec{\nabla}\times\vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{0}$$

Assuming solutions with harmonic time dependence $e^{-i\omega t}$, from which we can obtain an arbitrary solution using Fourier Analysis, we get:

$$\vec{\nabla}\cdot\vec{B} = 0 \quad (1) \qquad \vec{\nabla}\times\vec{E} - i\omega\vec{B} = \vec{0} \quad (3)$$

$$\vec{\nabla}\cdot\vec{D} = 0 \quad (2) \qquad \vec{\nabla}\times\vec{H} + i\omega\vec{D} = \vec{0} \quad (4)$$

Note that $(3) \Rightarrow (1)$ and $(4) \Rightarrow (2)$; so, for such solution, all the information is contained in (3) and (4).

For linear, isotropic, uniform media, we have $\vec{B} = \mu\vec{H}$ and $\vec{D} = \varepsilon\vec{E}$ where ε and μ are constant. $\rightarrow$

$$\begin{cases} \vec{\nabla}\times\vec{E} - i\omega\vec{B} = \vec{0} & (3) \\ \vec{\nabla}\times\vec{B} + i\omega\mu\varepsilon\vec{E} = \vec{0} & (4') \end{cases}$$

Taking the curl of both equations, we get

$$\begin{cases} -\nabla^2\vec{E} - i\omega\,\vec{\nabla}\times\vec{B} = \vec{0} \\ -\nabla^2\vec{B} + i\omega\mu\varepsilon\,\vec{\nabla}\times\vec{E} = \vec{0} \end{cases} \Rightarrow \begin{cases} \nabla^2\vec{E} + \mu\varepsilon\omega^2\vec{E} = \vec{0} \\ \nabla^2\vec{B} + \mu\varepsilon\omega^2\vec{B} = \vec{0} \end{cases}$$

$$\left[\begin{array}{l} \nabla^2\vec{E} - \mu\varepsilon\dfrac{\partial^2\vec{E}}{\partial t^2} = \vec{0} \\ \nabla^2\vec{B} - \mu\varepsilon\dfrac{\partial^2\vec{B}}{\partial t^2} = \vec{0} \end{array} \right]$$

$$\left[\begin{array}{l} \nabla^2\vec{E} - \mu\varepsilon \dfrac{\partial^2 \vec{E}}{\partial t^2} = \vec{0} \\ \nabla^2\vec{B} - \mu\varepsilon \dfrac{\partial^2 \vec{B}}{\partial t^2} = \vec{0} \end{array}\right]$$

We now consider an electromagnetic plane wave of frequency ω and wave number $\vec{k} = k\hat{e}_k$:

$$\vec{E}(\vec{r},t) = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$
$$\vec{B}(\vec{r},t) = \vec{B}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

that satisfies Maxwell's equations. Then

$$\nabla^2\vec{E} = -k^2\vec{E} \quad \text{and} \quad \nabla^2\vec{B} = -k^2\vec{B}$$

$$\rightarrow -k^2 + \mu\varepsilon\omega^2 = 0 \rightarrow k = \sqrt{\mu\varepsilon}\,\omega$$

$$\rightarrow \text{phase velocity} = v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\sqrt{\frac{\mu_0\varepsilon_0}{\mu\varepsilon}} = \frac{c}{n}$$

where $n = \sqrt{\dfrac{\mu\varepsilon}{\mu_0\varepsilon_0}}$ is the index of refraction of the medium which usually is a function of ω.

$$\left.\begin{array}{l} \vec{\nabla}\cdot\vec{E} = 0 \rightarrow i\vec{k}\cdot\vec{E} = 0 \rightarrow \vec{k} \perp \vec{E} \\ \vec{\nabla}\cdot\vec{B} = 0 \rightarrow i\vec{k}\cdot\vec{B} = 0 \rightarrow \vec{k} \perp \vec{B} \end{array}\right\} \rightarrow \text{transverse wave}$$

Moreover, $\vec{\nabla}\times\vec{E} = i\omega\vec{B} \rightarrow i\vec{k}\times\vec{E} = i\omega\vec{B}$

$$\rightarrow \vec{B} = \frac{\vec{k}}{\omega}\times\vec{E} = \frac{k}{\omega}\hat{e}_k\times\vec{E} = \sqrt{\mu\varepsilon}\,\hat{e}_k\times\vec{E}$$

$$\vec{B} = \frac{n}{c}\hat{e}_k\times\vec{E} \rightarrow |\vec{B}| = \frac{n}{c}|\vec{E}|$$

Let $(\hat{e}_1, \hat{e}_2, \hat{e}_k)$ form a right-handed system of orthogonal unit vectors.

So, for example,

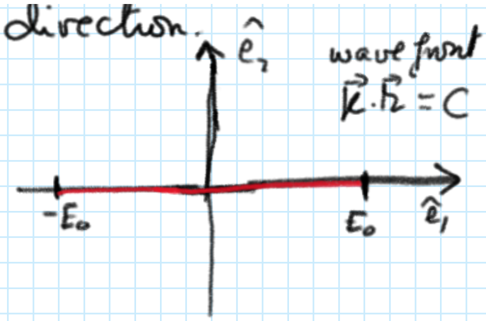
$$\left\{\begin{array}{l} \vec{E} = E_0 e^{i\alpha}\hat{e}_1\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \\ \quad = E_0\,\hat{e}_1\, e^{i(\vec{k}\cdot\vec{r} - \omega t + \alpha)} \end{array}\right.$$

linearly polarized in the $\hat{e}_1$ direction.

$\hat{e}_2$ wave front $\vec{k}\cdot\vec{r} = c$

$$\begin{cases} \vec{E} = E_0\,\hat{e}_1\, e^{i(\vec{k}\cdot\vec{r}-\omega t+\alpha)} \\ \vec{B} = B_0\,\hat{e}_2\, e^{i(\vec{k}\cdot\vec{r}-\omega t+\alpha)} \end{cases}$$

e_1 direction.



$$Re\left(\vec{E}(\vec{r},t)\right) = E_0\cos(\omega t - \vec{k}\cdot\vec{r}-\alpha)\,\hat{e}_1$$

$$\begin{cases} \vec{E} = E_0\,\hat{e}_2\, e^{i(\vec{k}\cdot\vec{r}-\omega t+\alpha)} \\ \vec{B} = -B_0\,\hat{e}_1\, e^{i(\vec{k}\cdot\vec{r}-\omega t+\alpha)} \end{cases}$$

linearly polarized in the $\hat{e}_2$ direction.

In general,

$$\vec{E}(\vec{r},t) = \left[E_0^{(1)} e^{i\alpha}\,\hat{e}_1 + E_0^{(2)} e^{i\beta}\,\hat{e}_2\right] e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= E_0^{(1)} e^{i(\vec{k}\cdot\vec{r}-\omega t+\alpha)}\,\hat{e}_1 + E_0^{(2)} e^{i(\vec{k}\cdot\vec{r}-\omega t+\beta)}\,\hat{e}_2$$

$$Re\left(\vec{E}(\vec{r},t)\right) = E_0^{(1)}\cos(\vec{k}\cdot\vec{r}-\omega t+\alpha)\,\hat{e}_1 + E_0^{(2)}\cos(\vec{k}\cdot\vec{r}-\omega t+\beta)\,\hat{e}_2$$

$$= E_1\,\hat{e}_1 + E_2\,\hat{e}_2$$

$$\left.\begin{aligned} \frac{E_1}{E_0^{(1)}} &= \cos(\vec{k}\cdot\vec{r}-\omega t)\cos\alpha - \sin(\vec{k}\cdot\vec{r}-\omega t)\sin\alpha \\ \frac{E_2}{E_0^{(2)}} &= \cos(\vec{k}\cdot\vec{r}-\omega t)\cos\beta - \sin(\vec{k}\cdot\vec{r}-\omega t)\sin\beta \end{aligned}\right\} \rightarrow$$

$$\frac{E_1}{E_0^{(1)}}\sin\beta - \frac{E_2}{E_0^{(2)}}\sin\alpha = \cos(\vec{k}\cdot\vec{r}-\omega t)\sin(\beta-\alpha)$$

$$\frac{E_1}{E_0^{(1)}}\cos\beta - \frac{E_2}{E_0^{(2)}}\cos\alpha = \sin(\vec{k}\cdot\vec{r}-\omega t)\sin(\beta-\alpha)$$

Squaring both sides of the two equations above and adding, we get adding, we get

$$\left(\frac{E_1}{E_0^{(1)}}\right)^2 - 2\left(\frac{E_1}{E_0^{(1)}}\right)\left(\frac{E_2}{E_0^{(2)}}\right)\cos(\beta-\alpha) + \left(\frac{E_2}{E_0^{(2)}}\right)^2 = \sin^2(\beta-\alpha).$$

- ellipses for $\sin(\beta-\alpha) \neq 0$.
- ellipses with axes along $\hat{e}_1$ and $\hat{e}_2$ if $\sin(\beta-\alpha) = \pm 1$
- Rotated ellipses for $\sin(\beta-\alpha) \neq 0, \pm 1$.

Special cases:

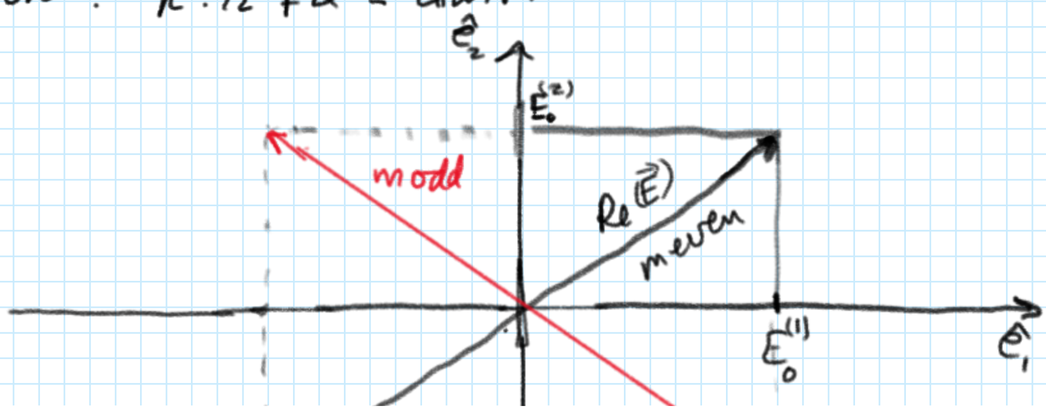
- If $\beta - \alpha = m\pi$, then $\cos(\beta-\alpha) = (-1)^m$ and $\sin(\beta-\alpha) = 0$

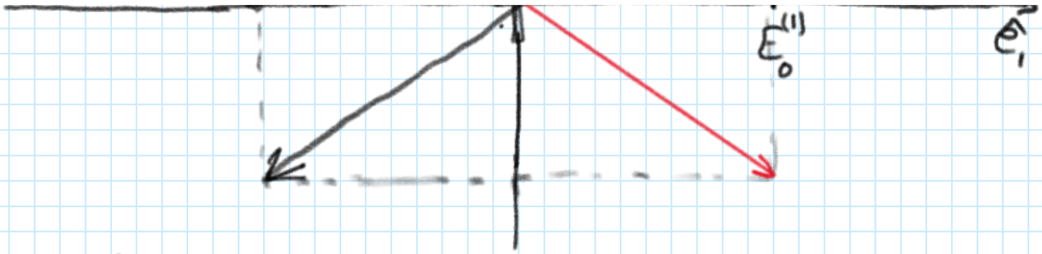
$$\rightarrow \left[\frac{E_2}{E_0^{(2)}} - (-1)^m \frac{E_1}{E_0^{(1)}}\right]^2 = 0 \rightarrow \frac{E_2}{E_0^{(2)}} = (-1)^m \frac{E_1}{E_0^{(1)}}$$

$\rightarrow \dfrac{E_2}{E_1} = (-1)^m \dfrac{E_0^{(2)}}{E_0^{(1)}}$: linearly polarized wave, with its polarization vector making an angle $\theta = (-1)^m \tan^{-1}\left(\dfrac{E_0^{(2)}}{E_0^{(1)}}\right)$ with $\hat{e}_1$.

$$\rightarrow Re\left(\vec{E}(\vec{r},t)\right) = E_0^{(1)}\cos(\omega t - \vec{k}\cdot\vec{r} - \alpha)\,\hat{e}_1 + E_0^{(2)}\cos(\omega t - \vec{k}\cdot\vec{r} - \beta)\,\hat{e}_2$$
$$= \left(E_0^{(1)}\hat{e}_1 + (-1)^m E_0^{(2)}\hat{e}_2\right)\cos(\omega t - \vec{k}\cdot\vec{r} - \alpha)$$

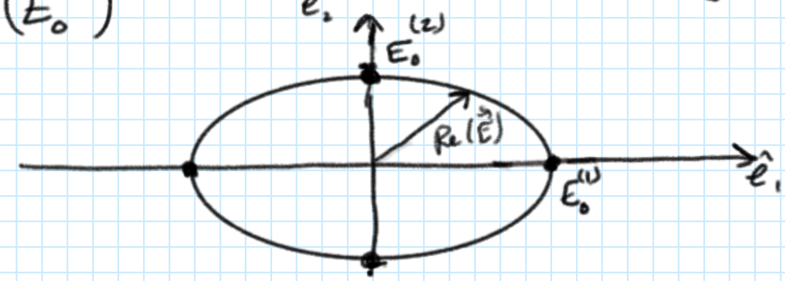
on a wave front: $\vec{k}\cdot\vec{r} + \alpha = \text{const}$.





• If $(\beta - \alpha) = (2m+1)\frac{\pi}{2}$ then $\cos(\beta - \alpha) = 0$ and $\sin^2(\beta - \alpha) = 1$

$$\rightarrow \quad \frac{E_1^2}{(E_0^{(1)})^2} + \frac{E_2^2}{(E_0^{(2)})^2} = 1 \quad \text{(Elliptically polarized).}$$



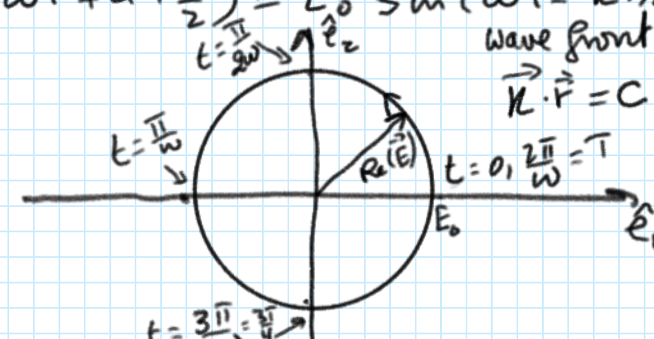
If, in addition, $E_0^{(1)} = E_0^{(2)} = E_0 \rightarrow$ the wave is circularly polarized.

(*) If $(\beta - \alpha) = \frac{\pi}{2}$ $(+2k\pi)$ $\rightarrow$

$$E_1 = E_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \alpha) = E_0 \cos(\omega t - \vec{k}\cdot\vec{r} - \alpha)$$

$$E_2 = E_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \alpha + \tfrac{\pi}{2}) = E_0 \sin(\omega t - \vec{k}\cdot\vec{r} - \alpha)$$

wave front: $\vec{k}\cdot\vec{r} = C$



$\rightarrow$ the wave is left-circularly polarized wave (+ helicity)

(**) If $(\beta - \alpha) = -\frac{\pi}{2}$ $(+2k\pi)$ $\longrightarrow$

$$\left.\begin{aligned} E_1 &= E_0 \cos(\omega t - \vec{k}\cdot\vec{r} - \alpha) \\ E_2 &= -E_0 \sin(\omega t - \vec{k}\cdot\vec{r} - \alpha) \end{aligned}\right\} \rightarrow \text{Right circularly polarized plane wave.}$$

$$E_2 = -E_0 \sin(\omega t - \vec{k}\cdot\vec{r} - \alpha) \Big\} \text{ polarized plane wave. (negative helicity)}$$

Poynting vector and electromagnetic energy density

$$\vec{E}(\vec{r},t) = \vec{E}(\vec{r})\, e^{-i\omega t} = (\vec{E}_1 + i\,\vec{E}_2)\, e^{-i\omega t}$$

$$\vec{B}(\vec{r},t) = \vec{B}(\vec{r})\, e^{-i\omega t} = (\vec{B}_1 + i\,\vec{B}_2)\, e^{-i\omega t}$$

$$\vec{\mathcal{E}}(\vec{r},t) = \mathrm{Re}\,(\vec{E}(\vec{r},t)) = \vec{E}_1(\vec{r})\cos\omega t + \vec{E}_2(\vec{r})\sin\omega t$$

$$\vec{\mathcal{B}}(\vec{r},t) = \mathrm{Re}\,(\vec{B}(\vec{r},t)) = \vec{B}_1(\vec{r})\cos\omega t + \vec{B}_2(\vec{r})\sin\omega t$$

$$\vec{S}(\vec{r},t) = \vec{\mathcal{E}} \times \vec{\mathcal{H}} = \frac{1}{\mu}(\vec{\mathcal{E}} \times \vec{\mathcal{B}})$$

$$= \frac{1}{\mu}\Big[\vec{E}_1(\vec{r}) \times \vec{B}_1(\vec{r})\cos^2\omega t + \vec{E}_2(\vec{r}) \times \vec{B}_2(\vec{r})\sin^2\omega t + \big(\vec{E}_1(\vec{r}) \times \vec{B}_2(\vec{r}) + \vec{E}_2(\vec{r}) \times \vec{B}_1(\vec{r})\big)\cos\omega t\sin\omega t\Big]$$

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2\mu}\Big[\vec{E}_1(\vec{r}) \times \vec{B}_1(\vec{r}) + \vec{E}_2(\vec{r}) \times \vec{B}_2(\vec{r})\Big]$$

On the other hand,

$$\Big[\vec{E}^*(\vec{r},t) \times \vec{B}(\vec{r},t)\Big] = \Big[\vec{E}_1(\vec{r},t) - i\,\vec{E}_2(\vec{r},t)\Big] \times \Big[\vec{B}_1(\vec{r},t) + i\,\vec{B}_2(\vec{r},t)\Big]$$

$$= \vec{E}_1(\vec{r},t) \times \vec{B}_1(\vec{r},t) + \vec{E}_2(\vec{r},t) \times \vec{B}_2(\vec{r},t) + i\Big[\vec{E}_1 \times \vec{B}_2 - \vec{E}_2 \times \vec{B}_1\Big] \longrightarrow$$

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2\mu}\,\mathrm{Re}\Big[\vec{E}^*(\vec{r},t) \times \vec{B}(\vec{r},t)\Big]$$

Recall: $\eta_e = \frac{1}{2}\varepsilon\, \vec{\mathcal{E}} \cdot \vec{\mathcal{E}}$ and hence

Recall: $\eta_e = \frac{1}{2}\,\varepsilon\,\mathcal{E}\cdot\mathcal{E}$ and hence

$$\langle \eta_e \rangle = \frac{1}{4}\,\varepsilon\left(\vec{E}_1(\vec{r})\cdot\vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r})\cdot\vec{E}_2(\vec{r})\right)$$

$$= \frac{1}{4}\,\varepsilon\left[\vec{E}^*(\vec{r},t)\cdot\vec{E}(\vec{r},t)\right]$$

Similarly, $\langle \eta_m \rangle = \frac{1}{2}\mu\langle \mathcal{H}\cdot\mathcal{H}\rangle = \frac{1}{2\mu}\langle \mathcal{B}\cdot\mathcal{B}\rangle$

$$= \frac{1}{4\mu}\left[\vec{B}^*(\vec{r},t)\cdot\vec{B}(\vec{r},t)\right]$$

For plane waves: $\vec{E}(\vec{r},t) = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$$\vec{B}(\vec{r},t) = \vec{B}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \sqrt{\mu\varepsilon}\;\hat{e}_k \times \vec{E}(\vec{r},t)$$

$$\rightarrow \langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2\mu}\,\mathrm{Re}\left[\sqrt{\mu\varepsilon}\;\vec{E}^*(\vec{r},t)\times(\hat{e}_k\times\vec{E}(\vec{r},t))\right]$$

$$= \frac{1}{2}\,\frac{\sqrt{\mu\varepsilon}}{\mu}\,\mathrm{Re}\left[(\vec{E}^*(\vec{r},t)\cdot\vec{E}(\vec{r},t))\,\hat{e}_k\right]$$

$$= \frac{1}{2}\,\frac{\sqrt{\mu\varepsilon}}{\mu}\,|\vec{E}_0|^2\,\hat{e}_k$$

$$= \left(\frac{1}{2}\,\varepsilon\,|\vec{E}_0|^2\,\hat{e}_k\right)\frac{1}{\sqrt{\mu\varepsilon}}$$

$$= v\,\langle \eta_{em}\rangle\,\hat{e}_k \text{ , where}$$

$$\langle \eta_{em}\rangle = \langle \eta_e\rangle + \langle \eta_m\rangle$$

$$= 2\langle \eta_e\rangle = 2\langle \eta_m\rangle \text{ since}$$

$$\langle \eta_m\rangle = \frac{1}{4\mu}\left[\vec{B}^*(\vec{r},t)\cdot\vec{B}(\vec{r},t)\right]$$

$$= \frac{1}{4\cancel{\mu}}\,(\varepsilon\cancel{\mu})\;\vec{E}^*(\vec{r},t)\cdot\vec{E}(\vec{r},t) = \langle \eta_e\rangle$$