Review of Bessel's Equation and Bessel Functions (Chapter 12, Sections 12-15 and Section 19 (Boas)

$$x^2 y'' + x y' + (x^2 - p^2) y = 0$$

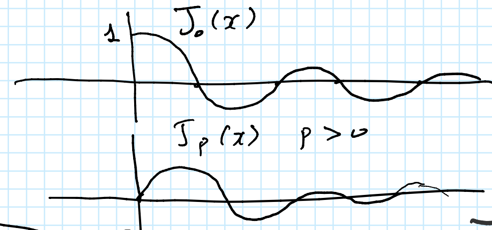
or

$$x \frac{d}{dx}\left[x \frac{dy}{dx}\right] + (x^2 - p^2) y = 0$$

We use generalized power series method (Frobenius method)

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$\Downarrow$$



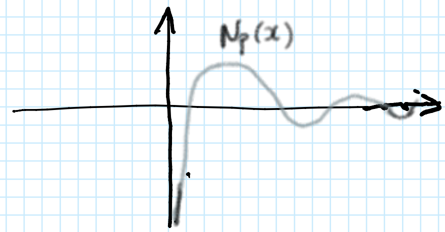
Bessel function of the first kind

$$\boxed{J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(p+k+1)} \left(\frac{x}{2}\right)^{2k+p} \qquad p \geq 0}$$

$$J_p(0) = \begin{cases} 0 & \text{if } p>0 \\ 1 & \text{if } p=0 \end{cases}$$

$N_p(x)$: Bessel function of the second kind: $|N_p(x)| \xrightarrow[x \to 0]{} \infty$

General solution of Bessel's Equation:



$$\boxed{y = A\, J_p(x) + B\, N_p(x)}$$

Modified Bessel's Equation:

$$x^2 y'' + x y' + (x^2 - p^2) y = 0$$

or

$$x \frac{d}{dx}\left[x \frac{dy}{dx}\right] + (x^2 - p^2) y = 0$$

Let $x = kr$. Then

$$r \frac{d}{dr}\left[r \frac{dy}{dr}\right] + (k^2 r^2 - p^2) y = 0$$

or $$r^2 \frac{d^2 y}{dr^2} + r \frac{dy}{dr} + (k^2 r^2 - p^2) y = 0$$

$$y = A\, J_p(x) + B\, N_p(x)$$

$$\Downarrow$$

$$\Downarrow$$

$$y = A J_p(kr) + B N_p(kr)$$

Useful Recursion Relations:

1) $$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$$

2) $$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$$

3) $$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$$

4) $$J_{p-1} - J_{p+1}(x) = 2 J_p'(x)$$

$\Leftrightarrow$

5) $$J_p'(x) = J_{p-1}(x) - \frac{p}{x} J_p(x)$$

6) $$J_p'(x) = \frac{p}{x} J_p(x) - J_{p+1}(x).$$

Orthogonality of the Bessel functions:

Let α and β be zeroes of $J_p(x)$ (i.e. $J_p(\alpha) = J_p(\beta) = 0$).

Then

$$\boxed{\int_0^1 x\, J_p(\alpha x) J_p(\beta x)\, dx = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \frac{1}{2} J_{p+1}^2(\alpha) & \text{if } \alpha = \beta \end{cases}}$$

Put $x = r/a$. Then $r = ax$. Thus,

$$\boxed{\int_0^a r\, J_p(\alpha r/a) J_p(\beta r/a)\, dr = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \frac{a^2}{2} J_{p+1}^2(\alpha) & \text{if } \alpha = \beta \end{cases}}.$$