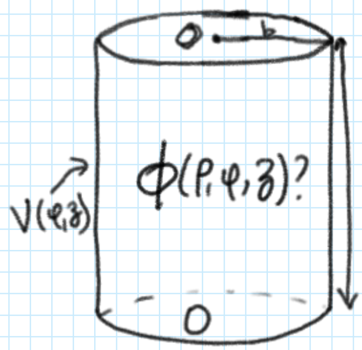
Assignment # 4

Problem 3.9



We need to solve the following boundary value problem:

$$\begin{cases} \nabla^2\phi = 0 \\ \phi(\rho,\varphi,0) = \phi(\rho,\varphi,L) = 0 \quad \text{for } 0 \le \rho < b,\ 0 \le \varphi \le 2\pi \\ \phi(b,\varphi,z) = V(\varphi,z) \quad \text{for } 0 \le \varphi \le 2\pi,\ 0 < z < L. \end{cases}$$

$$\nabla^2\phi = 0 \rightarrow \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad (*)$$

Let $\phi(\rho,\varphi,z) = R(\rho)\,Q(\varphi)\,Z(z)$. Then, substituting into $(*)$ and then dividing by $R(\rho)\,Q(\varphi)\,Z(z)$, we get:

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\rho^2 Q}\frac{d^2Q}{d\varphi^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0, \text{ or}$$

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\rho^2 Q}\frac{d^2Q}{d\varphi^2} = -\frac{1}{Z}\frac{d^2Z}{dz^2} \qquad (**)$$

The right-hand side is a function of z only, while the left-hand side is a function of ρ and φ only. The equality holds for all allowed values of ρ, φ, z which vary independently; the only way this is possible is for both sides of the equation $(**)$ to be equal to the same constant. The boundary conditions at $z=0$ and $z=L$ suggest that the separation constant be positive, say k^2. Thus,

$$-\frac{1}{Z}\frac{d^2Z}{dz^2} = k^2 \text{ or } \frac{d^2Z}{dz^2} + k^2 Z = 0 \rightarrow Z(z) = \begin{Bmatrix} \cancel{\cos kz} \\ \sin kz \end{Bmatrix}$$

Since $\phi(\rho,\varphi,0) = 0$ for all $0 \le \rho < b$ and $0 \le \varphi \le 2\pi$, it follows that $Z(0) = 0$, and hence $Z(z) \sim \sin kz$.

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Since $\phi(\rho,\varphi,L)=0$ for all $0\le\rho<b$ and $0\le\varphi\le 2\pi$, it follows that $Z(L)=0$ and $\sin(kL)=0$. Thus, $kL=m\pi$, where m is an integer. Hence $k=\frac{m\pi}{L}$ and $Z_m(z)\sim\sin\left(\frac{m\pi}{L}z\right)$.

Since $m=0$ and $m<0$ don't yield new linearly independent solutions, it is enough to take $m=1,2,3,\ldots$

Going back to $(**)$,

$$\frac{1}{\rho R}\frac{d}{d\rho}\left[\rho\frac{dR}{d\rho}\right]+\frac{1}{\rho^2 Q}\frac{d^2Q}{d\varphi^2}=\left(\frac{m\pi}{L}\right)^2 \xrightarrow{\times\rho^2}$$

$$\underbrace{\frac{1}{R}\frac{d}{d\rho}\left[\rho\frac{dR}{d\rho}\right]-\left(\frac{m\pi}{L}\right)^2\rho^2}_{\text{function of }\rho\text{ only}}=\underbrace{-\frac{1}{Q}\frac{d^2Q}{d\varphi^2}}_{\text{function of }\varphi\text{ only}}=\text{constant } \nu^2 \quad (***)$$

$$-\frac{1}{Q}\frac{d^2Q}{d\varphi^2}=\nu^2 \longrightarrow Q(\varphi)=\begin{Bmatrix}\cos\nu\varphi\\ \sin\nu\varphi\end{Bmatrix}$$

Since the full range of φ $(0\le\varphi\le 2\pi)$ is included, $Q(\varphi)$ must be periodic, of period $2\pi \rightarrow \nu=$ integer

Thus $\nu=n=0,1,2,3,\ldots$ [Negative integers don't give linearly independent solutions]. Thus,

$$Q_n(\varphi)=\begin{Bmatrix}\cos n\varphi\\ \sin n\varphi\end{Bmatrix} \quad \text{(linear combination of the two)}$$

with $\nu^2=n^2$, $(***)\rightarrow$

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)-\left(\frac{m\pi}{L}\right)^2\rho^2-n^2=0 \rightarrow$$

$$\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)-\left[\left(\frac{m\pi}{L}\right)^2\rho^2+n^2\right]R=0$$

$$\rho \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) - \left[\left(\frac{m\pi}{L}\right)^2\rho^2 + n^2\right]R = 0$$

$$\rightarrow R(\rho) = \left\{ \begin{matrix} I_n\left(\frac{m\pi}{L}\rho\right) \\ \cancel{K_n\left(\frac{m\pi}{L}\rho\right)} \end{matrix} \right\}, \text{ modified Bessel functions}$$

Since $\rho = 0$ (axis of the cylinder) is included in the region of interest, we throw out $K_n\left(\frac{m\pi}{L}\rho\right)$ which blows up as $\rho \to 0$. Thus, $R_{nm}(\rho) \sim I_n\left(\frac{m\pi}{L}\rho\right)$.

For each $n = 0, 1, 2, \ldots$ and for each $m = 1, 2, 3, \ldots$, we obtain the eigensolution

$$\phi_{nm}(\rho, \varphi, z) = I_n\left(\frac{m\pi}{L}\rho\right)\begin{bmatrix} \cos n\varphi \\ \text{or} \\ \sin n\varphi \end{bmatrix} \sin\left(\frac{m\pi}{L}z\right)$$

which is a solution of $\nabla^2\phi = 0$ inside the cylinder and satisfies the boundary conditions at the top ($z = L$) and bottom ($z = 0$) of the cylinder. The same is true for any linear combinations of the eigensolutions. Thus, to match the boundary condition on the curved surface of the cylinder ($\rho = b$), we take all possible linear combinations of the eigensolutions:

$$\phi(\rho, \varphi, z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} I_n\left(\frac{m\pi}{L}\rho\right)\left[A_{nm}\cos n\varphi + B_{nm}\sin n\varphi\right]\sin\left(\frac{m\pi}{L}z\right)$$

Since $\phi(b, \varphi, z) = V(\varphi, z)$, we get

$$V(\varphi, z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} I_n\left(\frac{m\pi}{L}b\right)\left[A_{nm}\cos n\varphi + B_{nm}\sin n\varphi\right]\sin\left(\frac{m\pi}{L}z\right)$$

which is a double Fourier series. Thus, for $n \geq 1, m \geq 1$, we have

$$\int_{z=0}^{L}\int_{\varphi=0}^{2\pi} V(\varphi, z)\sin n\varphi \sin\left(\frac{m\pi}{L}z\right) d\varphi\, dz$$

$$= \int^{L}\int^{2\pi} \sum^{\infty}\sum^{\infty} I_l\left(\frac{j\pi}{L}b\right)\left[A_{lj}\cos l\varphi + B_{lj}\sin l\varphi\right]\sin\left(\frac{j\pi}{L}z\right)\sin n\varphi \sin\left(\frac{m\pi}{L}z\right) d\varphi\, dz$$

$$= \int_{z=0}^{L}\int_{\varphi=0}^{2\pi} \underbrace{\sum_{\ell=0}^{\infty}\sum_{j=1}^{\infty} I_\ell\left(\frac{j\pi}{L}b\right)\left[A_{\ell j}\cos\ell\varphi + B_{\ell j}\sin\ell\varphi\right]\sin\left(\frac{j\pi}{L}z\right)}_{V(\varphi,z)} \sin n\varphi \sin\left(\frac{m\pi}{L}z\right) d\varphi\, dz$$

$$= \sum_{\ell=0}^{\infty}\sum_{j=1}^{\infty} I_\ell\left(\frac{j\pi}{L}b\right)\left[\overbrace{\int_{\varphi=0}^{2\pi}\left[A_{\ell j}\cos\ell\varphi + B_{\ell j}\sin\ell\varphi\right]\sin n\varphi\, d\varphi}^{\pi B_{\ell j}\,\delta_{\ell n}}\right] \cdot$$

$$\left[\underbrace{\int_{z=0}^{L}\left(\sin\frac{j\pi}{L}z\right)\sin\left(\frac{m\pi}{L}z\right) dz}_{\frac{L}{2}\delta_{jm}}\right]$$

$$= \frac{\pi L}{2}\sum_{\ell=0}^{\infty}\sum_{j=1}^{\infty} I_\ell\left(\frac{j\pi}{L}b\right) B_{\ell j}\,\delta_{\ell n}\,\delta_{jm}$$

$$= \frac{\pi L}{2} I_n\left(\frac{m\pi}{L}b\right) B_{nm}\text{, and hence}$$

$$\boxed{B_{nm} = \frac{2}{\pi L\, I_n\left(\frac{m\pi}{L}b\right)}\int_{z=0}^{L}\int_{\varphi=0}^{2\pi} V(\varphi,z)\sin n\varphi \sin\left(\frac{m\pi}{L}z\right) d\varphi\, dz}$$

for all $n \geq 1$ and for all $m \geq 1$.

Similarly, we show that

$$\boxed{A_{nm} = \frac{2}{\pi L\, I_n\left(\frac{m\pi}{L}b\right)}\int_{z=0}^{L}\int_{\varphi=0}^{2\pi} V(\varphi,z)\cos n\varphi \sin\left(\frac{m\pi}{L}z\right) d\varphi\, dz}$$

for all $n \geq 1$ and for all $m \geq 1$

and

and

$$\boxed{A_{0m} = \frac{1}{\pi L\, I_0\left(\frac{m\pi}{L} b\right)} \int_{z=0}^{L} \int_{\varphi=0}^{2\pi} V(\varphi, z) \sin\left(\frac{m\pi}{L} z\right) d\varphi\, dz \quad \text{for all } m \geq 1}$$

problem 3.10 : a) From the result of problem 3.9, we have that

$$\phi(\rho,\varphi,z) = \sum_{m=1}^{\infty} A_{0m}\, I_0\left(\frac{m\pi}{L}\rho\right) \sin\left(\frac{m\pi}{L} z\right) +$$

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} I_n\left(\frac{m\pi}{L}\rho\right) \left[A_{nm}\cos n\varphi + B_{nm}\sin n\varphi\right] \sin\left(\frac{m\pi}{L} z\right)$$

where B_{nm}, A_{nm} $(n \geq 1, m \geq 1)$ and A_{0m} $(m \geq 1)$ are as given at the end of the solution of problem 3.9.

using $V(\phi, z) = \begin{cases} V & \text{if } -\frac{\pi}{2} < \phi < \frac{\pi}{2} \\ -V & \text{if } \frac{\pi}{2} < \phi < \frac{3\pi}{2} \end{cases}$, we get:

$$B_{nm} = \frac{2}{\pi L\, I_n\left(\frac{m\pi}{L} b\right)} \left[\int_0^L \sin\left(\frac{m\pi}{L} z\right) dz\right] \cdot$$

$$\left[\int_{-\pi/2}^{\pi/2} +V \sin n\varphi\, d\varphi + \int_{\pi/2}^{3\pi/2} (-V) \sin n\varphi\, d\varphi\right]$$

$$= \frac{2V}{\pi L\, I_n\left(\frac{m\pi}{L} b\right)} \left[-\frac{L}{m\pi} \cos\left(\frac{m\pi}{L} z\right)\Bigg|_{z=0}^{L}\right] \cdot$$

$$\underbrace{\left[-\frac{\cos n\varphi}{n}\Bigg|_{\varphi=-\pi/2}^{+\pi/2} + \frac{\cos n\varphi}{n}\Bigg|_{\varphi=\pi/2}^{3\pi/2}\right]}_{0 \text{ for } n \text{ odd and for } n \text{ even!}}$$

Thus $B_{nm} = 0$ for all $n \geq 1$ and for all $m \geq 1$. Also, for all $m \geq 1$,

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Thus, $B_{nm} = 0$ for all $n \geq 1$ and for all $m \geq 1$. Also, for all $m \geq 1$,

$$A_{0m} = \frac{1}{\pi L\, I_0\left(\frac{m\pi}{L} b\right)} \left[\int_0^L \sin\left(\frac{m\pi}{L} z\right) dz\right] \cdot$$

$$\underbrace{\left[\int_{-\pi/2}^{\pi/2} +V\, d\varphi + \int_{\pi/2}^{3\pi/2} (-V)\, d\varphi\right]}_{\pi V - \pi V = 0} = 0 .$$

Finally, for $n \geq 1$ and $m \geq 1$, we obtain that

$$A_{nm} = \frac{2}{\pi L\, I_n\left(\frac{m\pi}{L} b\right)} \left[\int_0^L \sin\left(\frac{m\pi}{L} z\right) dz\right] \cdot$$

$$\left[\int_{-\pi/2}^{\pi/2} (+V) \cos n\varphi\, d\varphi + \int_{\pi/2}^{3\pi/2} (-V) \cos n\varphi\, d\varphi\right]$$

$$= \frac{2V}{\pi L\, I_n\left(\frac{m\pi}{L} b\right)} \left[-\frac{L}{m\pi} \cos\left(\frac{m\pi}{L} z\right)\Bigg|_{z=0}^{L}\right] \cdot \left[\frac{\sin n\varphi}{n}\Bigg|_{\varphi=-\pi/2}^{\pi/2} - \frac{\sin n\varphi}{n}\Bigg|_{\varphi=\pi/2}^{3\pi/2}\right]$$

$$= \frac{2V}{\pi L\, I_n\left(\frac{m\pi}{L} b\right)} \frac{L}{m\pi} (1 - \cos m\pi) \frac{1}{n}\left[3 \sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{3n\pi}{2}\right)\right]$$

Note that $1 - \cos m\pi = \begin{cases} 0 & \text{if } m = 2\ell \text{ (even)} \\ 2 & \text{if } m = 2\ell+1 \text{ (odd)} \end{cases}$

and $3\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{3n\pi}{2}\right) = \begin{cases} 0 & \text{if } n = 2k \text{ (even)} \\ (-1)^k 4 & \text{if } n = 2k+1 \text{ (odd)} \end{cases}$

Thus, $A_{nm} \neq 0$ if and only if both n and m are odd; and

$$A_{2k+1,\, 2\ell+1} = \frac{(-1)^k\, 16\, V}{(2k+1)(2\ell+1)\, \pi^2\, I_{2k+1}\left[(2\ell+1)\frac{\pi}{L} b\right]} .$$ Thus,

$$\boxed{\phi(\rho, \varphi, z) = \sum^{\infty} \sum^{\infty} A \quad I \left[(2\ell+1)\frac{\pi}{L}\rho\right] \cos[(2k+1)\varphi] \sin\left[(2\ell+1)\frac{\pi}{L} z\right]}$$

$$\boxed{\begin{aligned}\phi(\rho,\varphi,z) &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} A_{2k+1,2\ell+1}\, I_{2k+1}\!\left[(2\ell+1)\frac{\pi}{L}\rho\right]\cos[(2k+1)\varphi]\sin\left[(2\ell+1)\frac{\pi}{L}z\right] \\ &= \frac{16V}{\pi^2}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{(-1)^k}{(2k+1)(2\ell+1)}\,\frac{I_{2k+1}\left[(2\ell+1)\frac{\pi}{L}\rho\right]}{I_{2k+1}\left[(2\ell+1)\frac{\pi}{L}b\right]}\cos[(2k+1)\varphi]\sin\left[(2\ell+1)\frac{\pi}{L}z\right]\end{aligned}}$$

b) For $L \gg b$, we have that $\frac{\pi b}{L} \ll 1$ and $\frac{\pi\rho}{L} \ll 1$ for all $\rho < b$ (inside of the cylinder). Hence, by Equation (3.102) in the book,

$$\frac{I_{2k+1}\left[(2\ell+1)\frac{\pi}{L}\rho\right]}{I_{2k+1}\left[(2\ell+1)\frac{\pi}{L}b\right]} \sim \left(\frac{\rho}{b}\right)^{2k+1}.$$

Thus,

$$\phi\left(\rho,\varphi,\frac{L}{2}\right) \approx \frac{16V}{\pi^2}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{(-1)^k}{(2k+1)(2\ell+1)}\left(\frac{\rho}{b}\right)^{2k+1}\cos[(2k+1)\varphi]\,\overbrace{\sin\left[(2\ell+1)\frac{\pi}{2}\right]}^{(-1)^\ell}$$

$$= \frac{16V}{\pi^2}\left\{\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)}\left(\frac{\rho}{b}\right)^{2k+1}\cos[(2k+1)\varphi]\right\}\cdot\left[\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{2\ell+1}\right]$$

Recall that $\tan^{-1} z = \sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1} z^{2j+1}$. Thus,

$\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{2\ell+1} = \tan^{-1} 1 = \frac{\pi}{4}$ and hence, for $L \gg b$,

$$\begin{aligned}\phi\left(\rho,\varphi,\frac{L}{2}\right) &\approx \frac{4V}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)}\left(\frac{\rho}{b}\right)^{2k+1}\cos[(2k+1)\varphi] \\ &= \frac{4V}{\pi}\,\mathrm{Re}\left\{\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)}\left(\frac{\rho}{b}\right)^{2k+1}e^{i(2k+1)\varphi}\right\} \\ &= \frac{4V}{\pi}\,\mathrm{Re}\left[\tan^{-1}\left(\frac{\rho}{b}e^{i\varphi}\right)\right].\end{aligned}$$

Recall that $\tan(A+B) = \dfrac{\tan A + \tan B}{1-\tan A\tan B}$ and hence

$A + B = \tan^{-1}\left[\dfrac{\tan A + \tan B}{1-\tan A \tan B}\right]$. Letting $C = \tan A$ and $D = \tan B$, we get:

$$\tan^{-1} C + \tan^{-1} D = \tan^{-1}\left[\frac{C+D}{1-CD}\right].$$

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Moreover, for $z = x+iy$, we have that

$$\tan(z) = \tan(x+iy) = \frac{\tan x + \tan(iy)}{1-\tan x \tan(iy)} = \frac{\tan x + i\tanh y}{1 - i\tan x \tanh y}$$

and hence $\tan(\bar{z}) = \overline{\tan z}$, from which we readily obtain that $\overline{\tan^{-1}(\eta)} = \tan^{-1}\bar{\eta}$, where $\bar{\eta}$ denotes the complex conjugate of η.

$$\begin{aligned}
\mathrm{Re}\left[\tan^{-1}(\eta)\right] &= \frac{1}{2}\left[\tan^{-1}(\eta) + \overline{\tan^{-1}(\eta)}\right] \\
&= \frac{1}{2}\left[\tan^{-1}(\eta) + \tan^{-1}(\bar{\eta})\right] \\
&= \frac{1}{2}\tan^{-1}\left[\frac{\eta+\bar{\eta}}{1-\eta\cdot\bar{\eta}}\right] \\
&= \frac{1}{2}\tan^{-1}\left[\frac{2\,\mathrm{Re}(\eta)}{1-|\eta|^2}\right]
\end{aligned}$$

It follows that, for $L >> b$,

$$\begin{aligned}
\phi(\rho,\varphi,\tfrac{L}{2}) &\simeq \frac{4V}{\pi}\,\mathrm{Re}\left[\tan^{-1}\left(\frac{\rho}{b}e^{i\varphi}\right)\right] \\
&= \frac{2V}{\pi}\tan^{-1}\left[\frac{2\,\mathrm{Re}\left(\frac{\rho}{b}e^{i\varphi}\right)}{1-\left|\frac{\rho}{b}e^{i\varphi}\right|^2}\right] \\
&= \frac{2V}{\pi}\tan^{-1}\left[\frac{2(\rho/b)\cos\varphi}{1-\rho^2/b^2}\right] \\
&= \frac{2V}{\pi}\tan^{-1}\left[\frac{2b\rho\cos\varphi}{b^2-\rho^2}\right].
\end{aligned}$$

This agrees with the result of problem 2.13, if we set $V_1 = V$ and $V_2 = -V$ in that problem.

Problem 3.22: The Green function $G(\rho,\varphi;\rho',\varphi')$ is a solution to the Poisson equation: $\nabla^2 G = -\frac{4\pi}{\rho}\,\delta(\rho-\rho')\,\delta(\varphi-\varphi')$ where $G(\rho,\varphi;\rho',\varphi') = 0$ for (ρ,φ) or (ρ',φ') on the boundary.

Using the results of problem 2.24, the φ-dependence is of the

$G(\rho,\varphi;\rho',\varphi') = 0$ for (ρ,φ) or (ρ',φ') on the boundary.

Using the results of problem 2.24, the φ-dependence is of the form $Q_m(\varphi) \sim \sin\left(\frac{m\pi}{\beta}\varphi\right)$. The completeness relation for the orthonormal functions $\left\{\sqrt{\frac{2}{\beta}}\sin\left(\frac{m\pi}{\beta}\varphi\right)\right\}$, for $0 \le \varphi \le \beta$, is given by

$$\frac{2}{\beta}\sum_{m=1}^{\infty}\sin\left(\frac{m\pi}{\beta}\varphi\right)\sin\left(\frac{m\pi}{\beta}\varphi'\right) = \delta(\varphi-\varphi')$$

See Equation (2.35), with $U_m(\varphi) = \sqrt{\frac{2}{\beta}}\sin\left(\frac{m\pi}{\beta}\varphi\right)$. It follows that

$$\nabla^2 G(\rho,\varphi;\rho',\varphi') = -\frac{8\pi}{\beta\rho}\,\delta(\rho-\rho')\sum_{m=1}^{\infty}\sin\left(\frac{m\pi}{\beta}\varphi\right)\sin\left(\frac{m\pi}{\beta}\varphi'\right) \quad (i)$$

Letting $G(\rho,\varphi;\rho',\varphi') = \sum_{m=1}^{\infty} g_m(\rho,\rho')\sin\left(\frac{m\pi}{\beta}\varphi\right)\sin\left(\frac{m\pi}{\beta}\varphi'\right)$ and substituting into the Equation (i), we obtain:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial\varphi^2} = -\frac{8\pi}{\beta\rho}\,\delta(\rho-\rho')\sum_{m=1}^{\infty}\sin\left(\frac{m\pi}{\beta}\varphi\right)\sin\left(\frac{m\pi}{\beta}\varphi'\right).$$

$$\rightarrow \sum_{m=1}^{\infty}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial g_m(\rho,\rho')}{\partial\rho}\right]\sin\left(\frac{m\pi}{\beta}\varphi\right)\sin\left(\frac{m\pi}{\beta}\varphi'\right)$$

$$-\frac{1}{\rho^2}\sum_{m=1}^{\infty}\left(\frac{m\pi}{\beta}\right)^2 g_m(\rho,\rho')\sin\left(\frac{m\pi}{\beta}\varphi\right)\sin\left(\frac{m\pi}{\beta}\varphi'\right) =$$

$$-\frac{8\pi}{\beta\rho}\,\delta(\rho-\rho')\sum_{m=1}^{\infty}\sin\left(\frac{m\pi}{\beta}\varphi\right)\sin\left(\frac{m\pi}{\beta}\varphi'\right).$$ Thus, for each $m \ge 1$, we get:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial g_m(\rho,\rho')}{\partial\rho}\right] - \frac{1}{\rho^2}\left(\frac{m\pi}{\beta}\right)^2 g_m(\rho,\rho') = -\frac{8\pi}{\beta\rho}\,\delta(\rho-\rho') \quad (ii)$$

For $\rho \neq \rho'$, we have that $\delta(\rho-\rho') = 0$ and hence

$$\rho\frac{\partial}{\partial\rho}\left[\rho\frac{\partial g_m}{\partial\rho}\right] - \left(\frac{m\pi}{\beta}\right)^2 g_m = 0, \text{ or}$$

$$\rho^2\frac{\partial^2 g_m(\rho,\rho')}{\partial\rho^2} + \rho\frac{\partial g_m(\rho,\rho')}{\partial\rho} - \left(\frac{m\pi}{\beta}\right)^2 g_m(\rho,\rho') = 0$$

$$\rightarrow g_m(\rho,\rho') = A_m(\rho')\,\rho^{m\pi/\beta} + B_m(\rho')\,\rho^{-m\pi/\beta}$$

For $\rho < \rho'$, the boundary condition $[g_m(\rho,\rho') \to 0 \text{ as } \rho \to 0]$ entails that $B_m(\rho') = 0$. Thus, for $\rho < \rho'$, $g_m(\rho,\rho') = A_m(\rho')\,\rho^{\frac{m\pi}{\beta}}$.

For $\rho > \rho'$, the boundary condition $[g_m(\rho,\rho') \to 0]$ entails that $B_m(\rho') = 0$. Thus, for $\rho < \rho'$, $g_m(\rho, \rho') = A_m(\rho')\rho^{m\pi/\beta}$.

For $\rho > \rho'$, the boundary condition $[g_m(a, \rho') = 0]$ entails that $A_m(\rho')\, a^{m\pi/\beta} + B_m(\rho')\, a^{-m\pi/\beta} = 0$ and hence

$$A_m(\rho') = -\frac{B_m(\rho')}{a^{2m\pi/\beta}}.$$ Thus, for $\rho > \rho'$, we have that

$$g_m(\rho, \rho') = B_m(\rho')\left[\frac{1}{\rho^{m\pi/\beta}} - \frac{\rho^{m\pi/\beta}}{a^{2m\pi/\beta}}\right].$$

Since $g_m(\rho, \rho') = g_m(\rho', \rho)$, by symmetry of the Dirichlet Green function, $g_m(\rho, \rho')$ must be of the form:

$$g_m(\rho, \rho') = C_m \rho_<^{m\pi/\beta}\left[\frac{1}{\rho_>^{m\pi/\beta}} - \frac{\rho_>^{m\pi/\beta}}{a^{2m\pi/\beta}}\right]$$ where

$\rho_< = \min\{\rho, \rho'\}$ and $\rho_> = \max\{\rho, \rho'\}$.

To get C_m, we multiply Equation (ii) by ρ, integrate from $\rho = \rho' - \varepsilon$ to $\rho = \rho' + \varepsilon$ and then let $\varepsilon \to 0$ to get:

$$\lim_{\varepsilon \to 0}\left[\int_{\rho=\rho'-\varepsilon}^{\rho'+\varepsilon} \frac{\partial}{\partial \rho}\left[\rho \frac{\partial g_m}{\partial \rho}\right] d\rho - \left(\frac{m\pi}{\beta}\right)^2 \int_{\rho=\rho'-\varepsilon}^{\rho'+\varepsilon} \frac{1}{\rho} g_m \, d\rho\right]$$

$$= -\lim_{\varepsilon \to 0} \int_{\rho=\rho'-\varepsilon}^{\rho+\varepsilon} \frac{8\pi}{\beta}\, \delta(\rho - \rho')\, d\rho$$

$$\Rightarrow \lim_{\varepsilon \to 0}\left[\rho \frac{\partial g_m}{\partial \rho}\right]_{\rho=\rho'+\varepsilon} - \lim_{\varepsilon \to 0}\left[\rho \frac{\partial g_m}{\partial \rho}\right]_{\rho=\rho'-\varepsilon} =$$

$$= -\frac{8\pi}{\beta}$$

$$\Rightarrow \lim_{\varepsilon \to 0}\left[\frac{\partial g_m}{\partial \rho}\right]_{\rho=\rho'+\varepsilon} - \lim_{\varepsilon \to 0}\left[\frac{\partial g_m}{\partial \rho}\right]_{\rho=\rho'-\varepsilon} = -\frac{8\pi}{\rho'\beta}$$

$$\lim_{\varepsilon \to 0}\left[\frac{\partial g_m}{\partial \rho}\right]_{\rho=\rho'+\varepsilon} =$$

$$_{\varepsilon\to 0}\left[\partial\rho\right]_{\rho=\rho'+\varepsilon}$$

$$=\lim_{\varepsilon\to 0} C_m \frac{\partial}{\partial\rho}\left\{\rho'^{\,m\pi/\beta}\left[\frac{1}{\rho^{m\pi/\beta}}-\frac{\rho^{m\pi/\beta}}{a^{2m\pi/\beta}}\right]\right\}_{\rho=\rho'+\varepsilon}$$

$$=\lim_{\varepsilon\to 0} C_m\, \rho'^{\,m\pi/\beta}\left[\frac{-m\pi/\beta}{(\rho'+\varepsilon)^{m\pi/\beta+1}}-\frac{m\pi/\beta(\rho'+\varepsilon)^{m\pi/\beta-1}}{a^{2m\pi/\beta}}\right]$$

$$=-\frac{m\pi}{\beta}\frac{C_m}{\rho'}\left[1+\left(\frac{\rho'}{a}\right)^{2m\pi/\beta}\right]$$

$$\lim_{\varepsilon\to 0}\left[\frac{\partial g_m}{\partial\rho}\right]_{\rho=\rho'-\varepsilon}=$$

$$=\lim_{\varepsilon\to 0} C_m \frac{\partial}{\partial\rho}\left\{\rho^{m\pi/\beta}\left[\frac{1}{\rho'^{\,m\pi/\beta}}-\frac{\rho'^{\,m\pi/\beta}}{a^{2m\pi/\beta}}\right]\right\}_{\rho=\rho'-\varepsilon}$$

$$=\lim_{\varepsilon\to 0} C_m \frac{m\pi}{\beta}(\rho'+\varepsilon)^{m\pi/\beta-1}\left[\frac{1}{\rho'^{\,m\pi/\beta}}-\frac{\rho'^{\,m\pi/\beta}}{a^{2m\pi/\beta}}\right]$$

$$=\frac{m\pi}{\beta}\frac{C_m}{\rho'}\left[1-\left(\frac{\rho'}{a}\right)^{2m\pi/\beta}\right].$$ Thus,

$$-\frac{m\pi}{\beta}\frac{C_m}{\rho'}\left[1+\left(\frac{\rho'}{a}\right)^{2m\pi/\beta}\right]-\frac{m\pi}{\beta}\frac{C_m}{\rho'}\left[1-\left(\frac{\rho'}{a}\right)^{2m\pi/\beta}\right]=\frac{-8\pi}{\beta\rho'}$$

and hence

$$m C_m\left[1+\left(\frac{\rho'}{a}\right)^{2m\pi/\beta}\right]+m C_m\left[1-\left(\frac{\rho'}{a}\right)^{2m\pi/\beta}\right]=8.$$

Therefore, $2mC_m = 8$ and hence $\boxed{C_m=\frac{4}{m}}$.

It follows that $g_m(\rho,\rho')=\frac{4}{m}\rho_<^{m\pi/\beta}\left[\frac{1}{\rho_>^{m\pi/\beta}}-\frac{\rho_>^{m\pi/\beta}}{a^{2m\pi/\beta}}\right]$

for each $m\geq 1$; and hence

for each $m \geq 1$; and hence

$$\boxed{G(\rho,\varphi;\rho',\varphi') = \sum_{m=1}^{\infty} \frac{4}{m}\, \rho_<^{m\pi/\beta} \left[\frac{1}{\rho_>^{m\pi/\beta}} - \frac{\rho_>^{m\pi/\beta}}{a^{2m\pi/\beta}} \right] \sin\left(\frac{m\pi}{\beta}\varphi\right) \sin\left(\frac{m\pi}{\beta}\varphi'\right)}$$