phys 7590 Homework #3

3.1) Because of azimuthal symmetry, the solution inside the spherical shell $a \le r \le b$ has the form:

$$\phi(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)} \right] P_l(\cos\theta) \quad (*)$$

Note that $\phi(r,\theta)$ can be written as

$$\phi(r,\theta) = \frac{V}{2} + \phi_1(r,\theta)$$, where $\phi_1(r,\theta)$ is

the solution that corresponds to the boundary conditions:

$$\phi_1(a,\theta) = \begin{cases} +V/2 & \text{if } 0 \le \theta < \pi/2 \quad (\text{i.e. } \cos\theta > 0) \\ -V/2 & \text{if } \frac{\pi}{2} < \theta \le \pi \quad (\cos\theta < 0) \end{cases}$$ and

$$\phi_1(b,\theta) = \begin{cases} -V/2 & \text{if } 0 \le \theta < \pi/2 \quad (\cos\theta > 0) \\ V/2 & \text{if } \frac{\pi}{2} < \theta \le \pi \quad (\cos\theta < 0) \end{cases}$$

It follows that $\phi_1(r,\theta)$ is an odd function of $(\cos\theta)$; and hence only l odd (and $l=0$) ~~appear~~ contribute to $(*)$. Thus,

$$\phi(r,\theta) = \frac{V}{2} + \sum_{\substack{l=1 \\ l \text{ odd}}}^{\infty} \left[A_l r^l + B_l r^{-(l+1)} \right] P_l(\cos\theta)$$

Applying the boundary conditions at $r=a$ and at $r=b$, we get:

$$\sum_{\substack{\ell=1 \\ \ell \text{ odd}}}^{\infty} \left[A_\ell a^\ell + B_\ell a^{-(\ell+1)}\right] P_\ell(\cos\theta) = \phi_1(a,\theta)$$

$$= \begin{cases} V/2 & \text{if } \cos\theta > 0 \\ -V/2 & \text{if } \cos\theta < 0 \end{cases}$$

and

$$\sum_{\substack{\ell=1 \\ \ell \text{ odd}}}^{\infty} \left[A_\ell b^\ell + B_\ell b^{-(\ell+1)}\right] P_\ell(\cos\theta) = \phi_1(b,\theta)$$

$$= \begin{cases} -V/2 & \text{if } \cos\theta > 0 \\ V/2 & \text{if } \cos\theta < 0 \end{cases}$$

This is similar to the example in section 3.3 except that V is replaced by $\frac{V}{2}$ on the inner sphere and $-V/2$ on the outer sphere. Thus, for all odd ℓ, we obtain:

$$A_\ell a^\ell + B_\ell a^{-(\ell+1)} = (2\ell+1)\left(\frac{V}{2}\right) \int_0^1 P_\ell(x)\, dx \quad \text{and}$$

$$A_\ell b^\ell + B_\ell b^{-(\ell+1)} = (2\ell+1)\left(-\frac{V}{2}\right) \int_0^1 P_\ell(x)\, dx, \quad \text{where}$$

$$\int_0^1 P_\ell(x)\, dx = \left(-\frac{1}{2}\right)^{(\ell-1)/2} \frac{(\ell-2)!!}{2\left(\frac{\ell+1}{2}\right)!} \quad \text{(given in class!)} \quad \text{for } \ell = 3, 5, 7, \ldots$$

$$\left(-\frac{1}{2}\right)^{\frac{\ell-1}{2}}$$

and $\displaystyle\int_0^1 P_1(x)\, dx = \int_0^1 x\, dx = \frac{1}{2}$.

Thus, for each odd ℓ, we obtain a system of two equations in the two unknowns A_ℓ and B_ℓ:

$$A_\ell a^\ell + B_\ell a^{-(\ell+1)} = N_\ell \quad \text{and}$$

$$A_\ell b^\ell + B_\ell b^{-(\ell+1)} = -N_\ell \ , \text{ where}$$

$$N_\ell = \begin{cases} (-1)^{(\ell-1)/2} \dfrac{V(2\ell+1)(\ell-2)!!}{4\left(\frac{\ell+1}{2}\right)!} & \text{for } \ell = 3, 5, \ldots \\ \dfrac{3V}{4} & \text{for } \ell = 1 \end{cases}$$

(**). Solving for A_ℓ and B_ℓ, we get:

$$\begin{pmatrix} A_\ell \\ B_\ell \end{pmatrix} = \begin{pmatrix} a^\ell & a^{-(\ell+1)} \\ b^\ell & b^{-(\ell+1)} \end{pmatrix}^{-1} \begin{pmatrix} N_\ell \\ -N_\ell \end{pmatrix}$$

$$= N_\ell \left[\frac{-a^{\ell+1} b^{\ell+1}}{b^{2\ell+1} - a^{2\ell+1}} \right] \begin{pmatrix} b^{-(\ell+1)} & -a^{-(\ell+1)} \\ -b^\ell & a^\ell \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{N_\ell}{b^{2\ell+1} - a^{2\ell+1}} \begin{pmatrix} -a^{\ell+1} & b^{\ell+1} \\ +(ab)^{\ell+1} b^\ell & -(ab)^{\ell+1} a^\ell \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{N_\ell}{b^{2\ell+1} - a^{2\ell+1}} \begin{pmatrix} -(a^{\ell+1} + b^{\ell+1}) \\ (ab)^{\ell+1}(a^\ell + b^\ell) \end{pmatrix}$$

$$= \frac{N_\ell}{1-\left(\frac{a}{b}\right)^{2\ell+1}} \begin{pmatrix} -\left(1+\left(\frac{a}{b}\right)^{\ell+1}\right) b^{-\ell} \\ + a^{\ell+1}\left(1+\left(\frac{a}{b}\right)^{\ell}\right) \end{pmatrix}$$

So $A_\ell = \dfrac{-N_\ell}{1-\left(\frac{a}{b}\right)^{2\ell+1}} \left[1+\left(\frac{a}{b}\right)^{\ell+1}\right] b^{-\ell}$ and

$B_\ell = \dfrac{+N_\ell}{1-\left(\frac{a}{b}\right)^{2\ell+1}} \left[1+\left(\frac{a}{b}\right)^{\ell}\right] a^{\ell+1}$. Thus,

$$\phi(r,\theta) = \frac{V}{2} + \sum_{\substack{\ell=1 \\ \ell \text{ odd}}}^{\infty} \frac{N_\ell}{1-\left(\frac{a}{b}\right)^{2\ell+1}} \left\{ -\left[1+\left(\frac{a}{b}\right)^{\ell+1}\right]\left(\frac{r}{b}\right)^{\ell} + \left[1+\left(\frac{a}{b}\right)^{\ell}\right]\left(\frac{a}{r}\right)^{\ell+1} \right\} P_\ell(\cos\theta) \quad (***)$$, where

N_ℓ is given in (**). Note that

$N_1 = \frac{V}{2}\left(\frac{3}{2}\right)$ ~~and~~, $N_3 = \frac{V}{2}\left(\frac{-7}{8}\right)$, ~~etc~~

$N_5 = \frac{V}{2}\left(\frac{11}{16}\right)$, etc...

In the limit when $b \to \infty$, $\frac{a}{b} \to 0$ and $\left(\frac{r}{b}\right) \to 0$.

Thus, (***) becomes

$$\phi(r,\theta) = \frac{V}{2} + \sum_{\substack{l=1 \\ l \text{ odd}}}^{\infty} N_l \left(\frac{a}{r}\right)^{l+1} P_l(\cos\theta)$$

$$= \frac{V}{2} + \frac{V}{2}\left[\frac{3}{2}\left(\frac{a}{r}\right)^2 P_1(\cos\theta) - \frac{7}{8}\left(\frac{a}{r}\right)^4 P_3(\cos\theta) + \frac{11}{16}\left(\frac{a}{r}\right)^6 P_5(\cos\theta) - - - \right]$$

which agrees with the exterior solution for a sphere with oppositely charged hemispheres (except that here we have the average potential $\frac{V}{2}$ added and that V is replaced by $\frac{V}{2}$); see Eqn (3.36) with $\left(\frac{r}{a}\right)^l$ replaced by $\left(\frac{a}{r}\right)^{l+1}$ for the exterior problem. See also problem 2.22 (a).

Similarly, when $a \to 0$ then $\frac{a}{b}$ and $\frac{a}{r} \longrightarrow 0$. Thus, (**) now becomes:

$$\phi(r,\theta) = \frac{V}{2} - \sum_{\substack{l=1 \\ l \text{ odd}}}^{\infty} N_l \left(\frac{r}{b}\right)^{l} P_l(\cos\theta)$$

$$= \frac{V}{2} - \frac{V}{2}\left[\frac{3}{2}\left(\frac{r}{b}\right) P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{b}\right)^3 P_3(\cos\theta) + \frac{11}{16}\left(\frac{r}{b}\right)^5 P_5(\cos\theta) - - - \right]$$

which agrees with Eqn (3.36) for the interior solution for ~~oppos~~ a sphere with oppositely charged hemispheres – with the following adjustments to fit the problem at hand:

- The average potential $\frac{V}{2}$ is added
- V is replaced by $-\frac{V}{2}$ (the potential on the top hemisphere.)
- a is replaced by b (the radius of the sphere!)

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3.7) a) $$\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{-2}{r} + \frac{1}{|\vec{x} - a\hat{z}|} + \frac{1}{|\vec{x} + a\hat{z}|}\right]$$

where $r = |\vec{x}|$.

using equation (3.38), we get:

$$\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{-2}{r} + \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta) + \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos(\pi-\theta))\right]$$

where $r_< = \min\{r, a\}$ and $r_> = \max\{r, a\}$. Thus,

$$\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[-\frac{2}{r} + \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} \left(P_l(\cos\theta) + P_l(-\cos\theta)\right)\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[-\frac{2}{r} + \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} (1+(-1)^l) P_l(\cos\theta)\right]$$

$$= \frac{q}{2\pi\varepsilon_0}\left[-\frac{1}{r} + \sum_{l \text{ even}} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)\right]$$

$$= \frac{q}{2\pi\varepsilon_0}\left[-\frac{1}{r} + \frac{1}{r_>} + \sum_{l=2,4,6,\ldots}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)\right]$$

In the limit $a \to 0$ while $qa^2 = Q$ remains finite:
$r_< = a$ and $r_> = r$. So

$$\phi(\vec{x}) = \frac{q}{2\pi\varepsilon_0} \sum_{l=2,4,6,\ldots} \frac{a^l}{r^{l+1}} P_l(\cos\theta). \qquad (*)$$

As $a \to 0$, the $l=2$ term prevails over the other terms in the sum (*). Thus,

$$\phi(\vec{x}) \approx \frac{q}{2\pi\varepsilon_0}\frac{a^2}{r^3}P_2(\cos\theta)$$

$$= \frac{Q}{4\pi\varepsilon_0 r^3}(3\cos^2\theta - 1).$$

b) In the presence of a grounded conducting spherical shell of radius b, centered at the origin, the potential is obtained using the method of images:

$$\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{x}-a\hat{z}|} - \frac{b/a}{|\vec{x}-\frac{b^2}{a}\hat{z}|}\right]$$

$$+ \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{x}+a\hat{z}|} - \frac{b/a}{|\vec{x}+\frac{b^2}{a}\hat{z}|}\right]$$

$$+ \frac{q}{4\pi\varepsilon_0}\left[\frac{-2}{r} + \frac{2}{b}\right]. \qquad (**)$$

The image charge of the point charge $(-2q)$ at the origin is a point charge at infinity that produces the constant potential $\frac{q}{4\pi\varepsilon_0}\frac{2}{b}$. Substituting $r = |\vec{x}| = b$ into (**), we get $\phi = 0$.

Using Equation (3.38) again, we get the potential at any point inside the sphere $(r < b)$. Note that for $r < b$, $\frac{b^2}{a} > b > r = |\vec{x}|$. Thus,

$$\phi(\vec{x}) = \phi(r,\theta)$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta) - \frac{b}{a}\sum_{l=0}^{\infty} \frac{r^l}{(b^2/a)^{l+1}} P_l(\cos\theta)\right]$$

$$+ \frac{q}{4\pi\varepsilon_0}\left[\sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(-\cos\theta) - \frac{b}{a}\sum_{l=0}^{\infty} \frac{r^l}{(b^2/a)^{l+1}} P_l(-\cos\theta)\right]$$

$$+ \frac{q}{4\pi\varepsilon_0}\left[\frac{2}{b} - \frac{2}{r}\right], \text{ where } r_< = \min\{r, a\}, \; r_> = \max\{r, a\}.$$

Thus,

$$\phi(r,\theta) = \frac{q}{4\pi\varepsilon_0}\left\{\frac{2}{b} - \frac{2}{r} + \sum_{l=0}^{\infty}\left[\left(\frac{r_<^l}{r_>^{l+1}} - \frac{b}{a}\frac{r^l}{(b^2/a)^{l+1}}\right)\cdot \left(P_l(\cos\theta) + P_l(-\cos\theta)\right)\right]\right\}$$

$$= \frac{q}{2\pi\varepsilon_0}\left[\frac{1}{b} - \frac{1}{r} + \sum_{l \text{ even}}\left(\frac{r_<^l}{r_>^{l+1}} - \frac{1}{b}\left(\frac{ar}{b^2}\right)^l\right) P_l(\cos\theta)\right] \quad (***)$$

This is valid for both $r<a$ and $r>a$.

For $r>a$, $r_> = r$ and $r_< = a$; then

$$\phi(r,\theta) = \frac{q}{2\pi\varepsilon_0}\left[\frac{1}{b} - \frac{1}{r} + \sum_{l \text{ even}}\left(\frac{a^l}{r^{l+1}} - \frac{1}{b}\left(\frac{ar}{b^2}\right)^l\right) P_l(\cos\theta)\right]$$

$$= \frac{q}{2\pi\varepsilon_0}\left[\sum_{l=2,4,6,\ldots}\left(\frac{a^l}{r^{l+1}} - \frac{1}{b}\left(\frac{ar}{b^2}\right)^l\right) P_l(\cos\theta)\right]$$

$$= \frac{q}{2\pi\varepsilon_0}\sum_{l=2,4,6,\ldots}\frac{a^l}{r^{l+1}}\left(1 - \left(\frac{r}{b}\right)^{2l+1}\right) P_l(\cos\theta).$$

As $a \to 0$, the $l=2$ again dominates in (***). Thus,

$$\phi(r,\theta) \simeq \frac{q}{2\pi\varepsilon_0}\frac{a^2}{r^3}\left(1-\frac{r^5}{b^5}\right)P_2(\cos\theta)$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{a^2}{r^3}\left(1-\frac{r^5}{b^5}\right)(3\cos^2\theta-1).$$

$$= \frac{Q}{4\pi\varepsilon_0 r^3}\left(1-\frac{r^5}{b^5}\right)(3\cos^2\theta-1)$$

For $r<a$: $r_> = a$ and $r_< = r$; then

$$\phi(r,\theta) = \frac{q}{2\pi\varepsilon_0}\left[+\frac{1}{b}-\frac{1}{r}+\sum_{l\ \text{even}}\left(\frac{r^l}{a^{l+1}}-\frac{1}{b}\left(\frac{ar}{b^2}\right)^l\right)P_l(\cos\theta)\right]$$

$$= \frac{q}{2\pi\varepsilon_0}\left[\frac{1}{a}-\frac{1}{r}+\sum_{l=2,4,6\ldots}\left(\frac{r^l}{a^{l+1}}-\frac{1}{b}\left(\frac{ar}{b^2}\right)^l\right)P_l(\cos\theta)\right]$$

$$= \frac{q}{2\pi\varepsilon_0}\left[\frac{1}{a}-\frac{1}{r}+\sum_{l=2,4,\ldots}\frac{r^l}{a^{l+1}}\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)P_l(\cos\theta)\right]$$

3.13 The Dirichlet Green's function for a spherical ~~sp~~ shell of inner radius a and outer radius b is given by equation (3.125):

$$G(\vec{x},\vec{x}') = \sum_{l,m}\frac{4\pi Y^*_{lm}(\theta',\varphi')Y_{lm}(\theta,\varphi)}{(2l+1)\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]}\left(r_<^l-\frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\right)$$

where $r_< = \min\{r,r'\}$ and $r_> = \max\{r,r'\}$; $r=|\vec{x}|$, $r'=|\vec{x}'|$.

Since there are no charges ($\rho = 0$) between the two spheres, the potential ~~inside the spherical shell is given~~ at any point between the two concentric spheres is given by

$$\phi(\vec{x}) = -\frac{1}{4\pi}\oint_S \phi(\vec{x}\,')\frac{\partial G}{\partial n'}\,da'$$

$$= -\frac{1}{4\pi}\int_{r'=a}\phi(\vec{x}\,')\frac{\partial G}{\partial n'}\,a^2\,d\Omega' - \frac{1}{4\pi}\int_{r'=b}\phi(\vec{x}\,')\frac{\partial G}{\partial n'}\,b^2\,d\Omega' \qquad (*)$$

Recall that $\hat{n}'$ points outward from the volume of interest. Thus,

$\hat{n}' = \hat{r}'$ on the sphere of radius b and
$\hat{n}' = -\hat{r}'$ on the " of " a. Hence

$$\left.\frac{\partial G}{\partial n'}\right|_{r'=a} = -\left.\frac{\partial G}{\partial r'}\right|_{r'=a}$$

$$= -\frac{\partial}{\partial r'}\left\{\sum_{l,m}\frac{4\pi Y^*_{lm}(\theta',\varphi')Y_{lm}(\theta,\varphi)}{(2l+1)\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]}\left(r'^{\,l}-\frac{a^{2l+1}}{r'^{\,l+1}}\right)\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right)\right\}_{r'=a}$$

$$= -4\pi\sum_{l,m}\frac{Y^*_{lm}(\theta',\varphi')Y_{lm}(\theta,\varphi)}{\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]}\,a^{l-1}\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right)$$

$$= -\frac{4\pi}{a^2}\sum_{l,m}\frac{1}{\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]}\left[\left(\frac{a}{r}\right)^{l+1}-\left(\frac{a}{b}\right)^{l+1}\left(\frac{r}{b}\right)^{l}\right]Y^*_{lm}(\theta',\varphi')Y_{lm}(\theta,\varphi)$$

and

$$\frac{\partial G}{\partial n'}\bigg|_{r'=b} = \frac{\partial G}{\partial r'}\bigg|_{r'=b}$$

$$= \frac{\partial}{\partial r'}\left\{\sum_{l,m} \frac{4\pi Y^*_{lm}(\theta',\varphi')Y_{lm}(\theta,\varphi)}{(2l+1)\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}\right)\right\}_{r'=b}$$

$$= -\frac{4\pi}{b^2}\sum_{l,m}\frac{Y^*_{lm}(\theta',\varphi')Y_{lm}(\theta,\varphi)}{\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]}\left[\left(\frac{r}{b}\right)^l - \left(\frac{a}{b}\right)^l\left(\frac{a}{r}\right)^{l+1}\right].$$

Substituting into (*), we get:

$$\phi(\vec{x}) = \sum_{l,m}\frac{Y_{lm}(\theta,\varphi)}{1-\left(\frac{a}{b}\right)^{2l+1}}\cdot\left\{\left[\left(\frac{a}{r}\right)^{l+1} - \left(\frac{a}{b}\right)^{l+1}\left(\frac{r}{b}\right)^l\right]\int\phi(a,\theta')Y^*_{lm}(\theta',\varphi')\,d\Omega'\right.$$

$$\left. + \left[\left(\frac{r}{b}\right)^l - \left(\frac{a}{b}\right)^l\left(\frac{a}{r}\right)^{l+1}\right]\int\phi(b,\theta')Y^*_{lm}(\theta',\varphi')\,d\Omega'\right\}$$

Because of azimuthal symmetry, only the $m=0$ terms contribute; thus (with $Y_{l0}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$),

$$\phi(\vec{x}) = \sum_{l=0}^{\infty}\frac{(2l+1)P_l(\cos\theta)}{2\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]}\cdot\left\{\left[\left(\frac{a}{r}\right)^{l+1} - \left(\frac{a}{b}\right)^{l+1}\left(\frac{r}{b}\right)^l\right]\int_0^1 V P_l(\cos\theta')\,d(\cos\theta')\right.$$

$$\left. + \left[\left(\frac{r}{b}\right)^l - \left(\frac{a}{b}\right)^l\left(\frac{a}{r}\right)^{l+1}\right]\int_{-1}^0 V P_l(\cos\theta')\,d(\cos\theta')\right\}$$

$$\phi(\vec{x}) = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)V}{2\left[1-\left(\frac{a}{b}\right)^{2\ell+1}\right]} \cdot \left\{ \left[\left(\frac{a}{r}\right)^{\ell+1} - \left(\frac{a}{b}\right)^{\ell+1}\left(\frac{r}{b}\right)^{\ell}\right] \int_0^1 P_\ell(\xi)\, d\xi \right.$$

$$\left. + (-1)^\ell \left[\left(\frac{r}{b}\right)^{\ell} - \left(\frac{a}{b}\right)^{\ell}\left(\frac{a}{r}\right)^{\ell+1}\right] \int_0^1 P_\ell(\xi)\, d\xi \right\} P_\ell(\cos\theta)$$

where
$$\int_0^1 P_\ell(\xi)\, d\xi = \begin{cases} 1 & \text{if } \ell = 0 \\ 0 & \text{if } \ell = 2, 4, 6, \ldots \\ \left(-\frac{1}{2}\right)^{(\ell-1)/2} \dfrac{(\ell-2)!!}{2\left(\frac{\ell+1}{2}\right)!} & \text{if } \ell \text{ is odd} \end{cases}$$

Defining N_ℓ as in problem 3.1:

$$N_\ell = \frac{V(2\ell+1)}{2} \int_0^1 P_\ell(\xi)\, d\xi \text{, we get:}$$

$$\phi(\vec{x}) = \frac{V}{2} \overbrace{\frac{1}{\left[1-\frac{a}{b}\right]} \left\{\left(\frac{a}{r} - \frac{a}{b}\right) + \left(1 - \frac{a}{r}\right)\right\}}^{\to \ell = 0}$$

$$+ \sum_{\ell \text{ odd}} \frac{N_\ell}{\left[1-\left(\frac{a}{b}\right)^{2\ell+1}\right]} \cdot \left\{ \left[\left(\frac{a}{r}\right)^{\ell+1} - \left(\frac{a}{b}\right)^{\ell+1}\left(\frac{r}{b}\right)^{\ell}\right] \right.$$

$$\left. - \left[\left(\frac{r}{b}\right)^{\ell} - \left(\frac{a}{b}\right)^{\ell}\left(\frac{a}{r}\right)^{\ell+1}\right] \right\} P_\ell(\cos\theta)$$

$$\phi(r,\theta) = \frac{V}{2} + \sum_{\ell \text{ odd}} \frac{N_\ell}{\left[1-\left(\frac{a}{b}\right)^{2\ell+1}\right]} \left\{ -\left[1+\left(\frac{a}{b}\right)^{\ell+1}\right]\left(\frac{r}{b}\right)^{\ell} \right.$$

$$\left. + \left[1+\left(\frac{a}{b}\right)^{\ell}\right]\left(\frac{a}{r}\right)^{\ell+1} \right\} P_\ell(\cos\theta)$$

which is exactly the solution we got in Problem 3.1

3.26 a) Consider the boundary condition on the Neumann Green function: $\frac{\partial G}{\partial n'} = -\frac{4\pi}{S}$ for $\vec{x}'$ on the boundary.

Here $S = 4\pi a^2 + 4\pi b^2 = 4\pi(a^2+b^2)$, which is independent of the angles. Thus, we can write:

$$\left.\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\right|_{boundary} = \sum_{l=0}^{\infty} \frac{\partial g_l(r,r')}{\partial n'} P_l(\cos\gamma)\Big|_{boundary}$$

$$= -\frac{1}{a^2+b^2} P_0(\cos\gamma). \quad \text{Hence}$$

$$\left.\frac{\partial g_l}{\partial n'}\right|_{boundary} = -\frac{1}{a^2+b^2}\delta_{l0}.$$

On the outer sphere, $\hat{n}' = \hat{r}' \Rightarrow \left.\frac{\partial g_l}{\partial r'}\right|_{r'=b} = -\frac{1}{a^2+b^2}\delta_{l0}$ (*)

On the inner sphere, $\hat{n}' = -\hat{r}' \Rightarrow \left.\frac{\partial g_l}{\partial r'}\right|_{r'=a} = \frac{1}{a^2+b^2}\delta_{l0}$ (**).

Write $g_l(r,r') = \frac{r_<^l}{r_>^{l+1}} + f_l(r,r')$.

Since $G(\vec{x},\vec{x}') = \sum_{l=0}^{\infty} g_l(r,r') P_l(\cos\gamma)$, then

$$G(\vec{x},\vec{x}') = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma) + \sum_{l=0}^{\infty} f_l(r,r') P_l(\cos\gamma)$$

So $G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + \sum_{l=0}^{\infty} f_l(r,r')\, P_l(\cos\gamma)$.

Since $\nabla'^2 G(\vec{x},\vec{x}') = \nabla'^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\,\delta(\vec{x}-\vec{x}')$, it follows that $F(\vec{x},\vec{x}') := \sum_{l=0}^{\infty} f_l(r,r')\, P_l(\cos\gamma)$ is a solution of the Laplace equation: $\nabla'^2 F(\vec{x},\vec{x}') = 0$.

Thus, $f_l(r,r')$ must be of the form

$$f_l(r,r') = A_l\, r'^l + \frac{B_l}{r'^{l+1}}$$

, where A_l and B_l (constants w.r.t. r') may be functions of r. Hence

$$g_l(r,r') = \frac{r_<^l}{r_>^{l+1}} + A_l\, r'^l + \frac{B_l}{r'^{l+1}}$$

We use (*) and (**) to determine A_l and B_l.

For $l>0$:

$$(*) \Rightarrow \frac{\partial}{\partial r'}\left\{\frac{r^l}{r'^{l+1}} + A_l\, r'^l + \frac{B_l}{r'^{l+1}}\right\}\Bigg|_{r'=b} = 0$$

$$\rightarrow -\frac{(l+1)\, r^l}{b^{l+2}} + l\, A_l\, b^{l-1} - \frac{(l+1)\, B_l}{b^{l+2}} = 0$$

$$\rightarrow l\, b^{2l+1} A_l - (l+1)\, B_l = (l+1)\, r^l \qquad (***)$$

$$(**) \Rightarrow \frac{\partial}{\partial r'}\left\{\frac{r'^l}{r^{l+1}} + A_l\, r'^l + \frac{B_l}{r'^{l+1}}\right\}_{r'=a} = 0$$

$$\rightarrow l\, \frac{a^{l-1}}{r^{l+1}} + l\, A_l\, a^{l-1} - \frac{(l+1)\, B_l}{a^{l+2}} = 0$$

$$\longrightarrow \ell a^{2\ell+1} A_\ell - (\ell+1) B_\ell = -\ell\, a^{2\ell+1}/r^{\ell+1} \qquad (****)$$

From (***) and (****), we get

$$\begin{pmatrix} A_\ell \\ B_\ell \end{pmatrix} = \begin{bmatrix} \ell\, b^{2\ell+1} & -(\ell+1) \\ \ell\, a^{2\ell+1} & -(\ell+1) \end{bmatrix}^{-1} \begin{bmatrix} (\ell+1) r^\ell \\ -\ell\, a^{2\ell+1}/r^{\ell+1} \end{bmatrix}$$

$$= \frac{r^\ell}{b^{2\ell+1} - a^{2\ell+1}} \begin{bmatrix} \left(\frac{a}{r}\right)^{2\ell+1} + \frac{\ell+1}{\ell} \\ a^{2\ell+1} + \frac{\ell}{\ell+1}\left(\frac{ab}{r}\right)^{2\ell+1} \end{bmatrix}.$$

Thus,

$$g_\ell(r,r') = \frac{r_<^\ell}{r_>^{\ell+1}} + \frac{r^\ell}{b^{2\ell+1} - a^{2\ell+1}} \left\{ \left[\left(\frac{a}{r}\right)^{2\ell+1} + \frac{\ell+1}{\ell}\right] r'^\ell + \left[a^{2\ell+1} + \frac{\ell}{\ell+1}\left(\frac{ab}{r}\right)^{2\ell+1}\right] \frac{1}{r'^{\ell+1}} \right\}$$

$$= \frac{r_<^\ell}{r_>^{\ell+1}} + \frac{1}{b^{2\ell+1} - a^{2\ell+1}} \left\{ \frac{\ell+1}{\ell}(rr')^\ell + \frac{\ell}{\ell+1}\frac{(ab)^{2\ell+1}}{(rr')^{\ell+1}} + a^{2\ell+1}\left[\frac{r'^\ell}{r^{\ell+1}} + \frac{r^\ell}{r'^{\ell+1}}\right] \right\}$$

(which is symmetric in r and r').

b) For $\ell = 0$:

$$(*) \longrightarrow \frac{\partial}{\partial r'}\left\{\frac{1}{r'} + A_0 + \frac{B_0}{r'}\right\}\Bigg|_{r'=b} = -\frac{1}{a^2+b^2}$$

$$\rightarrow -(B_0+1)/b^2 = -\frac{1}{a^2+b^2}$$

$$\Rightarrow B_0 = -\frac{a^2}{a^2+b^2}$$

Also

$$(**) \rightarrow \frac{\partial}{\partial r'}\left\{\frac{1}{r} + A_0 + \frac{B_0}{r'}\right\}_{r'=a} = \frac{1}{a^2+b^2}$$

$$\rightarrow -\frac{B_0}{a^2} = \frac{1}{a^2+b^2}$$

$$\rightarrow B_0 = -\frac{a^2}{a^2+b^2}$$

So $(*)$ and $(**)$ both yield $B_0 = \frac{-a^2}{a^2+b^2}$ and leave A_0 arbitrary (an arbitrary function of the parameter r, say $A_0 = f(r)$). Thus,

$$g_0(r,r') = \frac{1}{r_>} - \frac{a^2}{a^2+b^2}\frac{1}{r'} + f(r).$$

To show that $f(r)$ does not contribute to $\phi(\vec{x})$ in equation (1.46), we need to show that

$$\frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')f(r)\,d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\phi(\vec{x}')}{\partial n'}f(r)\,da' = 0.$$ Thus,

$$\frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')f(r)\,d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\phi}{\partial n'}f(r)\,da' =$$

$$\frac{f(r)}{4\pi\varepsilon_0}\left[\int_V \rho(\vec{x}')\,d^3x' + \varepsilon_0\oint_S \frac{\partial\phi}{\partial n'}\,da'\right]$$

$$= \frac{f(r)}{4\pi\varepsilon_0}\left[\overbrace{q_{enc} - \varepsilon_0 \oint_S \vec{E}(\vec{x}')\cdot\hat{n}'\,da'}^{0 \text{ by Gauss' law}}\right]$$

$$= 0 .$$

We can use the freedom of $f(r)$ to make $g_0(r, r')$ symmetric in r and r'. Take $f(r) = \frac{-a^2}{a^2+b^2}\frac{1}{r}$; then

$$g_0(r, r') = \frac{1}{r_>} - \frac{a^2}{a^2+b^2}\left(\frac{1}{r} + \frac{1}{r'}\right).$$

With this choice of $f(r)$, $g_\ell(r, r')$ will be symmetric in r and r', for all $\ell \geq 0$; it follows that $G(\vec{x}, \vec{x}') = \sum_{\ell=0}^{\infty} g_\ell(r, r') P_\ell(\cos\gamma)$ is a symmetric Green's function for the Neumann boundary conditions.

Note: Recall problem 1.14