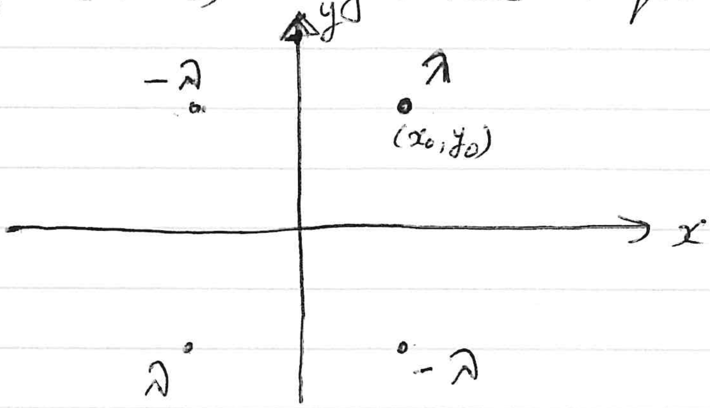
Phys 7590 Homework # 2

2.3

a) The image (line) charges needed for this problem are <u>three</u>:



$-\lambda$ at $(-x_0, y_0)$ and $(x_0, -y_0)$ and $+\lambda$ at $(-x_0, -y_0)$.

Adding up the contributions of these four charges yields:

$$\phi(x,y) = \frac{\lambda}{4\pi\varepsilon_0}\left[\ln\frac{R^2}{(x-x_0)^2+(y-y_0)^2} + \ln\frac{R^2}{(x+x_0)^2+(y+y_0)^2}\right.$$

$$\left. - \ln\frac{R^2}{(x-x_0)^2+(y+y_0)^2} - \ln\frac{R^2}{(x+x_0)^2+(y-y_0)^2}\right]$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{[(x-x_0)^2+(y+y_0)^2][(x+x_0)^2+(y-y_0)^2]}{[(x-x_0)^2+(y-y_0)^2][(x+x_0)^2+(y+y_0)^2]}$$

To check the validity of this solution, we evalute the potential on the boundary surfaces ($x=0, y\geq 0$ and $y=0, x\geq 0$).

$$\phi(0,y) = \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{[x_0^2+(y+y_0)^2][x_0^2+(y-y_0)^2]}{[x_0^2+(y-y_0)^2][x_0^2+(y+y_0)^2]}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln 1 = 0$$

Similarly, we show that $\phi(x, 0) = 0$.

For the tangential component of the electric field, we verify that $E_x(x, 0) = E_y(0, y) = 0$.

Since we need E_y for part b), we show that $E_y(0, y) = 0$, the verification of $E_x(x, 0) = 0$ is done similarly.

$$E_y(x, y) = -\frac{\partial}{\partial y}\phi(x, y)$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left[\frac{(y - y_0)}{(x - x_0)^2 + (y - y_0)^2} + \frac{(y + y_0)}{(x + x_0)^2 + (y + y_0)^2} - \frac{(y + y_0)}{(x - x_0)^2 + (y + y_0)^2} - \frac{(y - y_0)}{(x + x_0)^2 + (y - y_0)^2}\right] \quad (*)$$

$$E_y(0, y) = \frac{\lambda}{2\pi\varepsilon_0}\left[\frac{(y - y_0)}{x_0^2 + (y - y_0)^2} + \frac{(y + y_0)}{x_0^2 + (y + y_0)^2} - \frac{(y + y_0)}{x_0^2 + (y + y_0)^2} - \frac{y - y_0}{x_0^2 + (y - y_0)^2}\right]$$

$$= 0.$$

b) The surface charge density on the $y = 0$ plane is obtained from the normal component of the electric field: $\sigma(x) = \varepsilon_0 E_y(x, 0)$

Setting $y = 0$ in (*), we get:

$$\sigma(x) = \frac{\lambda}{2\pi}\left[\frac{-y_0}{(x-x_0)^2+y_0^2} + \frac{y_0}{(x+x_0)^2+y_0^2}\right.$$

$$\left. - \frac{y_0}{(x-x_0)^2+y_0^2} + \frac{y_0}{(x+x_0)^2+y_0^2}\right]$$

$$= \frac{\lambda y_0}{\pi}\left[\frac{1}{(x+x_0)^2+y_0^2} - \frac{1}{(x-x_0)^2+y_0^2}\right]$$

$$= -\frac{4\lambda x_0 y_0}{\pi}\,\frac{x}{[(x+x_0)^2+y_0^2]\,[(x-x_0)^2+y_0^2]} \qquad (**)$$

The expression in (**) shows that the surface charge density vanishes (linearly with x) at the origin; this is in agreement with Section 2.11 ($\beta = \pi/2$).

c) $$Q_x = \int_0^\infty \sigma(x)\,dx$$

$$= \frac{\lambda y_0}{\pi}\int_0^\infty \left[\frac{1}{(x+x_0)^2+y_0^2} - \frac{1}{(x-x_0)^2+y_0^2}\right] dx$$

$$= \frac{\lambda y_0}{\pi}\left[\frac{1}{y_0}\tan^{-1}\left(\frac{x+x_0}{y_0}\right) - \frac{1}{y_0}\tan^{-1}\left(\frac{x-x_0}{y_0}\right)\right]_{x=0}^{x=\infty}$$

$$= \frac{\lambda}{\pi}\left[\tan^{-1}\left(\frac{x+x_0}{y_0}\right) - \tan^{-1}\left(\frac{x-x_0}{y_0}\right)\right]_0^\infty$$

$$= -\frac{2\lambda}{\pi}\tan^{-1}\left(\frac{x_0}{y_0}\right).$$

The total charge Q_y (per unit length in z) on the plane $x=0$ $(y \geq 0)$ is obtained by ~~interchanging~~ interchanging the roles of x and y in Q_x (because of the symmetry of the roles of x and y in the problem). Thus,

$$Q_y = -\frac{2\lambda}{\pi} \tan^{-1}\left(\frac{y_0}{x_0}\right).$$

Hence the combined total charge (per unit length in z) is:

$$\begin{aligned} Q_{tot} &= Q_x + Q_y \\ &= -\frac{2\lambda}{\pi}\left[\tan^{-1}\left(\frac{x_0}{y_0}\right) + \tan^{-1}\left(\frac{y_0}{x_0}\right)\right] \\ &= -\frac{2\lambda}{\pi}\left[\frac{\pi}{2}\right] \\ &= -\lambda, \end{aligned}$$

as expected from the sum of the three image charges!

d) $$\phi(x,y) = \frac{\lambda}{4\pi\varepsilon_0} \ln \frac{\left[1 - \frac{2xx_0}{\rho^2} + \frac{2yy_0}{\rho^2} + \frac{\rho_0^2}{\rho^2}\right]\left[1 + \frac{2xx_0}{\rho^2} - \frac{2yy_0}{\rho^2} + \frac{\rho_0^2}{\rho^2}\right]}{\left[1 - \frac{2xx_0}{\rho^2} - \frac{2yy_0}{\rho^2} + \frac{\rho_0^2}{\rho^2}\right]\left[1 + \frac{2xx_0}{\rho^2} + \frac{2yy_0}{\rho^2} + \frac{\rho_0^2}{\rho^2}\right]}$$

$$\approx \frac{\lambda}{4\pi\varepsilon_0} \ln \frac{1 + 2\frac{\rho_0^2}{\rho^2} - \frac{4x^2x_0^2}{\rho^4} - \frac{4y^2y_0^2}{\rho^4} + \frac{8xyx_0y_0}{\rho^4} + \frac{\rho_0^4}{\rho^4}}{1 + 2\rho_0^2/\rho^2 - 4x^2x_0^2/\rho^4 - 4y^2y_0^2/\rho^4 - 8xyx_0y_0/\rho^4 + \frac{\rho_0^4}{\rho^4}}$$

$$\approx \frac{\lambda}{4\pi\varepsilon_0} \ln\left(1 + 16xyx_0y_0/\rho^4\right) \approx \frac{4\lambda xyx_0y_0}{\pi\varepsilon_0\rho^4}$$

This is basically a two-dimensional quadrupole potential, which is not ~~too~~ surprising as the space is divided into four quadrants with alternating positive and negative (image) charges. Using polar coordinates,

$$\phi(x, y) \simeq \frac{2\lambda x_0 y_0}{\pi \varepsilon_0} \frac{\sin 2\varphi}{\rho^2} .$$

Note that $4\lambda x_0 y_0$ is the quadrupole moment (charge times area) and that the $\sin 2\varphi$ angular behavior is obviously quadrupole (angular momentum $l = 2$).

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2.7 a) The Green's function can be obtained using the method of images, just as we did in Section 2.6:

$$G_D(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{1}{|\vec{x}-P\vec{x}'|}, \text{ where}$$

$P: (x,y,z) \longrightarrow (x,y,-z)$ is the mirror reflection operator through the plane $z=0$. Thus,

$$G_D(\vec{x},\vec{x}') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}$$

$$= \frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+(z-z')^2}} - \frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+(z+z')^2}}$$

b) Using Eqn. (1.44) with $\rho=0$:

$$\phi(\vec{x}) = -\frac{1}{4\pi}\oint_S \phi(\vec{x}')\,\frac{\partial G_D(\vec{x},\vec{x}')}{\partial n'}\,da'$$

$$= -\frac{1}{4\pi}\int_{z'=0} \phi(\vec{x}')\,\frac{\partial G_D}{\partial n'}\,da'$$

[the boundary at infinity has been ignored since $\phi=0$ at infinity!]

To compute $\frac{\partial G_D}{\partial n'}$, we note that $\hat{n}'$ is the unit normal pointing outward from the volume of interest ($z>0$). Thus, $\hat{n}' = -\hat{z}$; and hence

$$\frac{\partial G_D}{\partial n'}\bigg|_{z'=0} = -\frac{\partial G_D}{\partial z'}\bigg|_{z'=0}$$

$$= \left[\frac{z'-z}{[\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+(z-z')^2]^{3/2}} - \frac{z+z'}{[\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+(z+z')^2]^{3/2}}\right]_{z'=0}$$

$$= \frac{-2z}{[\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+z^2]^{3/2}}$$

Therefore,

$$\phi(\vec{x}) = \frac{z}{2\pi}\int_0^{2\pi}\int_0^{\infty}\phi(\rho',\varphi',0)\,\frac{1}{[\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+z^2]^{3/2}}\,\rho'\,d\rho'\,d\varphi'$$

$$= \frac{Vz}{2\pi}\int_0^{2\pi}\left[\int_0^{a}\frac{\rho'\,d\rho'}{[\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+z^2]^{3/2}}\right]d\varphi'$$

Since the boundary conditions are azimuthally symmetric, we expect $\phi(\vec{x})$ to be independent of φ and hence, for simplicity, we may set $\varphi = 0$. Thus,

$$\phi(\rho,z) = \frac{Vz}{2\pi}\int_0^{2\pi}\left[\int_0^a \frac{\rho' d\rho'}{[\rho^2+\rho'^2-2\rho\rho'\cos\varphi'+z^2]^{3/2}}\right]d\varphi' \quad (*)$$

c) To get ϕ on the z-axis, we set $\rho = 0$ in $(*)$; the integral is then easy to evaluate:

$$\phi(0,z) = \frac{Vz}{2\pi}\int_0^{2\pi}\left[\int_0^a \frac{\rho' d\rho'}{[\rho'^2+z^2]^{3/2}}\right]d\varphi'$$

$$= Vz\int_0^a \frac{\rho' d\rho'}{[\rho'^2+z^2]^{3/2}}$$

$$= Vz\left(\frac{1}{2}\right)(-2)\left[\rho'^2+z^2\right]^{-1/2}\Bigg|_{\rho'=0}^{\rho'=a}$$

$$= Vz\left[\frac{1}{|z|} - \frac{1}{\sqrt{z^2+a^2}}\right]$$

$$= V\left[1 - \frac{z}{\sqrt{z^2+a^2}}\right] \text{ since } z>0. \quad (**)$$

d) We may pull out the factor $r^2 = \rho^2+z^2$ in $(*)$ and expand $(r^2 \gg a^2 \geq \rho'^2)$:

$$\phi(\rho,z) = \frac{Vz}{2\pi r^3}\int_0^{2\pi}\left[\int_0^a \rho' d\rho'\left[1 - \frac{2\rho\rho'\cos\varphi'}{r^2} + \frac{\rho'^2}{r^2}\right]^{-3/2}\right]d\varphi'$$

$$= \frac{Vz}{2\pi r^3}\int_0^{2\pi}\left[\int_0^a \rho' d\rho'\left[1 - \tfrac{3}{2}r^{-2}(\rho'^2 - 2\rho\rho'\cos\varphi') + \tfrac{15}{8}r^{-4}(\rho'^2 - 2\rho\rho'\cos\varphi')^2 - + \cdots\right]\right]d\varphi'$$

Note that from

$$\int_0^{2\pi} \cos^n\varphi'\, d\varphi' = \frac{\cos^{n-1}\varphi' \sin\varphi'}{n}\Bigg|_0^{2\pi} + \frac{n-1}{n}\int_0^{2\pi}\cos^{n-2}\varphi'\, d\varphi'$$

$$= \frac{n-1}{n}\int_0^{2\pi}\cos^{n-2}\varphi'\, d\varphi' \quad \text{for } n \geq 2,$$

only even power of $\cos\varphi'$ will contribute to the integral expression for $\Phi(\rho, z)$ above. Thus,

$$\Phi(\rho,z) = \frac{Vz}{2\pi r^3}(2\pi)\int_0^a \rho'\, d\rho' \left[1 - \frac{3}{2} r^{-2}\rho'^2 + \frac{15}{8} r^{-4}(\rho'^4 + 2\rho^2\rho'^2) + \cdots\right]$$

$$= \frac{Vz}{\cancel{2\pi} r^3}\left[\frac{1}{2} a^2 - \frac{3}{2} r^{-2}\frac{1}{4} a^4 + \frac{15}{8} r^{-4}\left(\frac{1}{6} a^6 + 2\rho^2\left(\frac{1}{4} a^4\right)\right) + \cdots\right]$$

$$= \frac{Vz}{r^3}\left[\frac{1}{2} a^2 - \frac{3}{8}\frac{a^4}{r^2} + \frac{15}{8r^4}\left(\frac{1}{6} a^6 + \frac{1}{2}\rho^2 a^4\right) + \cdots\right]$$

$$= \frac{Vza^2}{2r^3}\left[1 - \frac{3}{4}\frac{a^2}{r^2} + \frac{5}{8r^4}(a^4 + 3\rho^2 a^2) + \cdots\right]$$

$$= \frac{Vza^2}{2[\rho^2+z^2]^{3/2}}\left[1 - \frac{3}{4}\frac{a^2}{\rho^2+z^2} + \frac{5}{8}\frac{a^2(a^2+3\rho^2)}{(\rho^2+z^2)^2} + \cdots\right]$$

Setting $\rho = 0$ in the series above gives:

$$\Phi(0,z) = \frac{Va^2}{2z^2}\left[1 - \frac{3}{4}\frac{a^2}{z^2} + \frac{5}{8}\frac{a^4}{z^4} + \cdots\right] \quad (***)$$

On the other hand, the exact potential on the z-axis (**) may be expanded for $z^2 \gg a^2$: ($z \gg a$)

$$\phi(0,z) = V \cdot \left[1 - \frac{z}{\sqrt{z^2+a^2}}\right]$$

$$= V\left[1 - \left(1 + \frac{a^2}{z^2}\right)^{-1/2}\right]$$

$$= V\left[1 - \left(1 - \frac{1}{2}\frac{a^2}{z^2} + \frac{3}{8}\frac{a^4}{z^4} - \frac{5}{16}\frac{a^6}{z^6} \cdots\right)\right]$$

$$= V\left[\frac{1}{2}\frac{a^2}{z^2} - \frac{3}{8}\frac{a^4}{z^4} + \frac{5}{16}\frac{a^6}{z^6} \cdots\right]$$

$$= \frac{Va^2}{2z^2}\left[1 - \frac{3}{4}\frac{a^2}{z^2} + \frac{5}{8}\frac{a^4}{z^4} \cdots\right]$$

which agrees with (***).

2.22 a) The Green's function for the interior conducting sphere problem is equivalent to that for the exterior problem.

The only difference we need to make is that, for the interior problem, the outward unit normal is indeed pointing away from the center of the sphere. Thus,

$$\phi(r,\theta,\varphi) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \phi(a,\theta',\varphi')\,\frac{a(a^2-r^2)}{(r^2+a^2-2ar\cos\gamma)^{3/2}}\,\sin\theta'\,d\theta'\,d\varphi'$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$

For the oppositely charged hemisphere problem, the integral takes the form:

$$\phi(r,\theta,\varphi) = \frac{Va(a^2-r^2)}{4\pi} \int_0^{2\pi} \left\{ \int_0^1 \left[(r^2+a^2-2ar\cos\gamma)^{-3/2} - (r^2+a^2+2ar\cos\gamma)^{-3/2} \right] d(\cos\theta') \right\} d\varphi' \quad (*)$$

This simplifies on the positive z-axis $(0<z<a)$, where

$\theta = 0 \rightarrow \cos\gamma = \cos\theta'$ and $r = z$. We find

$$\phi(z) = \frac{Va(a^2-z^2)}{4\pi} \int_0^{2\pi} \left\{ \int_0^1 \left[(z^2+a^2-2az\cos\theta')^{-3/2} - (z^2+a^2+2az\cos\theta')^{-3/2} \right] d(\cos\theta') \right\} d\varphi'$$

$$= \frac{V(a^2-z^2)}{2z} \left[(z^2+a^2-2az\cos\theta')^{-1/2} + (z^2+a^2+2az\cos\theta') \right]_{\cos\theta'=0}^{\cos\theta'=1}$$

$$= \frac{V(a^2-z^2)}{2z} \left[\frac{1}{\sqrt{(z-a)^2}} + \frac{1}{\sqrt{(z+a)^2}} - \frac{2}{\sqrt{z^2+a^2}} \right]$$

$$= \frac{V(a^2-z^2)}{2z} \left[\frac{1}{a-z} + \frac{1}{a+z} - \frac{2}{\sqrt{z^2+a^2}} \right]$$

(because $\sqrt{(z\pm a)^2} = |z \pm a| = a \pm z$)

$$= \frac{V}{2z} \left[a+z + a-z - \frac{2(a^2-z^2)}{\sqrt{z^2+a^2}} \right]$$

$$\phi(z) = \frac{V}{2z}\left[2a - \frac{2(a^2-z^2)}{\sqrt{z^2+a^2}}\right]$$

$$= \frac{Va}{z}\left[1 - \frac{a^2-z^2}{a\sqrt{z^2+a^2}}\right]$$

On the negative z-axis: $\theta = \pi \rightarrow \cos\gamma = -\cos\theta'$ and $r = -z$.

Substituting into $(*)$, we get the same expression for $\phi(z)$ as on the positive z-axis. Thus, for $|z| < a$,

$$\phi_{in}(z) = \frac{Va}{z}\left[1 - \frac{a^2-z^2}{a\sqrt{a^2+z^2}}\right]$$

This can be expanded in powers of $\frac{z}{a}$ as follows:

$$\phi_{in}(z) = V\frac{a}{z}\left[1 - (1 - z^2/a^2)(1 + z^2/a^2)^{-1/2}\right]$$

$$= V\frac{a}{z}\left[1 - (1 - z^2/a^2)\left(1 - \tfrac{1}{2}z^2/a^2 + \tfrac{3}{8}z^4/a^4 - \tfrac{5}{16}z^6/a^6 \cdots\right)\right]$$

$$= V\frac{a}{z}\left[1 - \left(1 - \frac{3}{2}\frac{z^2}{a^2} + \frac{7}{8}\frac{z^4}{a^4} - \frac{11}{16}\frac{z^6}{a^6} \cdots\right)\right]$$

$$= V\frac{a}{z}\left[\frac{3}{2}\frac{z^2}{a^2} - \frac{7}{8}\frac{z^4}{a^4} + \frac{11}{16}\frac{z^6}{a^6} \cdots\right]$$

$$= \frac{3V}{2}\left[\frac{z}{a} - \frac{7}{12}\left(\frac{z}{a}\right)^3 + \frac{11}{24}\left(\frac{z}{a}\right)^5 \cdots\right]$$

from which we infer that

$$\phi_{in}(r,\theta) = \frac{3V}{2}\left[\frac{r}{a}P_1(\cos\theta) - \frac{7}{12}\left(\frac{r}{a}\right)^3 P_3(\cos\theta) + \frac{11}{24}\left(\frac{r}{a}\right)^5 P_5(\cos\theta) - + \cdots\right]$$

$$= V\left[\frac{3}{2}\left(\frac{r}{a}\right)P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{a}\right)^3 P_3(\cos\theta) + \frac{11}{16}\left(\frac{r}{a}\right)^5 P_5(\cos\theta) - + \cdots\right]$$

while, from (2.22), we have that

$$\phi_{out}(r,\theta) = \frac{3V}{2}\left[\left(\frac{a}{r}\right)^2 P_1(\cos\theta) - \frac{7}{12}\left(\frac{a}{r}\right)^4 P_3(\cos\theta) + \frac{11}{24}\left(\frac{a}{r}\right)^6 P_5(\cos\theta) - + \cdots\right]$$

$$= V\left[\frac{3}{2}\left(\frac{a}{r}\right)^2 P_1(\cos\theta) - \frac{7}{8}\left(\frac{a}{r}\right)^4 P_3(\cos\theta) + \frac{11}{16}\left(\frac{a}{r}\right)^6 P_5(\cos\theta) - + \cdots\right].$$

Thus, $\phi_{in}(r,\theta)$ and $\phi_{out}(r,\theta)$ are identical up to the substitution:

$$\underset{\text{in}}{\left(\frac{r}{a}\right)^{\ell}} \longleftrightarrow \underset{\text{out}}{\left(\frac{a}{r}\right)^{\ell+1}}$$

b) On the positive z-axis, $r = z$ and hence

$$E_r(z) = -\frac{\partial\phi(r)}{\partial r}\bigg|_{r=z} = -\frac{\partial\phi(z)}{\partial z}.$$

For $-a<z<0$: $r=-z$ and $\phi(r) = -\phi(z)$; thus,

$$E_r(z) = -\frac{\partial\phi(r)}{\partial r}\bigg|_{r=-z} = -\left[\frac{-\partial\phi(z)}{-\partial z}\right] = -\frac{\partial\phi(z)}{\partial z}.$$

~~Alt~~ Altogether, $E_r(z) = -\frac{\partial\phi(z)}{\partial z}$ for $0<|z|<a$ and for $z>a$

Using $\phi(z) = \begin{cases} V\left[\dfrac{a}{z} - \dfrac{a^2-z^2}{z\sqrt{z^2+a^2}}\right] & \text{for } 0 < |z| \leq a \\ V\left[1 - \dfrac{z^2-a^2}{z\sqrt{z^2+a^2}}\right] & \text{for } z \geq a \end{cases}$, we get

$$E_z(z) = \begin{cases} -V\left[-\dfrac{a}{z^2} + \dfrac{a^2(a^2+3z^2)}{z^2(z^2+a^2)^{3/2}}\right] & \text{for } 0<|z|<a \\ -V\left[0 - \dfrac{a^2(a^2+3z^2)}{z^2(z^2+a^2)^{3/2}}\right] & \text{for } z>a \end{cases}$$

$$= \begin{cases} -\dfrac{V}{a}\left[\dfrac{3+(a/z)^2}{(1+(z/a)^2)^{3/2}} - \dfrac{a^2}{z^2}\right] & \text{for } 0<|z|<a \\ \dfrac{V}{a}\,\dfrac{3+(a/z)^2}{[1+(z/a)^2]^{3/2}} & \text{for } z>a \end{cases} \qquad (**)$$

which is the desired result.

As $z \to 0$, we may expand $E_z(z)$ in powers of (z/a):

$$E_z(z) = -\frac{V}{a}\left[\left(3+\left(\frac{a}{z}\right)^2\right)\left(1+\left(\frac{z}{a}\right)^2\right)^{-3/2} - \frac{a^2}{z^2}\right]$$

$$= -\frac{V}{a}\left[\left(3+\frac{a^2}{z^2}\right)\left(1-\frac{3}{2}\frac{z^2}{a^2}+\frac{15}{8}\frac{z^4}{a^4}-\frac{35}{16}\frac{z^6}{a^6}+\cdots\right) - \frac{a^2}{z^2}\right]$$

$$= -\frac{V}{a}\left[\frac{3}{2} - \frac{21}{8}\left(\frac{z}{a}\right)^2 + \frac{55}{16}\left(\frac{z}{a}\right)^4 - + \cdots\right]$$

$$= -\frac{3}{2}\frac{V}{a}\left[1 - \frac{7}{4}\left(\frac{z}{a}\right)^2 + \frac{55}{24}\left(\frac{z}{a}\right)^4 - + \cdots\right]$$

So $\lim_{z \to 0} E_z(z) = -\frac{3}{2}\frac{V}{a}$.

Finally $E_z(a_\pm)$ can be obtained from (**):

$$E_z(a_-) = \lim_{z \to a_-} E_z(z)$$

$$= \lim_{z \to a_-} -\frac{V}{a}\left[\frac{3 + (a/z)^2}{\left(1 + (z/a)^2\right)^{3/2}} - \frac{a^2}{z^2}\right]$$

$$= -\frac{V}{a}\left[\frac{3+1}{(1+1)^{3/2}} - 1\right]$$

$$= -\frac{V}{a}\left[\sqrt{2} - 1\right]$$

$$E_z(a_+) = \lim_{z \to a_+} \frac{V}{a} \frac{3 + (a/z)^2}{\left[1 + (z/a)^2\right]^{3/2}}$$

$$= \frac{V}{a} \frac{3+1}{(1+1)^{3/2}}$$

$$= \frac{V}{a}\sqrt{2}.$$

2.23 a) The potential may be obtained by superposition:

$$\phi_{in}(x,y,z) = \phi_{top}(x,y,z) + \phi_{bottom}(x,y,z)$$

where ϕ_{top} (ϕ_{bottom}) is the solution for a hollow cube with the top (bottom) held at constant potential V and all other sides at zero potential.

$\phi_{top}(x,y,z)$ is obtained, using Eqns (2.56) and (2.58) with $b=c=a$ and $V(x,y) = V$ substituted into those two equations. Thus,

$$\phi_{top}(x,y,z) = \sum_{n,m} A_{nm} \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{a}y\right) \sinh\left(\frac{\sqrt{n^2+m^2}\,\pi}{a}z\right)$$

where $$A_{nm} = \frac{4V}{a^2 \sinh\left(\sqrt{n^2+m^2}\,\pi\right)} \int_0^a \int_0^a \sin\left(\frac{n\pi}{a}\right)x \sin\left(\frac{m\pi}{a}y\right) dx\, dy$$

$$= \frac{4V}{a^2 \sinh\left(\sqrt{n^2+m^2}\,\pi\right)} \left(\int_0^a \sin\left(\frac{n\pi}{a}x\right) dx\right) \left(\int_0^a \sin\left(\frac{m\pi}{a}y\right) dy\right)$$

$$= \begin{cases} \dfrac{4V}{a^2 \sinh\left(\sqrt{n^2+m^2}\,\pi\right)} \left(\dfrac{2a}{n\pi}\right)\left(\dfrac{2a}{m\pi}\right) & \text{if } n \text{ and } m \text{ are odd} \\ 0 & \text{otherwise.} \end{cases}$$

$$A_{nm} = \begin{cases} \dfrac{16V}{\pi^2 mn \sinh(\sqrt{n^2+m^2}\,\pi)} & \text{if } n \text{ and } m \text{ are odd} \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\phi_{top}(x,y,z) = \frac{16V}{\pi^2} \sum_{n,m \text{ odd}} \left\{ \frac{1}{n\, m \sinh(\sqrt{n^2+m^2}\,\pi)} \cdot \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{a}y\right) \sinh\left(\frac{\sqrt{n^2+m^2}\,\pi}{a} z\right) \right\}$$

Since $\phi_{bottom}(x,y,z) - \phi_{top}(x,y,a-z)$ ~~sati~~ vanishes on <u>all</u> six sides of the cube and it is a solution of Laplace's equation inside the cube, it follows that $\phi_{bottom}(x,y,z) - \phi_{top}(x,y,a-z) = 0$ <u>everywhere</u> inside the cube. Thus,

$\phi_{bottom}(x,y,z) = \phi_{top}(x,y,a-z)$; and hence

$$\phi_{in}(x,y,z) = \phi_{top}(x,y,z) + \phi_{bottom}(x,y,z)$$

$$= \phi_{top}(x,y,z) + \phi_{top}(x,y,a-z)$$

$$= \frac{16V}{\pi^2} \sum_{n,m\,odd} \left\{ \frac{1}{nm \sinh\left(\sqrt{n^2+m^2}\,\pi\right)} \cdot \sin\left(\frac{n\pi}{a}x\right) \cdot \sin\left(\frac{m\pi}{a}y\right) \cdot \right.$$

$$\left. \left[\sinh\left(\frac{\sqrt{n^2+m^2}\,\pi}{a} z\right) + \sinh\left(\frac{\sqrt{n^2+m^2}\,\pi}{a}(a-z)\right) \right] \right\}$$

But $\sinh\left(\frac{\sqrt{n^2+m^2}\,\pi}{a} z\right) + \sinh\left(\frac{\sqrt{n^2+m^2}\,\pi}{a}(a-z)\right)$

$$= 2 \sinh\left(\sqrt{n^2+m^2}\,\pi/2\right) \cosh\left(\frac{\sqrt{n^2+m^2}\,\pi}{a}\left(z-\frac{a}{2}\right)\right)$$

and

$$\sinh\left(\sqrt{n^2+m^2}\,\pi\right) = 2 \sinh\left(\sqrt{n^2+m^2}\,\pi/2\right) \cosh\left(\sqrt{n^2+m^2}\,\pi/2\right).$$

Thus,

$$\phi(x,y,z) = \frac{16V}{\pi^2} \sum_{n,m\,odd} \left\{ \frac{1}{nm \cosh\left(\sqrt{n^2+m^2}\,\pi/2\right)} \cdot \sin\left(\frac{n\pi}{a}\right)x \cdot \right.$$

$$(*) \qquad \left. \sin\left(\frac{m\pi}{a}\right)y \cdot \cosh\left(\frac{\sqrt{n^2+m^2}\,\pi}{a}(z-a/2)\right) \right\}$$

c) $\sigma = -\varepsilon_0 \left.\frac{\partial \phi}{\partial n}\right|_{z=a} = \varepsilon_0 \left.\frac{\partial \phi}{\partial z}\right|_{z=a}$ (the normal unit vector pointing away from the top conductor is $\hat{n} = -\hat{z}$.)

Using $(*)$, we get:

$$\sigma_{(x,y)} = \frac{16\varepsilon_0 V}{a\pi} \sum_{n,m\,odd} \left\{ \frac{\sqrt{n^2+m^2}}{nm} \tanh\left(\sqrt{n^2+m^2}\,\pi/2\right) \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{a}y\right) \right\}$$