PHYS 7590 Homework Assignment # 1:

(1.4) Because of spherical symmetry, Gauss' law $\oint_S \vec{E}.\hat{n}\, da = \frac{q_{enc}}{\varepsilon_0}$, with S a sphere of radius r concentric with any one of the three spheres in the problem, yields:

$$E\, 4\pi r^2 = \frac{q_{enc}}{\varepsilon_0}. \text{ Thus,}$$

$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q_{enc}}{r^2}\hat{r}$; and this is valid for all three spheres and for all r.

i) For the conducting sphere, the charge Q is entirely on the surface of the sphere. Thus,

$$\vec{E}(r) = \begin{cases} 0 & \text{if } r < a \\ \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r} & \text{if } r \geq a \end{cases}$$

ii) For the sphere with uniform charge density:

$\rho = \frac{Q}{4\pi a^3/3}$ (a is the radius of the sphere). Thus,

$$q_{enc} = \begin{cases} Q\left(\frac{r}{a}\right)^3 & \text{if } r < a \\ Q & \text{if } r \geq a \end{cases}$$

Hence $\vec{E}(r) = \begin{cases} \frac{1}{4\pi\varepsilon_0} \frac{Q}{a^3} \vec{r} & \text{if } r < a \\ \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{r} & \text{if } r \geq a \end{cases}$

iii) For the the sphere with charge density $\rho \sim r^n$ $(n > -3)$, q_{enc} for $r < a$ can be obtained by integration:

$$q_{enc} = \int_0^r \rho \, 4\pi r'^2 \, dr'$$

$$= 4\pi\alpha \int_0^r r'^{n+2} \, dr'$$, where we have written $\rho = \alpha r^n$ (α constant)

$$q_{enc} = 4\pi\alpha \frac{r^{n+3}}{n+3} .$$

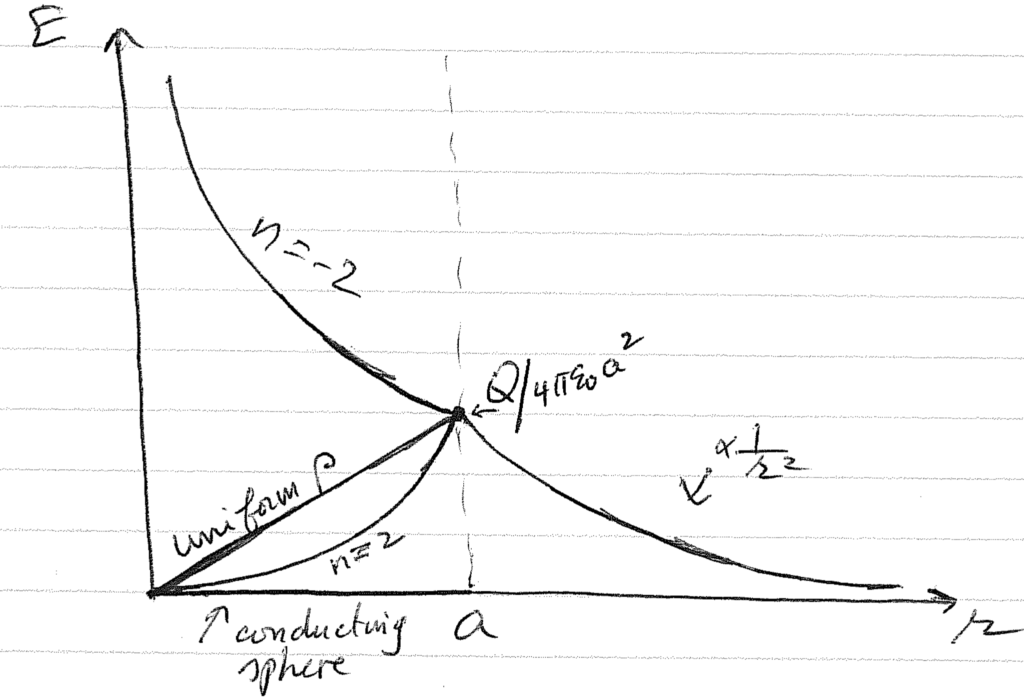
$$= 4\pi\alpha \frac{a^{n+3}}{n+3} \left(\frac{r}{a}\right)^{n+3}$$

$$= Q \left(\frac{r}{a}\right)^{n+3} .$$ Thus,

$$\vec{E} = \begin{cases} \frac{1}{4\pi\varepsilon_0} \frac{Q}{a^3} \left(\frac{r}{a}\right)^n \vec{r} & \text{if } r < a \\ \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{r} & \text{if } r \geq a \end{cases}$$

Clearly case iii) reduces to case ii) for $n = 0$, as expected!

Also, as expected from Gauss' law, the electric field outside the sphere is the same in all three cases.



(1.5) We can obtain the charge distribution ρ, using Poisson's Equation: $\nabla^2\phi = -\rho/\varepsilon_0$. But because $\lim_{r\to 0}\phi(r) = \infty$, we proceed with some caution:

- For $r>0$: $\rho = -\varepsilon_0 \nabla^2\phi = -\varepsilon_0 \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right)$

$$= \frac{-q}{4\pi r^2}\frac{d}{dr}\left[r^2 \frac{d}{dr}\left(e^{-\alpha r}\left(\frac{1}{r}+\frac{\alpha}{2}\right)\right)\right]$$

$$\rho(r) = -\frac{q}{4\pi r^2}\frac{d}{dr}\left[r^2 e^{-\alpha r}\left(-\frac{1}{r^2}-\frac{\alpha}{r}-\frac{\alpha^2}{2}\right)\right]$$

$$= \frac{q}{4\pi r^2}\frac{d}{dr}\left[e^{-\alpha r}\left(1+\alpha r+\frac{1}{2}\alpha^2 r^2\right)\right]$$

$$= \frac{q}{4\pi r^2} e^{-\alpha r}\left[\alpha+\alpha^2 r-\alpha-\alpha^2 r-\frac{1}{2}\alpha^3 r^2\right]$$

$$= -\frac{q\alpha^3}{8\pi} e^{-\alpha r}$$

- For $r \simeq 0$, on the other hand, we can expand ϕ:

$$\phi(r) = \frac{q}{4\pi\varepsilon_0 r}\left[1-\alpha r+\frac{1}{2}\alpha^2 r^2 - + \cdots\right]\left[1+\frac{\alpha r}{2}\right]$$

$$\simeq \frac{q}{4\pi\varepsilon_0 r}$$ which is the potential of a point charge q at the origin. Thus the charge distribution has a continuous component $\rho_1(\vec{r}) = -\frac{q\alpha^3}{8\pi} e^{-\alpha r} = \rho_1(r)$ and a discrete one $\rho_2(\vec{r}) = q\,\delta(\vec{r})$:

$$\rho(\vec{r}) = -\frac{q\alpha^3}{8\pi} e^{-\alpha r} + q\,\delta(\vec{r}).$$

The first term corresponds to the electron cloud around the proton (of charge q at the origin); and the second term corresponds to the charge of the proton itself.

One can easily check that the total charge is zero, as one would expect for a neutral hydrogen atom:

$$Q = \int_{\text{all space}} \rho(\vec{r})\, d^3x$$

$$= \int_{\text{all space}} \rho_1(\vec{r})\, d^3x + q \int_{\text{all space}} \delta(\vec{r})\, d^3x$$

$$= -\frac{q\alpha^3}{8\pi} \int_0^\infty e^{-\alpha r}\, 4\pi r^2\, dr + q$$

$$= -\frac{q}{2} \int_0^\infty e^{-\alpha r} (\alpha r)^2\, d(\alpha r) + q$$

$$= -\frac{q}{2} \int_0^\infty e^{-t}\, t^2\, dt + q$$

$$= -\frac{q}{2}(2!) + q = 0.$$

1.10) Let S be a sphere of radius a centered at the point $\vec{x}$. We want to show

$$\phi(\vec{x}) = \langle \phi \rangle_S = \frac{1}{4\pi a^2} \oint_S \phi(\vec{x}')\, da'$$

We can use Eqn (1.36) (with $\rho(\vec{x}') = 0$) or Eqn 1.44 (with $\rho(\vec{x}') = 0$ and $G_D(\vec{x}, \vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{1}{a}$)

Using (1.44), we have that

$$\phi(\vec{x}) = -\frac{1}{4\pi} \oint_S \phi(\vec{x}') \frac{\partial G_D(\vec{x},\vec{x}')}{\partial n'} da', \text{ where}$$

$$G_D(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{1}{a}$$ is the appropriate Green's function for this problem since

$$\nabla'^2 G_D(\vec{x},\vec{x}') = \nabla'^2 \left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\,\delta(\vec{x}-\vec{x}')$$

and $G_D(\vec{x},\vec{x}') = \frac{1}{a} - \frac{1}{a} = 0$ for $\vec{x}'$ on S.

$$\frac{\partial G_D}{\partial n'} = \vec{\nabla}' G_D \cdot \hat{n}'$$

$$= -\frac{(\vec{x}'-\vec{x})}{|\vec{x}-\vec{x}'|^3} \cdot \frac{(\vec{x}'-\vec{x})}{|\vec{x}'-\vec{x}|}$$

$$= -\frac{1}{|\vec{x}-\vec{x}'|^2}.$$ Hence

$$\left.\frac{\partial G_D}{\partial n'}\right|_{\vec{x}' \text{ on } S} = -\frac{1}{a^2}.$$ Thus,

$$\phi(\vec{x}) = \frac{1}{4\pi a^2} \oint_S \phi(\vec{x}')\, da' = \langle \phi \rangle_S .$$

1.12 using Green's Theorem applied to the potentials ϕ and ϕ', we get:

$$\int_V \left(\phi \nabla^2\phi' - \phi'\nabla^2\phi\right) d^3x = \oint_S \left(\phi \frac{\partial\phi'}{\partial n} - \phi' \frac{\partial\phi}{\partial n}\right) da \quad (*)$$

where $\nabla^2\phi' = -\rho'/\varepsilon_0$

$$\nabla^2\phi = -\rho/\varepsilon_0$$

$$\frac{\partial\phi'}{\partial n} = \vec{\nabla}\phi'\cdot\hat{n} = -\vec{E}'\cdot\hat{n} = -E_n'$$

and $\dfrac{\partial\phi}{\partial n} = -E_n$

where E_n (E_n') is the normal component of the electric field at the conducting surface. Since $\hat{n}$ is an outward pointing normal unit vector and since $\vec{E}$ ($\vec{E}'$) is the electric field on the interior of the surface, application of Gauss' law at the conducting surface yields: $E_n = -\dfrac{\sigma}{\varepsilon_0}$ and $E_n' = -\dfrac{\sigma'}{\varepsilon_0}$. Thus,

$$\frac{\partial\phi'}{\partial n'} = \frac{\sigma'}{\varepsilon_0} \quad \text{and} \quad \frac{\partial\phi}{\partial n} = \frac{\sigma}{\varepsilon_0}.$$ Now (*) becomes:

$$\frac{1}{\varepsilon_0}\int_V \left[\rho\phi' - \rho'\phi\right] d^3x = \frac{1}{\varepsilon_0}\oint_S \left[\sigma'\phi - \sigma\phi'\right] da$$

from which, we readily get:

$$\int_V \rho\phi' d^3x + \oint_S \sigma\phi' da = \int_V \rho'\phi d^3x + \oint_S \sigma'\phi da.$$

1.14) Applying Green's Theorem to $G(\vec{x},\vec{y})$ and $G(\vec{x}',\vec{y})$, we get:

$$\int_V \left[G(\vec{x},\vec{y})\nabla_y^2 G(\vec{x}',\vec{y}) - G(\vec{x}',\vec{y})\nabla_y^2 G(\vec{x},\vec{y})\right] d^3y$$

$$= \oint_S \left[G(\vec{x},\vec{y})\frac{\partial G(\vec{x}',\vec{y})}{\partial n_y} - G(\vec{x}',\vec{y})\frac{\partial G(\vec{x},\vec{y})}{\partial n_y}\right] da_y .$$

Thus, $\int_V \left[G(\vec{x},\vec{y})(-4\pi\delta(\vec{x}'-\vec{y})) - G(\vec{x}',\vec{y})(-4\pi\delta(\vec{x}-\vec{y}))\right] d^3y$

$$= \oint_S \left[G(\vec{x},\vec{y})\frac{\partial G(\vec{x}',\vec{y})}{\partial n_y} - G(\vec{x}',\vec{y})\frac{\partial G(\vec{x},\vec{y})}{\partial n_y}\right] da_y ,$$

from which we get (dividing both sides by -4π):

$$G(\vec{x},\vec{x}') - G(\vec{x}',\vec{x})$$

$$= \frac{1}{4\pi}\oint_S \left[G(\vec{x}',\vec{y})\frac{\partial G(\vec{x},\vec{y})}{\partial n_y} - G(\vec{x},\vec{y})\frac{\partial G(\vec{x}',\vec{y})}{\partial n_y}\right] da_y \quad (*)$$

a) For the Dirichlet Green's function, $G_D(\vec{x},\vec{y}) = 0$ for $\vec{y}$ on S. This means that the right hand side of (*) is equal to zero; and hence $G_D(\vec{x},\vec{x}') = G_D(\vec{x}',\vec{x})$.

b) We use the Neumann boundary condition

$\frac{\partial G_N(\vec{x},\vec{y})}{\partial n_y} = -\frac{4\pi}{S}$ for $\vec{y}$ on S. This means the right hand side of (*) becomes

$$\frac{1}{S}\oint_S \left[G_N(\vec{x},\vec{y}) - G_N(\vec{x}',\vec{y}) \right] da_y = f(\vec{x}) - f(\vec{x}')$$

where $f(\vec{x}) = \frac{1}{S}\oint_S G_N(\vec{x},\vec{y})\, da_y$. Hence

$$G_N(\vec{x},\vec{x}') - G_N(\vec{x}',\vec{x}) = f(\vec{x}) - f(\vec{x}') \quad (**)$$

, which shows that $G_N(\vec{x},\vec{x}')$ is not symmetric in general. But if we define

$$G_N^{new}(\vec{x},\vec{x}') = G_N(\vec{x},\vec{x}') - f(\vec{x}),$$

then

$$G_N^{new}(\vec{x},\vec{x}') = G_N^{new}(\vec{x}',\vec{x}),$$

which shows that $G_N^{new}(\vec{x},\vec{x}')$ is symmetric in $\vec{x}$ and $\vec{x}'$.

c) What we need to do is to show that the Neumann Green's function solution

$$\phi(\vec{x}) = \langle\phi\rangle_S + \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')\, G_N(\vec{x},\vec{x}')\, d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\phi(\vec{x}')}{\partial n'} G_N(\vec{x},\vec{x}')\, da' \quad (***)$$

is unchanged when we replace G_N by G_N^{new}. If we let ϕ^{new} denote the computation using G_N^{new} then

$$\phi^{new}(\vec{x}) = \langle\phi\rangle_S + \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')\, G_N^{new}(\vec{x},\vec{x}')\, d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\phi(\vec{x}')}{\partial n'} G_N^{new}(\vec{x},\vec{x}')\, da'$$

$$= \langle \phi \rangle_S + \frac{1}{4\pi\varepsilon_0} \int_V \rho(\vec{x}') \, G_N(\vec{x}, \vec{x}') \, d^3x'$$

$$+ \frac{1}{4\pi} \oint_S \frac{\partial \phi(\vec{x}')}{\partial n'} G_N(\vec{x}, \vec{x}') \, da'$$

$$- \frac{1}{4\pi\varepsilon_0} \int_V \rho(\vec{x}') \, F(\vec{x}) d^3x' - \frac{1}{4\pi} \oint_S \frac{\partial \phi(\vec{x}')}{\partial n'} F(\vec{x}) \, da'$$

$$= \phi(\vec{x}) - \frac{F(\vec{x})}{4\pi\varepsilon_0} \left[\int_V \rho(\vec{x}') d^3x' + \varepsilon_0 \oint_S \frac{\partial \phi(\vec{x}')}{\partial n'} da' \right]$$

$$= \phi(\vec{x}) - \frac{F(\vec{x})}{4\pi\varepsilon_0} \left[q_{enc} - \varepsilon_0 \oint_S \vec{E}(\vec{x}') \cdot \hat{n}' \, da' \right],$$ where

we used the fact that $\vec{E} \cdot \hat{n} = -\vec{\nabla}\phi \cdot \hat{n} = -\frac{\partial \phi}{\partial n}$.

A simple application of Gauss' Law then shows that $\phi^{new} = \phi$. Hence we have shown that the addition of $F(\vec{x})$ to $G_N(\vec{x}, \vec{x}')$ leaves the solution $\phi(\vec{x})$ unchanged.

This shows that we can always make G_N symmetric by appropriate modification with $F(\vec{x})$.

Remark: From (**), we could have ~~defined~~ ~~defined~~ the symmetric $G_N^{new}(\vec{x}, \vec{x}') = G_N(\vec{x}, \vec{x}') + F(\vec{x}')$. However, this is not a good choice as substitution of this $G_N^{new}(\vec{x}, \vec{x}')$ into (***) will generate an incorrect solution for ϕ.