SPECIAL RELATIVITY

1. Coordinate Rotations in 3-D

Consider two rectangular coordinate systems with a common origin O, each system being defined by a set of unit vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ and $(\hat{e}_1^{\prime}, \hat{e}_2^{\prime}, \hat{e}_3^{\prime})$ which differ only by a rotation of the coordinate axes. When the axes are orthogonal:

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$
$$\hat{e}_k^{\prime} \cdot \hat{e}_\ell^{\prime} = \delta_{k\ell} \tag{1}$$

Each unit vector $\hat{e}_i^{\prime}$ can be expressed in terms of its components along $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$

$$\hat{e}_i^{\prime} = \sum_{j=1}^{3} a_{ij}\hat{e}_j \tag{2}$$

in which the coefficients a_{ij} are the cosines of the angles θ_{ij} between $\hat{e}_i^{\prime}$ and $\hat{e}_j$

$$a_{ij} = \cos(\theta_{ij}) = \hat{e}_i^{\prime} \cdot \hat{e}_j \tag{3}$$

The inverse relationship: $a_{ji}^{\prime} = \hat{e}_j \cdot \hat{e}_i^{\prime} = a_{ij}$

$$\hat{e}_j = \sum_{i=1}^{3} a_{ji}^{\prime}\hat{e}_i^{\prime} = \sum_{i=1}^{3} a_{ij}\hat{e}_i^{\prime} \tag{4}$$

Thus,

$$\hat{e}_i \cdot \hat{e}_j = \left(\sum_{k=1}^{3} a_{ki}\hat{e}_k^{\prime}\right) \cdot \left(\sum_{l=1}^{3} a_{\ell j}\hat{e}_\ell^{\prime}\right) = \sum_{k=1}^{3}\sum_{l=1}^{3} a_{ki}a_{\ell j}\delta_{k\ell} = \sum_{k=1}^{3} a_{ki}a_{kj}$$

Using (1), we get:

$$\boxed{\sum_{k=1}^{3} a_{ki}a_{kj} = \delta_{ij}} \quad \text{CONDITION OF ORTHOGONALITY} \tag{5}$$

This gives the conditions which the cosines of the angles between the coordinate axes must satisfy in order that the axes be rectangular.

The position vector $\vec{r}$ of any point P is then:

$$\vec{r} = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3 = x_1^{/}\hat{e}_1^{/} + x_2^{/}\hat{e}_2^{/} + x_3^{/}\hat{e}_3^{/}$$

with

$$x_j^{/} = \vec{r}\,.\,\hat{e}_j^{/} = \sum_{k=1}^{3} x_k\hat{e}_k\,.\,\hat{e}_j^{/} = \sum_{k=1}^{3}\sum_{l=1}^{3} x_k\hat{e}_k\,.\,(a_{j\ell}\hat{e}_\ell) \quad \text{(from (2))}$$

$$= \sum_{k=1}^{3}\sum_{l=1}^{3} x_k a_{j\ell}\delta_{k\ell} \quad \text{(using 1)}$$

$$= \sum_{k=1}^{3} a_{jk}x_k$$

so that the coordinates of a point transform under an orthogonal coordinate rotation as:

$$\boxed{x_j^{/} = \sum_{k=1}^{3} a_{jk}x_k} \quad \text{LINEAR ORTHOGONAL TRANSFORMATION} \quad (7)$$

The characteristic property of an orthogonal transformation is that it leaves the sum of the squares of the coordinate *invariant*

$$\sum_{j=1}^{3}(x_j^{/})^2 = \sum_{j=1}^{3}\left(\sum_{k=1}^{3} a_{jk}x_k\right)\left(\sum_{\ell=1}^{3} a_{j\ell}x_\ell\right) = \sum_{k=1}^{3}\sum_{\ell=1}^{3} x_k x_\ell\left(\sum_{j=1}^{3} a_{jk}a_{j\ell}\right)$$

$$= \sum_{k=1}^{3}\sum_{l=1}^{3} x_k x_\ell \delta_{k\ell}$$

$$= \sum_{k=1}^{3} x_k^2$$

Therefore

$$\sum_{j=1}^{3}(x_j^{/})^2 = \sum_{k=1}^{3}(x_k)^2 \quad (8)$$

Further, for any fixed vector $\vec{A}$ in space

$$\vec{A} = \sum_{k=1}^{3} A_k\hat{e}_k = \sum_{k=1}^{3} A_k^{/}\hat{e}_k^{/}$$

with

$$A_j' = \vec{A} \cdot \hat{e}_j' = \sum_{k=1}^{3} A_k \hat{e}_k \cdot \hat{e}_j' = \sum_{k=1}^{3} a_{jk} A_k \tag{9}$$

so that the rectangular components of a fixed vector transform like the coordinates of a point under a rotation. The *scalar product* of two vectors, $\vec{A} \cdot \vec{B}$ is also *invariant* under an orthogonal transformation:

$$\vec{A} \cdot \vec{B} = \sum_{i=1}^{3} A_i B_i = \sum_{i=1}^{3} \left(\sum_{j=1}^{3} a_{ij}' A_j'\right)\left(\sum_{k=1}^{3} a_{ik}' B_k'\right) = \sum_{j=1}^{3} \sum_{k=1}^{3} A_j' B_k' \sum_{i=1}^{3} a_{ij}' a_{ik}'$$

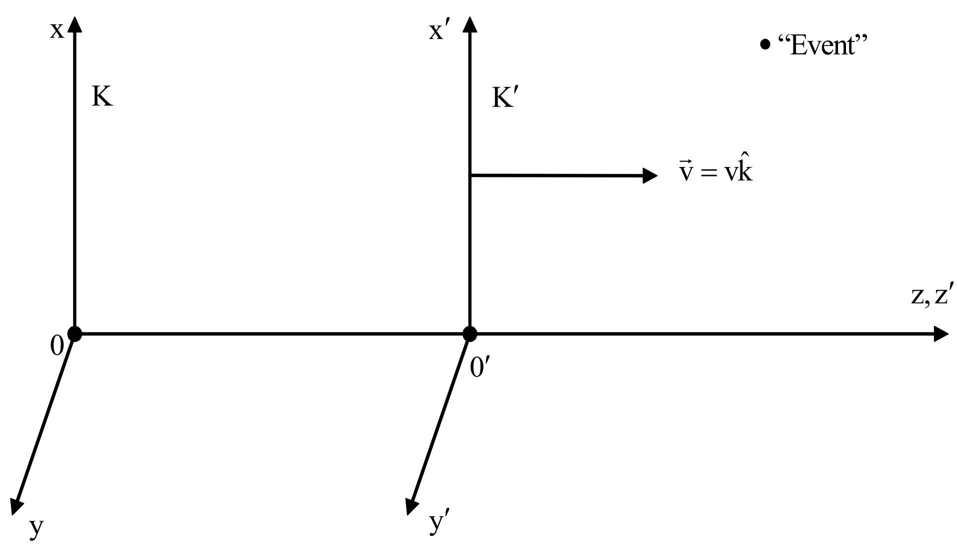
$$= \sum_{j=1}^{3} \sum_{k=1}^{3} A_j' B_k' \delta_{jk} = \sum_{j=1}^{3} A_j' B_j'$$

Therefore

$$\sum_{i=1}^{3} A_i B_i = \sum_{j=1}^{3} A_j' B_j' \tag{10}$$

2. Newtonian Relativity

Newtonian physics assumes that *space-time* is *four dimensional* so that any "event" can be located uniquely by 4 coordinates (3 spatial and 1 temporal); this requires that a *reference frame* be specified. For *inertial* reference frames, in which NEWTON'S FIRST LAW holds, the Galilean *principle of relativity* states: "the laws of classical physics (mechanics) have the same form (are COVARIANT) in all inertial reference frames which are in uniform translational motion relative to each other."



Consider two parallel inertial cartesian reference frames K and K' with a common z-axis, with system K' moving with constant velocity $\vec{v} = v\hat{k}$ with respect to K. It is convenient to imagine that each reference frame has an *infinite* array of stationary observers (one for each point in space if you like) with synchronized clocks. An "event" is characterised by the four coordinates (x, y, z, t) in K and (x', y', z', t') in K'. It is also convenient to assume that K and K' (i.e. 0 and $0'$) coincide at $t = t' = 0$.

The coordinates of the event as measured in the two reference frames are related by the *Galilean transformation equations*

$$\boxed{\begin{aligned} & x' = x \;;\quad y' = y \;;\quad t' = t \\ & z' = z - vt \end{aligned}} \qquad \text{GALILEAN TRANSFORMATION} \quad (11)$$

The geometry of Newtonian space-time thus consists of two disjoint Euclidean geometries for space and for time; i.e.: –

The length interval $\Delta\ell$, given by

$$(\Delta\ell)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \equiv (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

at a given time t (dt = 0), is *INVARIANT* (same in both frames) as is the time interval $dt = dt'$ $(\Delta t = \Delta t')$.

Newton's Laws of classical mechanics are *COVARIANT* under the Galilean transformation. However Maxwell's equations are *not* covariant, in the sense that the speed of light in *free space* is not preserved. This can be seen from the Galilean velocity transformation, from (11):

$$\boxed{\begin{aligned} \frac{dx'}{dt'} &= \frac{dx}{dt} & \text{or} \quad u_x' &= u_x \\ \frac{dy'}{dt'} &= \frac{dy}{dt} & \text{or} \quad u_y' &= u_y \\ \frac{dz'}{dt'} &= \frac{dz}{dt} - v & \text{or} \quad u_z' &= u_z - v \end{aligned}} \qquad (12)$$

3. Einstein's Relativity

Einstein became convinced that Maxwell's Equations represented the proper description of electromagnetic phenomena in all inertial reference frames, and thus stated the two basic *postulates* of special relativity

(i) The laws of physics are identical in any two reference frames which move at constant relative velocity.

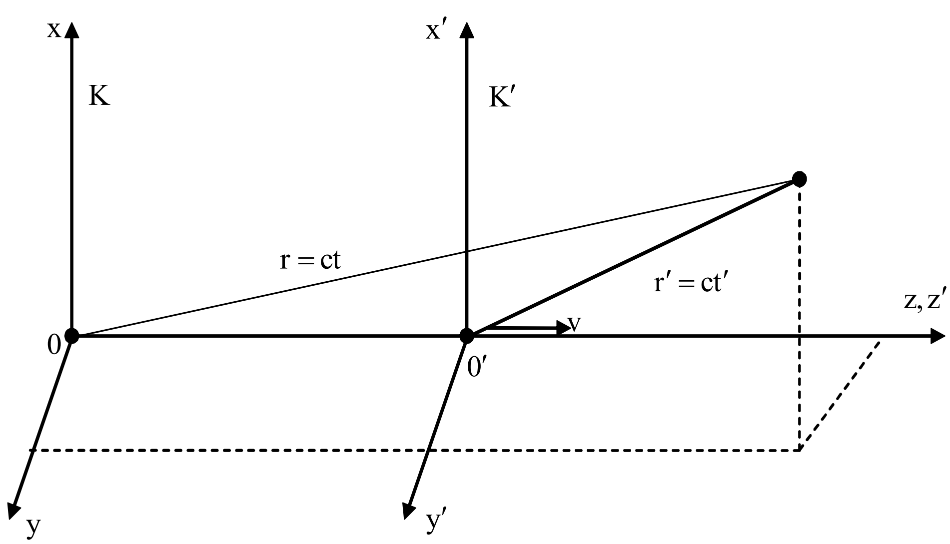
(ii) All observers in *uniform* relative motion will measure the same value for the speed of light (c) in free space.

4. The Lorentz Transformation

Consider the same two cartesian coordinate systems K and K' described previously. Imagine that a single photon is emitted at the instant $t = t' = 0$ from either 0 or $0'$, which are coincident at that instant. Suppose subsequently that this photon is absorbed at point A in space by some detector; an observer in K will assign space and time coordinates (x, y, z, t) to this event, while one in K' will designate its coordinates as (x', y', z', t'). *If* the speed of light c is to have the same measured value in both frames, then

$$c = \frac{r}{t} = \frac{(x^2 + y^2 + z^2)^{1/2}}{t} \Rightarrow x^2 + y^2 + z^2 - c^2t^2 = 0 \qquad (K)$$

$$c = \frac{r'}{t'} = \frac{\left((x')^2 + (y')^2 + (z')^2\right)^{1/2}}{t'} \Rightarrow (x')^2 + (y')^2 + (z')^2 - c^2(t')^2 = 0 \qquad (K')$$



Hence:

$$\boxed{x^2 + y^2 + z^2 - c^2t^2 = (x')^2 + (y')^2 + (z')^2 - c^2(t')^2} \qquad (13)$$

– thus the transformation which relates the spatial and temporal coordinates in the frames K and $K^{/}$, if the Einstein second postulate is true, is one for which

$$x^2 + y^2 + z^2 - c^2t^2 \qquad \text{is INVARIANT.}$$

To determine the appropriate transformation, define a *four dimensional cartesian coordinate system* (x_1, x_2, x_3, x_4) with $x_1 = x$, $x_2 = y$, $x_3 = z$ and $x_4 = ict$ with a corresponding four dimensional position vector $\vec{R}$:

$$\vec{R} = (x_1, x_2, x_3, x_4) = (\vec{r}, ict)$$

Now the invariance condition – equation (13) – can be written:

$$\sum_{j=1}^{4} (x_j^{/})^2 = \sum_{k=1}^{4} (x_k)^2 \qquad (14)$$

and the associated transformation is a *four dimensional coordinate rotation*

$$x_j^{/} = \sum_{k=1}^{4} a_{jk} x_k \qquad j = 1,2,3,4 \qquad (15)$$

i.e.

$$\begin{bmatrix} x_1^{/} \\ x_2^{/} \\ x_3^{/} \\ x_4^{/} \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad (16)$$

in which the matrix A has 16 elements. However the invariance condition (14) requires

$$\sum_{j=1}^{4} (x_j^{/})^2 = \sum_{j=1}^{4} \Big(\sum_{k=1}^{4} a_{jk} x_k\Big)\Big(\sum_{l=1}^{4} a_{j\ell} x_\ell\Big) = \sum_{k=1}^{4} \sum_{l=1}^{4} x_k x_\ell \Big(\sum_{j=1}^{4} a_{jk} a_{j\ell}\Big)$$

since this must equal $\sum_{k=1}^{4} (x_k)^2$, we require that these 16 elements or coefficients satisfy

$$\boxed{\sum_{j=1}^{4} a_{jk} a_{j\ell} = \delta_{k\ell}} \qquad k,\ell = 1,2,3,4 \qquad (17)$$

$$a_{11}a_{11} + a_{21}a_{21} + a_{31}a_{31} + a_{41}a_{41} = 1 \quad : \quad k = 1,\ \ell = 1$$

$$a_{12}a_{11} + a_{22}a_{21} + a_{32}a_{31} + a_{42}a_{41} = 0 \quad : \quad k = 2,\ \ell = 1 \ \textit{or}\ \ell = 2,\ k = 1$$

$$a_{13}a_{11} + a_{23}a_{21} + a_{33}a_{31} + a_{43}a_{41} = 0 \quad : \quad k = 3,\ \ell = 1 \text{ or } \ell = 3,\ k = 1$$

$$a_{14}a_{11} + a_{24}a_{21} + a_{34}a_{31} + a_{44}a_{41} = 0 \quad : \quad k = 4,\ \ell = 1 \text{ or } \ell = 4,\ k = 1$$

$$a_{12}a_{12} + a_{22}a_{22} + a_{32}a_{32} + a_{42}a_{42} = 1 \quad : \quad k = 2,\ \ell = 2$$

$$a_{13}a_{12} + a_{23}a_{22} + a_{33}a_{32} + a_{43}a_{42} = 0 \quad : \quad k = 3,\ \ell = 2 \text{ or } \ell = 3,\ k = 2$$

$$a_{14}a_{12} + a_{24}a_{22} + a_{34}a_{32} + a_{44}a_{42} = 0 \quad : \quad k = 4,\ \ell = 2 \text{ or } \ell = 4,\ k = 2$$

$$a_{13}a_{13} + a_{23}a_{23} + a_{33}a_{33} + a_{43}a_{43} = 1 \quad : \quad k = 3,\ \ell = 3$$

$$a_{14}a_{13} + a_{24}a_{23} + a_{34}a_{33} + a_{44}a_{43} = 0 \quad : \quad k = 4,\ \ell = 3 \text{ or } \ell = 4,\ k = 3$$

$$a_{14}a_{14} + a_{24}a_{24} + a_{34}a_{34} + a_{44}a_{44} = 1 \quad : \quad k = 4,\ \ell = 4$$

which actually leads to 10 independent equations for these 16 coefficients

4 equations with $k = \ell$
6 equations with $k \neq \ell$ (since the interchange $k \leftrightarrow \ell$ is symmetric i.e. $a_{jk}a_{j\ell} = a_{j\ell}a_{jk}$).

Recall however that the two coordinate systems are rectangular and *spatially parallel*, thus *each* primed *spatial* axis will only have a projection onto the corresponding unprimed *spatial* axis, so in the matrix A we can set

$$a_{jk} = 0 \qquad \text{for} \qquad j \neq k, \text{ when } \ j,k = 1, 2 \text{ or } 3$$

i.e.

$$A = \begin{bmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

– leaving 10 coefficients to be determined.

Now return to these 10 equations resulting from the requirement of equation (17); and use the above conditions

$k = 2, \ell = 1$ yields

$$a_{12}a_{11} + a_{22}a_{21} + a_{32}a_{31} + a_{42}a_{41} = 0 \rightarrow a_{42}a_{41} = 0$$

$k = 2, \ell = 3$ yields

$$a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} + a_{42}a_{43} = 0 \rightarrow a_{42}a_{43} = 0$$

$k = 1, \ell = 3$ yields

$$a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} + a_{41}a_{43} = 0 \rightarrow a_{41}a_{43} = 0$$

and this requires two of a_{41}, a_{42}, a_{43} be zero; actually

$$a_{41} = 0$$

$$a_{42} = 0$$

but $a_{43} \neq 0$, as will become clear later.

Again,

$k = 1, \ell = 4$ yields

$$a_{11}a_{14} + a_{21}a_{24} + a_{31}a_{34} + a_{41}a_{44} = 0 \rightarrow a_{11}a_{14} + a_{41}a_{44} = 0$$

$$\rightarrow a_{11}a_{14} = 0 \Rightarrow a_{14} = 0$$

$k = 2, \ell = 4$ yields

$$a_{12}a_{14} + a_{22}a_{24} + a_{32}a_{34} + a_{42}a_{44} = 0 \rightarrow a_{22}a_{24} + a_{42}a_{44} = 0$$

$$\rightarrow a_{22}a_{24} = 0 \Rightarrow a_{24} = 0$$

$k = 3, \ell = 4$ yields

$$a_{13}a_{14} + a_{23}a_{24} + a_{33}a_{34} + a_{43}a_{44} = 0 \rightarrow a_{33}a_{34} + a_{43}a_{44} = 0$$

$$\Rightarrow a_{43} = -a_{34} \cdot \frac{a_{33}}{a_{44}} \neq 0$$

Notice that if the transformation equations (15) are to agree with the Galilean transformation (11) for low velocities then we require that

$$a_{11},\ a_{22},\ a_{33},\ a_{44} \text{ and } a_{34} \neq 0$$

Thus to satisfy the latter and equation (17) the 6 “off diagonal” equations yield:

$$\boxed{a_{14} = a_{41} = a_{24} = a_{42} = 0 \ ; \quad a_{43} = -a_{34} \cdot \frac{a_{33}}{a_{44}} \neq 0} \tag{18}$$

while the 4 “diagonal” equations yield

$$\boxed{a_{11}^2 = 1;\ a_{22}^2 = 1;\ a_{33}^2 + a_{43}^2 = 1;\ a_{34}^2 + a_{44}^2 = 1} \tag{19}$$

Next, substitute

$$a_{43} = -a_{34} \cdot \frac{a_{33}}{a_{44}}$$

into the 3rd equation of (19)

$$a_{33}^2 + a_{33}^2 \cdot \frac{a_{34}^2}{a_{44}^2} = 1 \quad : \quad a_{33}^2 a_{44}^2 + a_{33}^2 a_{34}^2 = a_{44}^2$$

and multiply the 4th equation in (19) by a_{33}^2:

$$a_{33}^2 a_{34}^2 + a_{33}^2 a_{44}^2 = a_{33}^2 \quad \text{i.e.} \quad a_{44}^2 = a_{33}^2$$

Hence the diagonal elements of matrix A satisfy:

$$a_{11}^2 = a_{22}^2 = 1 \quad : \quad a_{33}^2 = a_{44}^2$$

If we adopt the sign convention that *all* the diagonal elements $a_{kk} > 0$, then:

$$\boxed{\begin{aligned} a_{11} &= a_{22} = +1 \\ a_{33} &= a_{44} > 0 \end{aligned}} \tag{20}$$

and then (18) indicates that the only surviving off-diagonal elements are related by $a_{43} = -a_{34}$ and hence

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_{44} & a_{34} \\ 0 & 0 & -a_{34} & a_{44} \end{bmatrix}$$

and $a_{34}^2 = 1 - a_{44}^2$ from (19).

Now use equation (15)

$$x_j' = \sum_{k=1}^{4} a_{jk} x_k$$

with j = 3 and the above conditions

i.e.

$$\begin{aligned} x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 \rightarrow a_{33}x_3 + a_{34}x_4 \\ &= a_{44}x_3 + a_{34}x_4 \end{aligned}$$

i.e.

$$z' = a_{44}z + a_{34}(ict).$$

When $z' = 0$ we have

$$z = -ic \,.\, \frac{a_{34}}{a_{44}} t$$

however, an observer in K will describe the motion of the origin $0'$ in system $K'(x' = y' = z' = 0)$ by

$$z = vt$$

hence

$$\boxed{a_{34} = \frac{iv}{c} \,.\, a_{44}} \qquad (21)$$

and using this in $a_{34}^2 = 1 - a_{44}^2$ yields

$$-\frac{v^2}{c^2} \,.\, a_{44}^2 = 1 - a_{44}^2$$

or

$$\boxed{a_{44} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv \gamma} \qquad (22)$$

Thus the transformation equations (15) become:

$$\boxed{\begin{aligned} x' &= x \\ y' &= y \qquad\qquad \text{with } \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \\ z' &= \gamma(z - vt) \\ t' &= \gamma\left(t - \frac{vz}{c^2}\right) \end{aligned}} \qquad \text{LORENTZ TRANSFORMATION} \quad (23)$$

from which we can also solve for the inverse transformation

$$\boxed{\begin{aligned} x &= x' \\ y &= y' \\ z &= \gamma(z' + vt') \quad : \quad \gamma = (1 - \beta^2)^{-1/2} \quad : \quad \beta = \frac{v}{c} \\ t &= \gamma\left(t' + \frac{vz'}{c^2}\right) \end{aligned}} \qquad (24)$$

i.e.

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & \dfrac{iv\gamma}{c} \\ 0 & 0 & -\dfrac{iv\gamma}{c} & \gamma \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\Uparrow \qquad\qquad \Uparrow \qquad\qquad \Uparrow$$

$$\vec{R}' \qquad\qquad A \qquad\qquad \vec{R} = (\vec{r}, ict)$$

5. Consequences of the Lorentz Transformation

(a) The relative velocity v of the two inertial frames must be $\leq$c, otherwise

$\gamma(v) = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ becomes imaginary; and that would contradict the fact that z, t, z' and t' in equations (23) or (24) are all real.

(b) ***Simultaneity***

Consider two events with coordinates (x_1, y_1, z_1, t_1) and $(x_2, y_2, z_2 t_2)$ in K. The corresponding coordinates of the two events in K' are given by:

$$\left[x_1, y_1, \gamma(z_1 - vt_1),\ \gamma\left(t_1 - \frac{vz_1}{c^2}\right)\right]$$

and

$$\left[x_2, y_2, \gamma(z_2 - vt_2),\ \gamma\left(t_2 - \frac{vz_2}{c^2}\right)\right]$$

so that the time interval between the two events in K' is

$$t_2' - t_1' = \gamma\left[(t_2 - t_1) - \frac{v(z_2 - z_1)}{c^2}\right] \tag{25}$$

Clearly two events that are simultaneous in K $(t_2 = t_1)$ are NOT simultaneous in K' *unless* they occur at the same z-coordinate in K $(z_1 = z_2)$: simultaneity is therefore *not absolute*.

(c) ***Causality***

Suppose some process results in event 1 in K causing a subsequent event 2 also in K, so $t_2 > t_1$. From (25) it follows however that if

$$\frac{v(z_2 - z_1)}{c^2} > (t_2 - t_1) > 0$$

then $t_2' < t_1'$ in which case event 2 *precedes* event 1 in K'. This violates *causality*, and so we require

$$\frac{z_2 - z_1}{t_2 - t_1} \le c \left(\le \frac{c^2}{v} \right)$$

so that

$$\frac{v(z_2 - z_1)}{c^2} \le (t_2 - t_1).$$

Thus, the "interaction" responsible for this cause and effect relationship must propagate from z_1 to z_2 with speed $\le c$. So "c" represents the upper limit for the speed of all particles and physical "signals".

(d) ***Length Contraction***

Consider two *fixed* points on the z'-axis in K' at z_1' and z_2', with $z_2' - z_1' = L_0$ their separation in K'. What do observers in K measure for L, their separation? Since points 1 and 2 are moving relative to K, the measurements of z_1 and z_2 MUST be performed simultaneously in K. Thus:

$$t_1 = t_2$$

As a result:

$$\gamma\left(t_1' + \frac{v z_1'}{c^2}\right) = \gamma\left(t_2' + \frac{v z_2'}{c^2}\right) \quad \text{so} \quad t_2' - t_1' = -\frac{v(z_2' - z_1')}{c^2} \ne 0$$

$$(t_2' < t_1')$$

so that K's measurements will *NOT* be performed simultaneously according to observers in K'.

Furthermore,

$$L = z_2 - z_1 = \gamma(z_2' + vt_2') - \gamma(z_1' + vt_1')$$

$$= \gamma(z_2' - z_1') + \gamma v(t_2' - t_1') = \gamma(z_2' - z_1') - \frac{\gamma v^2}{c^2}(z_2' - z_1')$$

$$= \gamma\left(1 - \frac{v^2}{c^2}\right)(z_2' - z_1') = \frac{1}{\gamma}(z_2' - z_1') = \frac{L_0}{\gamma}$$

Therefore

$$L = \frac{L_0}{\gamma} = L_0\sqrt{1 - \frac{v^2}{c^2}} < L_0 \qquad (26)$$

– a uniformly moving body has its greatest length (L_0) in its *rest frame*, L_0 called *rest-length* or *proper-length* of body. L is called Lorentz-Fitzgerald contracted length.

(e) ***Time Dilation***

Consider two successive events in K' which *both* occur at the *same point* ($z_1' = z_2'$) but at *different times* t_1' and t_2' so that $T_0 = t_2' - t_1'$ is the time interval between them as measured in K'.

In system K these two events occur at *different* points ($z_1 \neq z_2$) and in principle require two observers (each with a synchronized clock) to measure $T = t_2 - t_1$ in K

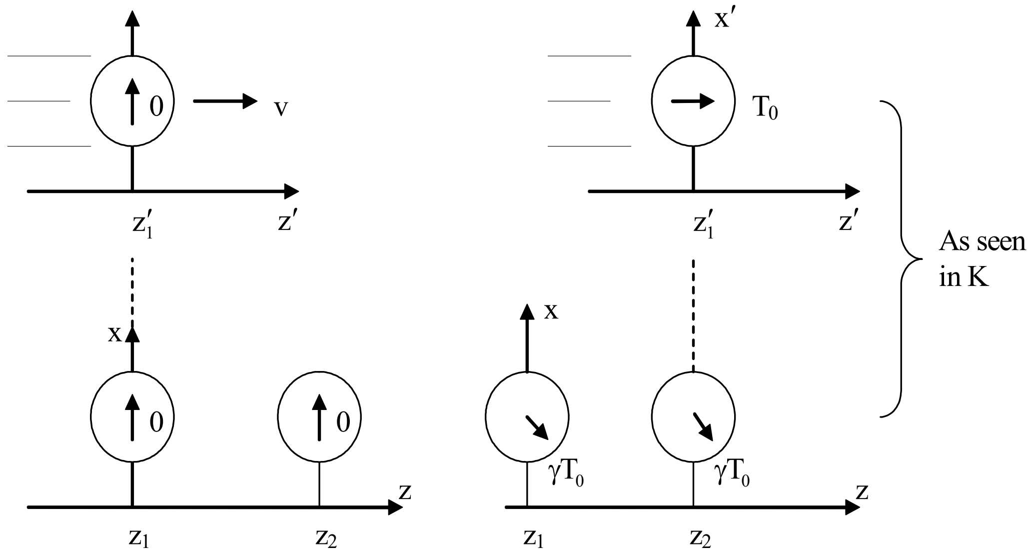
$$T = t_2 - t_1 = \gamma\left(t_2' + \frac{vz_2'}{c^2}\right) - \gamma\left(t_1' + \frac{vz_1'}{c^2}\right)$$

$$= \gamma(t_2' - t_1'), \quad \text{as} \quad z_2' = z_1'$$

Therefore

$$T = \gamma T_0 = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}} > T_0 \qquad (27)$$

i.e. a clock moving uniformly through an inertial frame K (e.g. in K') runs slow by a factor of γ^{-1} relative to clocks stationary in K. Thus a clock runs at its fastest rate in *its* rest frame and this is called the *proper rate* and T_0 the *proper time*.



(f) Events in one space and one time dimension are visually well described on an (x-t) graph. This continues to be true in 3 dimensions for relativistic mechanics, but here it is more convenient to use the relativistic time scale ct, leading to a 4 dimensional

$$(x, y, z, ct)$$

"Minkowski" space-time diagram.

The 4-dimensional space time "interval"

$$\vec{S}_{AB} = \vec{R}_A - \vec{R}_B = [(x_A - x_B), (y_A - y_B), (z_A - z_B), ic(t_A - t_B)]$$

between two events A and B in "Minkowski" space has components

$$(S_{AB})_\mu = \left((\chi_A)_\mu - (\chi_B)_\mu\right) \quad ; \quad \mu = 1, \ldots, 4;$$

and (squared) magnitude

$$S_{AB}^2 = \sum_{\mu=1}^{4} \left((\chi_A)_\mu - (\chi_B)_\mu\right)^2 = (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 - c^2(t_A - t_B)^2$$

$$= |\vec{r}_A - \vec{r}_B|^2 - c^2(t_A - t_B)^2 \tag{28}$$

which is *invariant* $[S_{AB}^2 = (S_{AB}^{\prime})^2]$.

Clearly if $S_{AB}^2 > 0$ then $|\vec{r}_A - \vec{r}_B|^2 > c^2(t_A - t_B)^2$ and the two events *cannot* be connected by an object or signal traveling with speed $v \leq c$, then S_{AB} is said to be *space-like*. However, if

$$S_{AB}^2 < 0$$

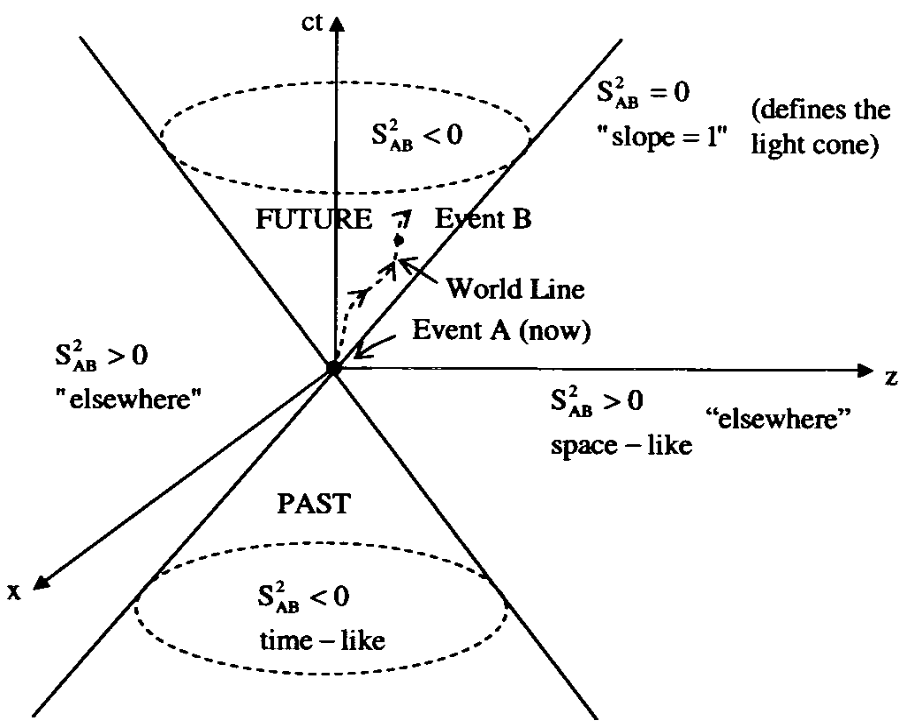
then

$$|\vec{r}_A - \vec{r}_B|^2 < c^2(t_A - t_B)^2$$

so that it is possible to bridge the distance between the two events with an object or signal moving with speed v < c, and now S_{AB} is said to be *time-like*. Finally if

$$S_{AB}^2 = 0 \text{ then } |\vec{r}_A - \vec{r}_B|^2 = c^2(t_A - t_B)^2$$

and the two events can only be connected by a light signal (v = c), then S_{AB} is said to be *light-like*. Suppressing the y-dimension on a "Minkowski" diagram:



The motion of a particle ($v \leq c$) may be viewed as a sequence of events and thus can be represented by some line on a Minkowski diagram; such a line or curve is called the world-line of the particle (and its slope must be greater than or equal to 1).

6. 4-Vectors

With the notation $x_1 = x;\ x_2 = y;\ x_3 = z;\ x_4 = ict$, the Lorentz Transformation becomes

$$x_k' = \sum_{j=1}^{4} a_{kj} x_j \quad : \quad k = 1,2,3,4$$

so

$$
\boxed{\begin{aligned}
x_1' &= x_1 \\
x_2' &= x_2 \\
x_3' &= \gamma\left(x_3 + \frac{iv}{c}x_4\right) \\
x_4' &= \gamma\left(x_4 - \frac{iv}{c}x_3\right)
\end{aligned}} \tag{29}
$$

where

$$
\gamma(v) = \left(1 - \frac{v^2}{c^2}\right)^{-1/2},
$$

and the transformation matrix A is:

$$
A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & +\frac{iv\gamma}{c} \\ 0 & 0 & -\frac{iv\gamma}{c} & \gamma \end{bmatrix}
$$

The *inverse* Lorentz transformation

$$
x_j = \sum_{k=1}^{4} a_{jk}' x_k' \quad ; \quad j = 1,2,3,4
$$

can be obtained by inverting equations (29):

$$
\boxed{
\begin{aligned}
x_1 &= x_1' \\
x_2 &= x_2' \\
x_3 &= \gamma\left(x_3' - \frac{iv}{c}\,x_4'\right) \\
x_4 &= \gamma\left(x_4' + \frac{iv}{c}\,x_3'\right)
\end{aligned}
}
$$

and the corresponding transformation matrix (the transpose of A)

$$
A' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -\dfrac{iv\gamma}{c} \\ 0 & 0 & +\dfrac{iv\gamma}{c} & \gamma \end{bmatrix}
$$

so that $a_{jk}' = a_{kj}$, and hence the pair of transformation equations can be written

$$
x_k' = \sum_{j=1}^{4} a_{kj} x_j \quad ; \quad x_j = \sum_{k=1}^{4} a_{kj} x_k'
$$

as expected for an orthogonal (linear) transformation. Equations (29) describe the transformation properties of the 4-D position vector $\vec{R} = (x_1, x_2, x_3, x_4) = (\vec{r}, ict)$.

Any set of 4 quantities

$$
\vec{M} = (m_1, m_2, m_3, m_4)
$$

which transform in the same way as the 4-coordinates i.e.

$$m_k^{/} = \sum_{j=1}^{4} a_{kj} m_j \qquad j = 1,2,3,4 \tag{30}$$

with the same set of coefficients a_{kj} above, *is known as a 4-vector*.

In the case where two frames K and $K^{/}$ move at constant velocity $\vec{v}$ (= $v\hat{k}$ in our case) relative to each other, the differential

$$d\vec{R} = (dx_1, dx_2, dx_3, dx_4) = (d\vec{r}, icdt)$$

is a 4-vector (as can be seen directly from equations (29): $\vec{v}$ = constant $\Rightarrow a_{jk}$ are constant too). The corresponding invariance condition (14) becomes

$$\sum_{\mu=1}^{4} (dx_\mu)^2 = \sum_{\nu=1}^{4} (dx_\nu^{/})^2$$

so:

$$\sum_{i=1}^{3} (dx_i)^2 - c^2(dt)^2 = \sum_{j=1}^{3} (dx_j^{/})^2 - c^2(dt^{/})^2$$

thus

$$(dt)^2 \left\{ 1 - \frac{1}{c^2} \sum_{i=1}^{3} \left(\frac{dx_i}{dt} \right)^2 \right\} = (dt^{/})^2 \left\{ 1 - \frac{1}{c^2} \sum_{j=1}^{3} \left(\frac{dx_j^{/}}{dt^{/}} \right)^2 \right\}$$

so that the quantity

$$d\alpha = dt \left[1 - \frac{1}{c^2} \sum_{i=1} \left(\frac{dx_i}{dt} \right)^2 \right]^{1/2} \tag{31}$$

is INVARIANT between frames; dα is actually the PROPER TIME (dα is the TIME interval measure when all the $\frac{dx_i}{dt} = 0$).

Since dα is invariant (i.e. a constant between frames) then the ratio $\frac{d\vec{R}}{d\alpha}$ is also a 4-vector since $d\vec{R}$ is a 4-vector.

The ratio $\frac{d\vec{R}}{d\alpha}$ defines the 4-vector velocity $\vec{V}$, with

$$\frac{d\vec{R}}{d\alpha} = \vec{V} = \left(\frac{d\vec{r}}{d\alpha},\ ic\frac{dt}{d\alpha} \right)$$

Thus, if a particle undergoes a displacement $d\vec{r}$ in a time dt measured in the unprimed frame K, then the ordinary 3-D velocity $\vec{u}$ of this particle is given by

$$\vec{u} = \frac{d\vec{r}}{dt} = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt} \right) = (u_1, u_2, u_3) \tag{32}$$

Using this, (31) becomes

$$d\alpha = dt \left[1 - \frac{1}{c^2}(u_1^2 + u_2^2 + u_3^2) \right]^{1/2} = dt \sqrt{1 - \frac{u^2}{c^2}} = \frac{dt}{\gamma(u)}$$

(in agreement with (27); dt measured by observers in K is *not* the proper time, observer on particle measures it). Thus the *4-vector velocity* can be rewritten as

$$\vec{V} = \left(\frac{\vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}}, \frac{ic}{\sqrt{1 - \frac{u^2}{c^2}}} \right) = \gamma(u)\,[\vec{u}, ic] \tag{33}$$

Therefore

$$V_1 = \frac{u_1}{\sqrt{1 - \frac{u^2}{c^2}}},\ V_2 = \frac{u_2}{\sqrt{1 - \frac{u^2}{c^2}}},\ V_3 = \frac{u_3}{\sqrt{1 - \frac{u^2}{c^2}}},\ V_4 = \frac{ic}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$V_1' = \frac{u_1'}{\sqrt{1 - \frac{u'^2}{c^2}}},\ V_2' = \frac{u_2'}{\sqrt{1 - \frac{u'^2}{c^2}}},\ V_3' = \frac{u_3'}{\sqrt{1 - \frac{u'^2}{c^2}}},\ V_4' = \frac{ic}{\sqrt{1 - \frac{u'^2}{c^2}}}$$

This yields:

$$V_1 = V_1' \qquad \text{i.e.} \qquad \frac{u_1}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{u_1'}{\sqrt{1 - \frac{u'^2}{c^2}}}$$

$$u_1 = \frac{dx_1}{dt} \quad \text{and} \quad dt\sqrt{1 - \frac{u^2}{c^2}} = d\alpha = dt'\sqrt{1 - \frac{u'^2}{c^2}}$$

Therefore

$$\frac{dx_1}{d\alpha} = \frac{dx_1^{/}}{d\alpha} \quad \text{i.e.} \quad dx_1 = dx_1^{/}$$

as expected from the Lorentz Transformation. Similarly, we get:

$$dx_2 = dx_2^{/}$$

Furthermore, we expect:

$$V_3^{/} = \gamma\left(V_3 + \frac{iv}{c}V_4\right)$$

where

$$V_3^{/} = \frac{u_3^{/}}{\sqrt{1 - \frac{u^{/2}}{c^2}}} = \frac{dx_3^{/}}{d\alpha}$$

and

$$\gamma\left(V_3 + \frac{iv}{c}V_4\right) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left(\frac{u_3}{\sqrt{1 - \frac{u^2}{c^2}}} + \frac{iv}{c} \cdot \frac{ic}{\sqrt{1 - \frac{u^2}{c^2}}}\right) = \frac{u_3 - v}{\sqrt{1 - \frac{v^2}{c^2}}\sqrt{1 - \frac{u^2}{c^2}}}$$

$$= \gamma(v)\left\{\frac{dx_3}{dt\sqrt{1 - \frac{u^2}{c^2}}} - \frac{v}{\sqrt{1 - \frac{u^2}{c^2}}}\right\} = \gamma(v)\left\{\frac{dx_3}{d\alpha} - \frac{v\,dt}{d\alpha}\right\}$$

Therefore

$$dx_3^{/} = \gamma\{dx_3 - v\,dt\}$$

as expected from the Lorentz Transformation.

Notice $\vec{V}$ can also yield the velocity transformation between frames:

$$V_1 = V_1^{\prime} \qquad \text{yields} \qquad \frac{u_1}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{u_1^{\prime}}{\sqrt{1 - \frac{u^{\prime 2}}{c^2}}}$$

while

$$V_4^{\prime} = \gamma\left(V_4 - \frac{iv}{c} V_3\right)$$

yields

$$\frac{ic}{\sqrt{1 - \frac{u^{\prime 2}}{c^2}}} = \gamma\left\{\frac{ic}{\sqrt{1 - \frac{u^2}{c^2}}} - \frac{iv}{c} \cdot \frac{u_3}{\sqrt{1 - \frac{u^2}{c^2}}}\right\}$$

Therefore

$$\frac{1}{\sqrt{1 - \frac{u^{\prime 2}}{c^2}}} = \frac{\gamma}{\sqrt{1 - \frac{u^2}{c^2}}}\left(1 - \frac{v u_3}{c^2}\right)$$

Substituting into the equation relating u_1 and $u_1^{\prime}$ above, we get:

$$\frac{u_1}{\sqrt{1 - \frac{u^2}{c^2}}} = u_1^{\prime} \frac{\gamma}{\sqrt{1 - \frac{u^2}{c^2}}}\left(1 - \frac{v u_3}{c^2}\right)$$

Therefore

$$u_1^{\prime} = \frac{u_1}{\gamma\left(1 - \frac{v u_3}{c^2}\right)}$$

– fully worked out later.

As a natural corollary, the *4-vector acceleration* $\vec{A}$ can be obtained from

$$\vec{A} = \frac{d\vec{V}}{d\alpha} = \frac{d}{d\alpha}\left(\frac{\vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}}, \frac{ic}{\sqrt{1 - \frac{u^2}{c^2}}} \right)$$

Notice that

$$\frac{d}{d\alpha}\left(\frac{\vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}} \right) = \frac{1}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}} \cdot \frac{d}{dt}\left(\frac{u_1}{\sqrt{1 - \frac{(u_1^2 + u_2^2 + u_3^2)}{c^2}}}, \dots, \dots \right)$$

$$= \frac{1}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}} \left[\frac{du_1}{dt} \cdot \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} + u_1\left(-\frac{1}{2}\right) \frac{\left(-\frac{2u_1}{c^2}\frac{du_1}{dt} - \frac{2u_2}{c^2}\frac{du_2}{dt} - \frac{2u_3}{c^2}\frac{du_3}{dt} \right)}{\left(1 - \frac{u^2}{c^2}\right)^{3/2}}, \dots, \dots \right]$$

$$= \frac{1}{\left(1 - \frac{u^2}{c^2}\right)} \left[a_1 + \frac{u_1\left(\frac{u_1 a_1}{c^2} + \frac{u_2 a_2}{c^2} + \frac{u_3 a_3}{c^2} \right)}{\left(1 - \frac{u^2}{c^2}\right)}, \dots, \dots \right]$$

where

$$\vec{a} := (a_1, a_2, a_3) = \left(\frac{du_1}{dt}, \frac{du_2}{dt}, \frac{du_3}{dt} \right)$$

is the ordinary 3-D acceleration

Therefore

$$\frac{d}{d\alpha}\left(\frac{\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}\right) = \left(\frac{\vec{a}}{\left(1-\frac{u^2}{c^2}\right)} + \frac{\vec{u}(\vec{u}\,.\,\vec{a})}{c^2\left(1-\frac{u^2}{c^2}\right)^2}\right)$$

Finally:

$$\frac{d}{d\alpha}\left(\frac{ic}{\sqrt{1-\frac{u^2}{c^2}}}\right) = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\cdot\frac{d}{dt}\left(\frac{ic}{\sqrt{1-\frac{u^2}{c^2}}}\right)$$

$$= \frac{ic}{\sqrt{1-\frac{u^2}{c^2}}}\frac{d}{dt}\left(\left(1-\frac{[u_1^2+u_2^2+u_3^2]}{c^2}\right)^{-1/2}\right)$$

$$= \frac{ic}{\left(1-\frac{u^2}{c^2}\right)^{1/2}}\cdot\left(-\frac{1}{2}\right)\cdot\frac{\left(-\frac{2u_1}{c^2}\frac{du_1}{dt}-\frac{2u_2}{c^2}\frac{du_2}{dt}-\frac{2u_3}{c^2}\frac{du_3}{dt}\right)}{\left(1-\frac{u^2}{c^2}\right)^{3/2}}$$

$$= \frac{ic}{c^2\left(1-\frac{u^2}{c^2}\right)^2}(u_1a_1+u_2a_2+u_3a_3) = \frac{i(\vec{u}\,.\,\vec{a})}{c\left(1-\frac{u^2}{c^2}\right)^2}$$

Therefore

$$\vec{A} = \left(\frac{\vec{a}}{\left(1-\frac{u^2}{c^2}\right)} + \frac{(\vec{u}\,.\,\vec{a})\vec{u}}{c^2\left(1-\frac{u^2}{c^2}\right)^2}, \frac{i(\vec{u}\,.\,\vec{a})}{c\left(1-\frac{u^2}{c^2}\right)^2}\right) \qquad (34)$$

and the four "components" of $\vec{A}$ transform according to the Lorentz transformation.

Notice that the 4-vector formalism means

$$A_1 = \frac{a_1}{\left(1 - \frac{u^2}{c^2}\right)} + \frac{u_1(\vec{u} \cdot \vec{a})}{c^2\left(1 - \frac{u^2}{c^2}\right)^2} \rightarrow A_1^{/} = \frac{a_1^{/}}{\left(1 - \frac{u^{/2}}{c^2}\right)} + \frac{u_1^{/}(\vec{u}^{\,/} \cdot \vec{a}^{\,/})}{c^2\left(1 - \frac{u^{/2}}{c^2}\right)}$$

etc. Thus:

$$\frac{a_1}{\left(1 - \frac{u^2}{c^2}\right)} + \frac{u_1(\vec{u} \cdot \vec{a})}{c^2\left(1 - \frac{u^2}{c^2}\right)^2} = \frac{a_1^{/}}{\left(1 - \frac{u^{/2}}{c^2}\right)} + \frac{u_1^{/}(\vec{u}^{\,/} \cdot \vec{a}^{\,/})}{c^2\left(1 - \frac{u^{/2}}{c^2}\right)}$$

However, we have already shown

$$\frac{dV_1}{d\alpha} = \frac{a_1}{\left(1 - \frac{u^2}{c^2}\right)} + \frac{u_1(\vec{u} \cdot \vec{a})}{c^2\left(1 - \frac{u^2}{c^2}\right)^2} \quad \text{and} \quad V_1 = \frac{dx_1}{d\alpha}$$

We could similarly show

$$\frac{dV_1^{/}}{d\alpha} = \frac{d}{d\alpha}\left(\frac{dx_1^{/}}{d\alpha}\right) = \frac{a_1^{/}}{\left(1 - \frac{u^{/2}}{c^2}\right)} + \frac{u_1^{/}(\vec{u}^{\,/} \cdot \vec{a}^{\,/})}{c^2\left(1 - \frac{u^{/2}}{c^2}\right)^2}$$

i.e. $\frac{d^2x_1}{d\alpha^2} = \frac{d^2x_1^{/}}{d\alpha^2}$ or $d^2x_1 = d^2x_1^{/}$ as expected for a Lorentz Transformation.

$\vec{A}$ will, under suitable manipulation, again yield the acceleration transformation between frames, although this will (probably) be quite complicated.

The mass of a particle measured in the frame of reference in which it is at rest is called the *rest* *mass* m_0; it is an invariant scalar quantity and can be used to define the *4-vector momentum* $\vec{P}$

$$\vec{P} = m_0\vec{V} = \left(\frac{m_0\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}, \frac{i m_0 c}{\sqrt{1-\frac{u^2}{c^2}}}\right) \tag{35}$$

The momentum 4-vector $\vec{P}$ can be rewritten in the form

$$\vec{P} = (\vec{p}, i m c) \tag{36}$$

where

$$\vec{p} = \frac{m_0\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}} = \gamma(u) m_0\vec{u} = m\vec{u} \tag{37}$$

is called the *3-D relativistic momentum*, while

$$m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} = \gamma(u) m_0 \tag{38}$$

is called the *relativistic (inertial) mass* (yielding $m_0 = \frac{m}{\gamma(u)}$ as the invariant scalar rest mass).

Since the 4-D momentum $\vec{P}$ is a four-vector, it follows that $d\vec{P}$ is also a four-vector, and this leads to the definition of the *4-D force* $\vec{F}$ as

$$\vec{F} = \frac{d\vec{P}}{d\alpha} = \gamma(u)\frac{d\vec{P}}{dt} \quad ; \quad \gamma(u) = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

so

$$\vec{F} = \left(\gamma(u) \frac{d}{dt}\left[\frac{m_0 \vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}}\right], \quad \gamma(u) \frac{d}{dt}\left[\frac{i m_0 c}{\sqrt{1 - \frac{u^2}{c^2}}}\right] \right) = m_0 \vec{A}$$

and this is also a 4-vector. If we then define the *3-D relativistic force* $\vec{f}$ by

$$\vec{f} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left[\frac{m_0 \vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}}\right] \tag{39}$$

which also specifies a *3-D relativistic equation of motion* then we have

$$\vec{F} = \left(\gamma(u)\vec{f} \ , \ \frac{i\gamma(u)}{c} \frac{d}{dt}[mc^2] \right) \tag{40}$$

From (39) it follows that the 3-D relativistic momentum $\vec{p}$ is conserved for a "free particle" i.e. when the 3-D relativistic force $\vec{f} = 0$; this also holds for an *isolated* system of mutually interacting objects when no *external* forces are present:

$$\vec{f} = 0 \Rightarrow \sum_{\substack{\ell \\ \text{(particles)}}} \frac{d\vec{p}_\ell}{dt} = 0$$

Therefore

$$\sum_{\substack{\ell \\ \text{(particles)}}} \vec{p}_\ell = \text{constant (independent of time)}$$

However, if we examine the transformation properties of the (total) 4-vector momentum

$$\vec{P} = \sum_\ell \vec{P}_\ell$$

for several particles:

$$(\sum_\ell p'_{\ell 1}) = (\sum_\ell p_{\ell 1})$$

$$(\sum_\ell p'_{\ell 2}) = (\sum_\ell p_{\ell 2})$$

$$(\sum_\ell p'_{\ell 3}) = \gamma\left\{(\sum_\ell p_{\ell 3}) + \frac{iv}{c}(\sum_\ell i m_\ell c)\right\}$$

$$(\sum_\ell i m'_\ell c) = \gamma\left\{(\sum_\ell i m_\ell c) - \frac{iv}{c}(\sum_\ell p_{\ell 3})\right\}$$

so that the conservation of the 3-D relativistic momentum:

$$\sum_\ell p_{\ell\mu} \; (\mu = 1,2,3) = C_{\mu,S}$$ (independent of time for interacting particles, in S)

$$\sum_\ell p'_{\ell\mu} \; (\mu = 1,2,3) = C_{\mu,S'}$$ (which may be $\neq C_S$, independent of time for interacting particles, as seen in S')

requires the conservation of relativistic mass, i.e.

$$\sum_\ell m_\ell = \mathbb{C}_S \quad ; \quad \sum_\ell m'_\ell = \mathbb{C}'_S \qquad (\mathbb{C}_S \neq \mathbb{C}'_S)$$

(i.e. $\sum_\ell p_{\ell 3}$ = constant and $\sum_\ell p'_{\ell 3}$ = constant $\Rightarrow$ $\sum_\ell i m_\ell c$ *MUST* = constant).

Correspondingly, the *4-vector momentum* must be *conserved* for an isolated system.

$$\sum_\ell \vec{P}_\ell = \text{Constant (independent of time)} \quad \text{(ISOLATED SYSTEM)} \tag{41}$$

(the “constant” will however be *different* in S and S').

Thus:

$\sum_\ell \vec{p}_\ell$ = constant: CONSERVATION OF 3D RELATIVISTIC MOMENTUM $\sum_\ell m_\ell$ = constant: CONSERVATION OF RELATIVISTIC MASS	ISOLATED SYSTEM

(In the more general case, either *both* $\sum_\ell \vec{p}_\ell$ and $\sum_\ell m_\ell$ are conserved *or* both are not conserved.)

In analogy with Newtonian mechanics, the relativistic kinetic energy T of an object moving with velocity $\vec{u}$ calculates the work done by the 3-D relativistic force $\vec{f}$ in accelerating it from rest to its final velocity.

If the work done by $\vec{f}$ appears as kinetic energy alone

$$\frac{dT}{dt} = \text{power delivered by } \vec{f} = \vec{f} \cdot \vec{u}(t)$$

where $\vec{u}(t)$ is the instantaneous velocity.

Therefore

$$\frac{dT}{dt} = \vec{u}(t) \cdot \frac{d}{dt}\left[\frac{m_0 \vec{u}(t)}{\sqrt{1 - \frac{u^2(t)}{c^2}}}\right] \qquad \text{(done previously)}$$

$$= \vec{u}(t) \cdot \left[\frac{m_0 \dot{\vec{u}}(t)}{\sqrt{1 - \frac{u^2(t)}{c^2}}} + \frac{m_0(\vec{u}(t) \cdot \dot{\vec{u}}(t))}{\left(1 - \frac{u^2(t)}{c^2}\right)^{3/2}} \frac{\vec{u}(t)}{c^2}\right]$$

$$= \frac{m_0}{\sqrt{1 - \frac{u^2(t)}{c^2}}}\left\{\vec{u}(t) \cdot \dot{\vec{u}}(t) + \frac{(\vec{u}(t) \cdot \dot{\vec{u}}(t))}{\left(1 - \frac{u^2(t)}{c^2}\right)} \frac{\vec{u}(t) \cdot \vec{u}(t)}{c^2}\right\}$$

$$= \frac{m_0 (\vec{u}(t) \cdot \dot{\vec{u}}(t))}{\left(1 - \frac{u^2(t)}{c^2}\right)^{3/2}}$$

On the other hand,

$$\frac{d}{dt}\left\{\frac{m_0 c^2}{\sqrt{1 - \frac{u^2(t)}{c^2}}}\right\} = m_0 c^2 \left(-\frac{1}{2}\right)\left(1 - \frac{u^2(t)}{c^2}\right)^{-3/2} \frac{d}{dt}\left(1 - \frac{u^2(t)}{c^2}\right)$$

where

$$\frac{d}{dt}\left(1 - \frac{u^2(t)}{c^2}\right) = \frac{d}{dt}\left(1 - \frac{\vec{u}(t)\,.\,\vec{u}(t)}{c^2}\right)$$

$$= -\frac{\dot{\vec{u}}(t)\,.\,\vec{u}(t)}{c^2} - \frac{\vec{u}(t)\,.\,\dot{\vec{u}}(t)}{c^2} = \frac{-2\vec{u}(t)\,.\,\dot{\vec{u}}(t)}{c^2}$$

Therefore

$$\frac{d}{dt}\left\{\frac{m_0 c^2}{\sqrt{1 - \frac{u^2(t)}{c^2}}}\right\} = \frac{m_0 \vec{u}(t)\,.\,\dot{\vec{u}}(t)}{\left(1 - \frac{u^2(t)}{c^2}\right)^{3/2}}.$$

Thus,

$$\frac{dT}{dt} = \frac{d}{dt}\left\{\frac{m_0 c^2}{\sqrt{1 - \frac{u^2(t)}{c^2}}}\right\}.$$

Integrating both sides, taking m_0 *to be a constant* and T = 0 when u = 0, we get:

$$T = m_0 c^2 \left\{\frac{1}{\sqrt{1 - \frac{u^2(t)}{c^2}}}\right\}_{u=0}^{u_{\text{"final"}}}$$

Therefore

$$T = m_0 c^2 \left\{\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - 1\right\}$$

Therefore

$$T = \frac{m_0 c^2}{\sqrt{1 - \frac{u^2}{c^2}}} - m_0 c^2 .$$

Notice however that this is:

$$T = mc^2 - m_0 c^2$$

or

$$m = m_0 + \frac{T}{c^2} \tag{42}$$

Thus according to (42) the kinetic energy T *contributes* to the total relativistic mass of a particle, and a change ΔT in the kinetic energy is *accompanied* by a *proportional* change in the relativistic mass attributed to the particle, i.e.

$$\Delta m = m - m_0 = \frac{\Delta T}{c^2}$$

For u << c

$$T = m_0 c^2 \left\{ \left(1 - \frac{u^2}{c^2} \right)^{-1/2} - 1 \right\} \simeq m_0 c^2 \left\{ 1 + \frac{1}{2} \frac{u^2}{c^2} - 1 \right\} = \frac{1}{2} m_0 u^2$$

(in agreement with Newtonian definition).

We also define the total relativistic energy E by:

$$E = mc^2 = T + m_0 c^2 \tag{43}$$

KINETIC (MOTIONAL) ENERGY	REST-MASS (INERTIAL) ENERGY

For a *free* body ($\vec{f} = 0$) or an isolated system of mutually (internally) interacting objects

$$\frac{dT}{dt} = 0$$

Therefore T is conserved (independent of time) and since m_0 is fixed, *E is also conserved*. [Since $\vec{p}$ is conserved we know the relativistic mass m is conserved. Therefore E must be conserved by this argument also!]

We have shown (equation 42) that the inertial mass m of a moving particle exceeds its rest mass m_0 by T/c^2, so the kinetic energy T contributes to mass (i.e. total energy). Since all energy in principle is exchangeable with kinetic energy, Einstein postulated that *all* energy has mass and all mass is *equivalent* to energy, according to (43)

$$\Delta m = \frac{\Delta E}{c^2} \qquad \text{PRINCIPLE OF MASS-ENERGY EQUIVALENCE} \tag{44}$$

Implicit in (44) is the assertion that *all the mass* of a particle can be transmuted into available energy (a bold step in Einstein's time), and amply confirmed by experience:

decay of neutral mesons into photon pairs: $\pi^\circ \rightarrow \gamma + \gamma$ (neutral meson) → (two γ-ray photons)

pair annihilation of an elementary particle & anti-particle: $e^+ + e^- \rightarrow \gamma + \gamma$ (positron) + (electron)

and in collisions in which different elementary particles with different rest masses emerge than went in.

$$\gamma + p \rightarrow \pi^\circ + p$$

In view of this equivalence, expressed in (43) and (44), the 4-vector momentum (36) becomes

$$\vec{P} = \left(\vec{p}, \frac{iE}{c}\right) \tag{45}$$

while the 4-vector force $\vec{F}$ from (40) becomes

$$\vec{F} = \left(\gamma(u)\vec{f}, \frac{i\gamma(u)}{c}\frac{dE}{dt}\right) \qquad \text{in general} \tag{46}$$

If the rest mass(es) are *constant* then

$$\frac{dE}{dt} = \frac{d}{dt}(mc^2) = \frac{d}{dt}(T) + \frac{d}{dt}(m_0c^2) = \vec{u}\,.\,\vec{f}$$

and hence

$$\vec{F} = \left(\gamma(u)\vec{f}, \frac{i\gamma(u)}{c}\vec{u}\,.\,\vec{f}\right) \qquad \textit{constant } m_0 \tag{47}$$

From (35)

$$\vec{P} = m_0 \vec{V}$$

Therefore

$$\sum_{\mu=1}^{4} P_\mu^2 = m_0^2 \sum_{\mu=1}^{4} V_\mu^2$$

Therefore

$$p^2 + \left(\frac{iE}{c}\right)^2 = m_0^2 \left(\frac{u^2}{\left(1 - \frac{u^2}{c^2}\right)} + \frac{(ic)^2}{\left(1 - \frac{u^2}{c^2}\right)} \right)$$

Therefore

$$p^2 - \frac{E^2}{c^2} = m_0^2 \left(\frac{u^2 - c^2}{1 - \frac{u^2}{c^2}} \right) = -m_0^2 c^2$$

Therefore

$$E^2 = p^2 c^2 + m_0^2 c^4 \qquad (48)$$

Comment 1: It is usual to distinguish between the *kinetic* energy T which a particle possesses in virtue of its motion

$$T = (m - m_0)c^2$$

and its *internal* energy (inertial rest energy) $m_0 c^2$. All changes in the *internal energy* of a body appear as changes in the *rest mass* m_0.

For "ordinary matter" this internal energy is equal to 9×10^{20} ergs per gram of mass; it is "stored" as

(i) mass of the ultimate particles (99%)
(ii) thermal motion (heat energy) of the atoms/molecules
(iii) intermolecular/interatomic cohesive forces
(iv) nuclear bonds (quite large)
(v) excited atoms (which can radiate)

Comment 2: Suppose a particle of constant rest mass m_0 is acted upon by a *conservative* force $\vec{\mathscr{F}} = -\vec{\nabla}V(\vec{r})$, then

$$\frac{dE}{dt} = \frac{d}{dt}(mc^2) = \vec{u}\,.\,\vec{\mathscr{F}} = -\vec{u}\,.\,\vec{\nabla}V(\vec{r}) = -\frac{d\vec{r}}{dt}\,.\,\vec{\nabla}V(\vec{r}) = -\frac{d}{dt}(V(\vec{r}))$$

Therefore

$$\frac{d}{dt}(mc^2 + V(\vec{r})) = 0$$

Therefore

$$\text{Total energy } W = mc^2 + V(\vec{r}) = \text{constant}$$

Thus, the potential energy *of position* does NOT contribute to mass.

[In classical mechanics a particle moving in an em (or gravitational) field is said to possess potential energy, and the sum of its kinetic energy and potential energy is constant. Energy conservation then attributes any increase in kinetic energy of the particle to a decrease in the potential energy of the particle, whereas the "correct" description would be to debit the *field*.]

Notice that if Comment 2 is the "correct" description, the total energy (particle + field) is conserved, however the kinetic energy (and hence the relativistic mass m) is increased if a particle "falls" in such a field. By contrast, if the potential energy of position did contribute to the relativistic mass (i.e. $mc^2 = m_0c^2 + T + V(\vec{r})$), then since $T + V(\vec{r})$ is conserved, the relativistic mass $m\ (\equiv E/c^2)$ would be the *same* everywhere.

Recall the experiment with photons ($m_0 \equiv 0$)

$$m = \frac{p}{c} = \frac{T}{c^2} = \frac{h\nu}{c^2}$$

If $V(\vec{r})$ contributed to m, since

$$T + V(\vec{r}) = mc^2 = h\nu$$

is conserved (in a conservative field), ν would *not* change; actually ν is observed to *increase* when photons "fall" in a gravitational field.

Comment 3: Special relativity admits the possibility of entities traveling with the speed of light but having necessarily zero *rest* mass (but non-zero relativistic mass) since

$$m_0 = m\sqrt{1 - \frac{u^2}{c^2}} \quad \text{and} \quad m = \frac{E}{c^2} = \text{finite}$$

Clearly for such an entity:

$$u = c; \quad m_0 = 0; \quad E = T = pc = mc^2.$$

(This provides a clear example of a massless field – actually the em field – which nevertheless possesses momentum and energy (density)).

Comment 4: Suppose particles 1 and 2 "interact" (collide) and produce two new particles "a" and "b". The conservation of the 4-vector momentum (an isolated system) requires

$$P_{1\mu} + P_{2\mu} = P_{a\mu} + P_{b\mu} \qquad \mu = 1, 2, 3, 4$$

For simplicity assume all motion is confined to the z ($\mu = 3$) direction:

$$\frac{m_{01}u_{1z}}{\left(1 - \frac{u_1^2}{c^2}\right)^{1/2}} + \frac{m_{02}u_{2z}}{\left(1 - \frac{u_2^2}{c^2}\right)^{1/2}} = \frac{m_{0a}u_{az}}{\left(1 - \frac{u_a^2}{c^2}\right)^{1/2}} + \frac{m_{0b}u_{bz}}{\left(1 - \frac{u_b^2}{c^2}\right)^{1/2}}$$

and

$$\frac{m_{01}c^2}{\left(1 - \frac{u_1^2}{c^2}\right)^{1/2}} + \frac{m_{02}c^2}{\left(1 - \frac{u_2^2}{c^2}\right)^{1/2}} = \frac{m_{0a}c^2}{\left(1 - \frac{u_a^2}{c^2}\right)^{1/2}} + \frac{m_{0b}c^2}{\left(1 - \frac{u_b^2}{c^2}\right)^{1/2}}$$

While the initial kinetic energy is

$$T_i = \frac{m_{01}c^2}{\left(1 - \frac{u_1^2}{c^2}\right)^{1/2}} - m_{01}c^2 + \frac{m_{02}c^2}{\left(1 - \frac{u_2^2}{c^2}\right)^{1/2}} - m_{02}c^2,$$

the final kinetic energy is

$$T_f = \frac{m_{0a}c^2}{\left(1 - \frac{u_a^2}{c^2}\right)^{1/2}} - m_{0a}c^2 + \frac{m_{0b}c^2}{\left(1 - \frac{u_b^2}{c^2}\right)^{1/2}} - m_{0b}c^2$$

Hence

$$\Delta T = T_f - T_i = (m_{01} + m_{02})c^2 - (m_{0a} + m_{0b})c^2$$

Therefore

$$\Delta T = -\Delta M_0 \,.\, c^2$$

where $\Delta M_0 = \Sigma\, m_{0f} - \Sigma\, m_{0i}$; and in *inelastic* collisions or reactions (kinetic energy is *not* conserved) motional energy is converted into rest mass or vice versa.

7. Transformation of 3-D Relativistic Quantities

(i) ***Transformation for the 3-D velocity $\vec{u}$***

Recall the velocity 4-vector:

$$\vec{V} = (V_1, V_2, V_3, V_4) = \left(\frac{u_1}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}}, \frac{u_2}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}}, \frac{u_3}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}}, \frac{ic}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}} \right)$$

Equations (29), applied to this 4-vector, then yield:

$$V_1' = V_1$$

$$V_2' = V_2$$

$$V_3' = \gamma\left(V_3 + \frac{iv}{c} V_4 \right)$$

$$V_4' = \gamma\left(V_4 - \frac{iv}{c} V_3 \right)$$

Thus:

$$\frac{u_1^{/}}{\left(1 - \frac{(u^{/})^2}{c^2}\right)^{1/2}} = \frac{u_1}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}}; \quad \frac{u_2^{/}}{\left(1 - \frac{(u^{/})^2}{c^2}\right)^{1/2}} = \frac{u_2}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}}$$

$$\frac{u_3^{/}}{\left(1 - \frac{(u^{/})^2}{c^2}\right)^{1/2}} = \gamma\left[\frac{u_3}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}} + \frac{iv}{c}\frac{ic}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}}\right]$$

$$\frac{ic}{\left(1 - \frac{(u^{/})^2}{c^2}\right)^{1/2}} = \gamma\left[\frac{ic}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}} - \frac{iv}{c}\frac{u_3}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}}\right]$$

ie.

$$\frac{ic}{\left(1 - \frac{(u^{/})^2}{c^2}\right)^{1/2}} = \frac{ic\gamma}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}}\left[1 - \frac{vu_3}{c^2}\right]$$

from which the expression

$$\frac{1}{\left(1 - \frac{(u^{/})^2}{c^2}\right)^{1/2}} = \frac{\gamma}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}}\left[1 - \frac{vu_3}{c^2}\right]$$

can be substituted back into the other three equations to get:

	INVERSE TRANSFORMATION
$u_1' = \dfrac{u_1}{\gamma\left(1 - \dfrac{v u_3}{c^2}\right)}$	$u_1 = \dfrac{u_1'}{\gamma\left(1 + \dfrac{v u_3'}{c^2}\right)}$
$u_2' = \dfrac{u_2}{\gamma\left(1 - \dfrac{v u_3}{c^2}\right)}$	$u_2 = \dfrac{u_2'}{\gamma\left(1 + \dfrac{v u_3'}{c^2}\right)}$
$u_3' = \dfrac{u_3 - v}{\left(1 - \dfrac{v u_3}{c^2}\right)}$	$u_3 = \dfrac{u_3' + v}{\left(1 + \dfrac{v u_3'}{c^2}\right)}$

The 3-D momentum , and energy E can similarly be obtained from the 4-momentum

$$\vec{P} = m_0 \vec{V} = \left(\frac{m_0 \vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}}, \frac{i m_0 c}{\sqrt{1 - \frac{u^2}{c^2}}} \right) = \left(\vec{p}, \frac{iE}{c} \right)$$

$p_1' = p_1$	$p_1 = p_1'$
$p_2' = p_2$	$p_2 = p_2'$
$p_3' = \gamma\left(p_3 - \frac{vE}{c^2}\right)$	$p_3 = \gamma\left(p_3' + \frac{vE'}{c^2}\right)$
$E' = \gamma(E - vp_3)$	$E = \gamma(E' + vp_3')$

with $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$.

And for 3-D force $\vec{f}$ and power $\mathcal{P} = \frac{dE}{dt}$

$$f_1' = \frac{f_1}{\gamma\left(1 - \frac{vu_3}{c^2}\right)} \qquad f_1 = \frac{f_1'}{\gamma\left(1 + \frac{vu_3'}{c^2}\right)}$$

$$f_2' = \frac{f_2}{\gamma\left(1 - \frac{vu_3}{c^2}\right)} \qquad f_2 = \frac{f_2'}{\gamma\left(1 + \frac{vu_3'}{c^2}\right)}$$

$$f_3' = \frac{\left(f_3 - \frac{v\mathcal{P}}{c^2}\right)}{\left(1 - \frac{vu_3}{c^2}\right)} \qquad f_3 = \frac{\left(f_3' + \frac{v\mathcal{P}'}{c^2}\right)}{\left(1 + \frac{vu_3'}{c^2}\right)}$$

$$\mathcal{P}' = \frac{(\mathcal{P} - vf_3)}{\left(1 - \frac{vu_3}{c^2}\right)} \qquad \mathcal{P} = \frac{(\mathcal{P}' + vf_3')}{\left(1 + \frac{vu_3'}{c^2}\right)}$$

$$\vec{F} = \left(\gamma(u)\vec{f},\ \frac{i\gamma(u)}{c}\mathcal{P} \right)$$

Lorentz transformation:

$$\gamma(u')f_1' = \gamma(u)f_1$$

$$\gamma(u')f_2' = \gamma(u)f_2$$

$$\gamma(u')f_3' = \gamma(v)\left[\gamma(u)f_3 + \frac{iv}{c}\frac{i\gamma(u)}{c}\mathcal{P}\right]$$

$$\frac{i\gamma(u')}{c}\mathcal{P}' = \gamma(v)\left[\frac{i\gamma(u)}{c}\mathcal{P} - \frac{iv}{c}\gamma(u)f_3\right]$$

Notice that

$$f_1' = \frac{\gamma(u)}{\gamma(u')}f_1;$$

and recall the velocity transformation equations

$$\frac{\gamma(u)}{\gamma(u')} = \frac{\sqrt{1 - \frac{(u')^2}{c^2}}}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{1}{\gamma(v)}\frac{1}{\left[1 - \frac{vu_3}{c^2}\right]}$$

Therefore

$$f_1' = \frac{f_1}{\gamma\left(1 - \frac{vu_3}{c^2}\right)};$$

and the same for f_2'.

Also

$$f_3^{/} = \frac{\gamma(v)\gamma(u)}{\gamma(u^{/})}\left[f_3 - \frac{v\mathcal{P}}{c^2}\right] = \frac{f_3 - \dfrac{v\mathcal{P}}{c^2}}{\left(1 - \dfrac{v u_3}{c^2}\right)},$$

with a similar result for $\mathcal{P}^{/}$.

8. 4-Vectors in Electrodynamics

Begin by defining a *4-dimensional gradient operator* $\vec{\Box}$ with components:

$$\vec{\Box} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\right) = \left(\vec{\nabla}, \frac{1}{ic}\frac{\partial}{\partial t}\right) \tag{49}$$

here $\vec{\nabla}$ is the usual 3-dimensional gradient. The transformation properties for the components of $\vec{\Box}$ can be obtained by applying the familiar chain rule for differentiation:

$$\frac{\partial}{\partial x_j^{/}} = \sum_{k=1}^{4} \frac{\partial x_k}{\partial x_j^{/}} \frac{\partial}{\partial x_k}$$

However we know from the inverse Lorentz Transformation (equations (15), (29)):

$$x_k = \sum_{j=1}^{4} a_{jk} x_j^{/} \quad \text{thus:} \quad \frac{\partial x_k}{\partial x_j^{/}} = a_{jk}$$

hence:

$$\frac{\partial}{\partial x_j^{/}} = \sum_{k=1}^{4} a_{jk} \frac{\partial}{\partial x_k}$$

which indicates directly that the four components of $\vec{\Box}$ transform in exactly the same way as the components of $\vec{R}$; this is the basic definition of a 4-vector, hence the operator $\vec{\Box}$ is a *4-vector*.

The dot product $\vec{\Box} . \vec{\Box}$ forms the *4-D Laplacian operator* $\Box^2$ (– called the D'Alembertian):

$$\vec{\Box} . \vec{\Box} = \Box^2 = \sum_{k=1}^{4} \frac{\partial^2}{\partial x_k^2} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{50}$$

Here ∇^2 is the usual 3-D Laplacian; the operator $\Box^2$ is a *Lorentz invariant*.

$$(\Box^{\prime})^2 = \frac{\partial^2}{\partial x_1^{\prime 2}} + \frac{\partial^2}{\partial x_2^{\prime 2}} + \frac{\partial^2}{\partial x_3^{\prime 2}} + \frac{\partial^2}{\partial x_4^{\prime 2}} = \sum_{i=1}^{4} \frac{\partial^2}{\partial x_i^{\prime 2}}$$

Write

$$\frac{\partial^2}{\partial x_i^{\prime 2}} = \frac{\partial}{\partial x_i^{\prime}} \frac{\partial}{\partial x_i^{\prime}}$$

and recall that

$$\frac{\partial}{\partial x_i^{\prime}} = \sum_{j=1}^{4} a_{ij} \frac{\partial}{\partial x_j} \qquad \Box \text{ a 4-vector!}$$

so

$$\frac{\partial}{\partial x_i^{\prime}} \frac{\partial}{\partial x_i^{\prime}} = \sum_{j=1}^{4} a_{ij} \frac{\partial^2}{\partial x_i^{\prime} \partial x_j} = \sum_{j=1}^{4} \sum_{k=1}^{4} a_{ij} a_{ik} \frac{\partial^2}{\partial x_k \partial x_j}$$

Therefore

$$(\Box^{\prime})^2 = \sum_{i=1}^{4} \sum_{j=1}^{4} \sum_{k=1}^{4} a_{ij} a_{ik} \frac{\partial^2}{\partial x_k \partial x_j} = \sum_{j=1}^{4} \sum_{k=1}^{4} \Big(\underbrace{\sum_{i=1}^{4} a_{ij} a_{ik}}_{\delta_{jk}} \Big) \frac{\partial^2}{\partial x_k \partial x_j}$$

$$= \sum_{j=1}^{4} \sum_{k=1}^{4} \delta_{jk} \frac{\partial^2}{\partial x_k \partial x_j} = \sum_{j=1}^{4} \frac{\partial^2}{\partial x_j^2} = \Box^2$$

The equation of continuity – which describes charge conservation – relates the current density $\vec{J}$ to the charge density ρ according to:

$$\vec{\nabla} . \vec{J} + \frac{\partial \rho}{\partial t} = 0.$$

Rewrite the above equation as:

$$\vec{\nabla} . \vec{J} + \frac{1}{ic} \frac{\partial}{\partial t} (i c \rho) = 0 \qquad (51)$$

and hence define a 4-D current density $\vec{J}$ by

$$\vec{J} = (\vec{J}, i c \rho) \qquad (52)$$

[This is dimensionally consistent: $\vec{J}$ = charge/unit area-sec; $ic\rho$ = (charge/unit volume) × length/sec = charge/unit area-sec].

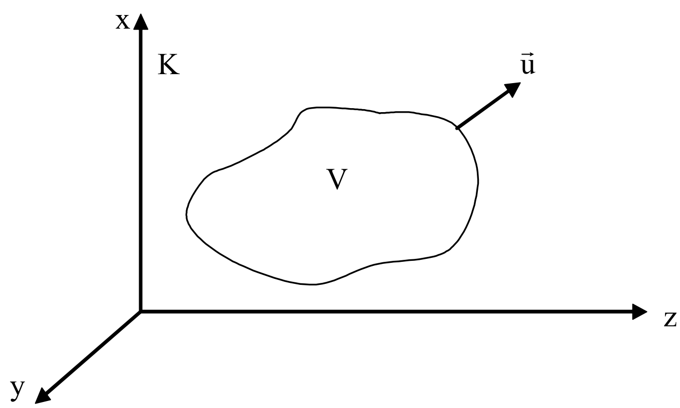
Here $\vec{J}$ is the usual 3-D current density, and (51) becomes

$$\frac{\partial J_1}{\partial x_1} + \frac{\partial J_2}{\partial x_2} + \frac{\partial J_3}{\partial x_3} + \frac{\partial J_4}{\partial x_4} = 0$$

so that the 4-D form for the equation of continuity becomes

$$\vec{\Box} \cdot \vec{J} = 0 \tag{53}$$

To help visualize this connection, consider a cloud of charge with volume V and velocity $\vec{u}$ as measured by an observer in the system K:



Since objects are contracted along the *direction of relative motion* with respect to observers in system K, the volume V measured in K is related to the proper volume V_0 in the rest frame of the cloud by

$$V = V_0 \sqrt{1 - \frac{u^2}{c^2}}$$

The charge density measured in the two frames is then

$$\rho_0 = \frac{q_0}{V_0} \qquad \text{(in the rest frame of the cloud)}$$

$$\rho = \frac{q}{V} \qquad \text{(in K)}$$

– and since the total charge is *invariant* (the total charge q_0 is an integer multiple N_0 of elementary charges, so if the basic unit of charge is invariant, then $q = q_0$, since counting (N_0) is an invariant process).

Therefore

$$\rho_0 V_0 = \rho V = \rho V_0 \sqrt{1 - \frac{u^2}{c^2}}$$

Therefore

$$\rho = \frac{\rho_0}{\sqrt{1 - \frac{u^2}{c^2}}} \qquad (54)$$

The 3-D current density in K is $\vec{J} = \rho \vec{u}$, and hence

$$\vec{\mathcal{J}} = (\vec{J}, ic\rho) = (\rho\vec{u}, ic\rho)$$

Thus:

$$\vec{\mathcal{J}} = \rho_0 \left(\frac{\vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}}, \frac{ic}{\sqrt{1 - \frac{u^2}{c^2}}} \right) = \rho_0 \vec{V} \qquad (55)$$

where $\vec{V}$ is the 4-vector velocity of equation (33). Since ρ_0 is a scalar invariant, then it is clear from (55) that $\vec{\mathcal{J}}$ is also a 4-vector.

Applying the usual 4-vector transformation equations – (29) – to the 4-vector

$$\vec{\mathcal{J}} = (\vec{J}, ic\rho) = (J_x, J_y, J_z, ic\rho) = (J_1, J_2, J_3, ic\rho),$$

we get:

	INVERSE
$J_1' = J_1$	$J_1 = J_1'$
$J_2' = J_2$	$J_2 = J_2'$
$J_3' = \gamma(J_3 - v\rho)$	$J_3 = \gamma(J_3' + v\rho')$
$\rho' = \gamma\left(\rho - \frac{v}{c^2}J_3\right)$	$\rho = \gamma\left(\rho' + \frac{v}{c^2}J_3'\right)$

$$J_3' = \gamma\left(J_3 + \frac{iv}{c}J_4\right) = \gamma\left(J_3 + \frac{iv}{c}.\,ic\rho\right) = \gamma(J_3 - v\rho)$$

$$J_4' = \gamma\left(J_4 - \frac{iv}{c}J_3\right)$$

Therefore

$$ic\rho' = \gamma\left(ic\rho - \frac{iv}{c}J_3\right);$$

and hence

$$\rho' = \gamma\left(\rho - \frac{v}{c^2}J_3\right)$$

Recall that in free space; $\mu = \mu_0$, $\varepsilon = \varepsilon_0$, the magnetic vector potential $\vec{A}$ and the scalar potential V_c obeyed two uncoupled though inhomogeneous wave equations in the Lorentz Gauge

$$\nabla^2 V_c - \frac{1}{c^2}\frac{\partial^2 V_c}{\partial t^2} = -\frac{\rho_f}{\varepsilon_0} \quad (56)$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}_f \quad (57)$$

(having replaced $\mu_0\varepsilon_0$ by $\frac{1}{c^2}$), with the Lorentz Gauge being one in which

$$\vec{\nabla}.\vec{A} + \frac{1}{c^2}\frac{\partial V_c}{\partial t} = 0$$

which can be rewritten:

$$\vec{\nabla}.\vec{A} + \frac{1}{ic}\frac{\partial}{\partial t}\left(\frac{iV_c}{c}\right) = 0 \quad (58)$$

This latter equation suggests that we can define a *4-D potential* $\vec{\Phi}$ with components

$$\vec{\Phi} = \left(\vec{A}, i\,\frac{V_c}{c} \right) \qquad \text{(free space)} \qquad (59)$$

for the Lorentz condition (58) to get:

$$\frac{\partial \Phi_1}{\partial x_1} + \frac{\partial \Phi_2}{\partial x_2} + \frac{\partial \Phi_3}{\partial x_3} + \frac{\partial \Phi_4}{\partial x_4} = 0$$

i.e.

$$\vec{\Box} \,.\, \vec{\Phi} = 0 \qquad \text{(free space)} \qquad (60)$$

Now the wave-equations (56) and (57) can also be expressed in terms of $\Box^2$ as follows:

$$\Box^2 \vec{A} = -\mu_0 \vec{J}_f$$

$$\Box^2 \left(\frac{i V_c}{c} \right) = -\frac{i \rho_f}{\varepsilon_0 c} = -\mu_0 (ic\,\rho_f) \qquad \text{since } \frac{1}{\varepsilon_0} = \mu_0 c^2$$

which can be combined into:

$$\Box^2 \vec{\Phi} = -\mu_0 \vec{J}_f \qquad (61)$$

Since $\vec{J}_f$ is a 4-vector and $\Box^2$ is a Lorentz invariant operator, it follows that the 4-D potential must also be a 4-vector.

Using the usual 4-vector transformation equations – (29) – then we get:

	INVERSE
$A_1^{\prime} = A_1$	$A_1 = A_1^{\prime}$
$A_2^{\prime} = A_2$	$A_2 = A_2^{\prime}$
$A_3^{\prime} = \gamma \left(A_3 - \frac{v}{c^2} V_c \right)$	$A_3 = \gamma \left(A_3^{\prime} + \frac{v}{c^2} V_c^{\prime} \right)$
$V_c^{\prime} = \gamma \left(V_c - v A_3 \right)$	$V_c = \gamma \left(V_c^{\prime} + v A_3^{\prime} \right)$

$$\vec{\Phi} = (\Phi_1, \Phi_2, \Phi_3, \Phi_4) = \left(A_1, A_2, A_3, \frac{i V_c}{c} \right)$$

$$\Phi_1^{/} = \Phi_1 \quad \Rightarrow \quad A_1^{/} = A_1$$

$$\Phi_2^{/} = \Phi_2 \quad \Rightarrow \quad A_2^{/} = A_2$$

$$\Phi_3^{/} = \gamma\left(\Phi_3 + \frac{i v}{c}\Phi_4 \right) \quad \Rightarrow \quad A_3^{/} = \gamma\left(A_3 + \frac{i v}{c} . \frac{i V_c}{c} \right) = \gamma\left(A_3 - \frac{v}{c^2} V_c \right)$$

$$\Phi_4^{/} = \gamma\left(\Phi_4 - \frac{i v}{c}\Phi_3 \right) \quad \Rightarrow \quad \frac{i V_c^{/}}{c} = \gamma\left(\frac{i V_c}{c} - \frac{i v}{c} A_3 \right)$$

Therefore

$$V_c^{/} = \gamma\left(V_c - v A_3 \right)$$

and the inverse transformation follows directly.

9. The Electromagnetic Field Tensor $\tilde{\mathbf{F}}$

Recall the Lorentz Transformation equations (29) of section 6:

$$x_k^{/} = \sum_{j=1}^{4} a_{kj} x_j \; ; \quad k = 1, 2, 3, 4$$

with $x_1 = x; \; x_2 = y; \; x_3 = z; \; x_4 = i c t$

The coefficients a_{kj} are elements of the transformation matrix A, where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & \frac{i v \gamma}{c} \\ 0 & 0 & -\frac{i v \gamma}{c} & \gamma \end{bmatrix} ; \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Furthermore, we showed

$$\sum_{j=1}^{4} a_{jk} a_{j\ell} = \delta_{\ell k} \quad ; \quad k,\ell = 1,2,3,4$$

We have already defined a 4-vector $\vec{M}$ [a vector in the four dimensional space (x_1, x_2, x_3, x_4)] as one that transforms as

$$m_k^{/} = \sum_{j=1}^{4} a_{kj} m_j \quad ; \quad k = 1,2,3,4 \quad ; \quad \vec{M} = (m_1, m_2, m_3, m_4)$$

with the a_{kj} given previously.

Whereas the 4-D current density

$$\vec{\mathcal{J}} = (\vec{J}, ic\rho)$$

and the 4-D (free space) potential

$$\vec{\Phi} = \left(\vec{A}, i\frac{V_c}{c} \right)$$

are 4-vectors (i.e. they obey Lorentz transformation equations), $\vec{E}$ and $\vec{B}$ cannot be cast in 4-vector form (i.e. they transform differently). One can get some idea for why this happens by looking at the non-relativistic relationships:

$$\vec{E}(\vec{R},t) = -\vec{\nabla} V_c(\vec{R},t) - \frac{\partial \vec{A}(\vec{R},t)}{\partial t} \quad ; \quad \vec{B}(\vec{R},t) = \vec{\nabla} \times \vec{A}(\vec{R},t)$$

If we try to write the first of these in a 4-vector form we might "guess":

$$"\vec{E}_{4V}" = -\vec{\Box} \,.\, \vec{\Phi} = -\left(\vec{\nabla}, \frac{1}{ic} \frac{\partial}{\partial t} \right) . \left(\vec{A}, \frac{iV_c}{c} \right)$$

$$= -\vec{\nabla} \,.\, \vec{A} - \frac{1}{c^2} \frac{\partial V_c}{\partial t}$$

– not only is this *not* a vector, but it also vanishes in free space (in the Lorentz Gauge) and the "derivatives" are the wrong way around.

This "reversal" of the derivatives is reminiscent of a cross-product i.e.

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \rightarrow \quad B_i = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}$$

and this hints at how we proceed.

Furthermore, even if the above scheme worked, what would the "4th component" of $\vec{E}$ and $\vec{B}$ become?

$$\vec{E}(\vec{R},t) = -\vec{\nabla} V_c(\vec{R},t) - \frac{\partial \vec{A}}{\partial t}(\vec{R},t)$$

$$\vec{B}(\vec{R},t) = \vec{\nabla} \times \vec{A}(\vec{R},t)$$

– give the $\vec{E}$ and $\vec{B}$ fields in, say, frame K with respect to derivatives $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}$ of the "coordinates" (space and time) measured in that frame. Similarly we would define

$$\vec{E}' = -\vec{\nabla}' V_c' - \frac{\partial \vec{A}'}{\partial t'} \quad \text{and} \quad \vec{B}' = \vec{\nabla}' \times \vec{A}'$$

i.e.

$$E_x' = -\frac{\partial V_c'}{\partial x'} - \frac{\partial A_x'}{\partial t'} = -\frac{\partial V_c'}{\partial x}\frac{\partial x}{\partial x'} - \frac{\partial A_x'}{\partial t}\frac{\partial t}{\partial t'} - \frac{\partial A_x'}{\partial z}\frac{\partial z}{\partial t'}$$

(using the Chain Rule.)

Lorentz transformation:

$$x = x' \qquad\qquad A_x' = A_x$$

$$y = y' \qquad\qquad A_y' = A_y$$

$$z = \gamma(z' + vt') \qquad\qquad A_z' = \gamma\left(A_z - \frac{v}{c^2}V_c\right)$$

$$t = \gamma\left(t' + \frac{vz'}{c^2}\right) \qquad\qquad V_c' = \gamma(V_c - vA_z)$$

Therefore

$$\frac{\partial x}{\partial x'} = 1 \; : \quad \frac{\partial t}{\partial t'} = \gamma ; \quad \frac{\partial z}{\partial t'} = \gamma v$$

Therefore

$$E_x' = -1 \,.\, \frac{\partial}{\partial x}[\gamma V_c - \gamma v A_z] - \gamma \frac{\partial}{\partial t}(A_x) - \gamma v \frac{\partial}{\partial z}(A_x)$$

$$= -\gamma \frac{\partial V_c}{\partial x} - \gamma \frac{\partial A_x}{\partial t} - \gamma v \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right] = +\gamma E_x - \gamma v B_y$$

Therefore $$E_x' = \gamma(E_x - v B_y)$$

Similarly,

$$E_y' = -\frac{\partial V_c'}{\partial y'} - \frac{\partial A_y'}{\partial t'} = -\frac{\partial}{\partial y}[\gamma V_c - \gamma v A_z] - \gamma \frac{\partial}{\partial t}(A_y) - \gamma v \frac{\partial}{\partial z}(A_y)$$

$$= \gamma \left[-\frac{\partial V_c}{\partial y} - \frac{\partial A_y}{\partial t} \right] + \gamma v \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right]$$

Therefore

$$E_y' = \gamma [E_y + v B_x]$$

Also,

$$E_z' = -\frac{\partial V_c'}{\partial z'} - \frac{\partial A_z'}{\partial t'}$$

Recall that

$$\frac{\partial z}{\partial z'} = \gamma \; ; \quad \frac{\partial t}{\partial z'} = \frac{\gamma v}{c^2}$$

Therefore

$$E_z' = -\gamma \frac{\partial V_c'}{\partial z} - \frac{\gamma v}{c^2}\frac{\partial V_c'}{\partial t} - \gamma \frac{\partial}{\partial t} A_z' - \gamma v \frac{\partial}{\partial z} A_z'$$

$$= -\gamma \frac{\partial}{\partial z}[\gamma V_c - \gamma v A_z] - \frac{\gamma v}{c^2}\frac{\partial}{\partial t}[\gamma V_c - \gamma v A_z] - \gamma \frac{\partial}{\partial t}\left[\gamma A_z - \frac{\gamma v}{c^2} V_c\right]$$

$$- \gamma v \frac{\partial}{\partial z}\left[\gamma A_z - \frac{\gamma v}{c^2} V_c\right]$$

$$= -\gamma^2 \frac{\partial V_c}{\partial z} + \gamma^2 v \frac{\partial A_z}{\partial z} - \frac{\gamma^2 v}{c^2}\frac{\partial V_c}{\partial t} + \frac{\gamma^2 v^2}{c^2}\frac{\partial A_z}{\partial t} - \gamma^2 \frac{\partial A_z}{\partial t} + \frac{\gamma^2 v}{c^2}\frac{\partial V_c}{\partial t}$$

$$- \gamma^2 v \frac{\partial A_z}{\partial z} + \frac{\gamma^2 v^2}{c^2}\frac{\partial V_c}{\partial z}$$

$$= -\gamma^2 \frac{\partial V_c}{\partial z}\left[1 - \frac{v^2}{c^2}\right] - \gamma^2 \frac{\partial A_z}{\partial t}\left[1 - \frac{v^2}{c^2}\right]$$

$$= -\frac{\partial V_c}{\partial z} - \frac{\partial A_z}{\partial t} \qquad \text{as } \gamma^2\left[1 - \frac{v^2}{c^2}\right] = 1$$

Therefore $\qquad E_z' = E_z$

Next examine

$$B_x' = \frac{\partial A_z'}{\partial y'} - \frac{\partial A_y'}{\partial z'} = \frac{\partial}{\partial y}(A_z') - \gamma \frac{\partial}{\partial z}(A_y') - \frac{\gamma v}{c^2}\frac{\partial}{\partial t}(A_y')$$

$$= \frac{\partial}{\partial y}\left(\gamma A_z - \frac{\gamma v}{c^2} V_c\right) - \gamma \frac{\partial}{\partial z}(A_y) - \frac{\gamma v}{c^2}\frac{\partial}{\partial t}(A_y)$$

$$= \gamma\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\gamma v}{c^2}\left(-\frac{\partial V_c}{\partial y} - \frac{\partial A_y}{\partial t}\right) = \gamma B_x + \frac{\gamma v}{c^2} E_y$$

Therefore

$$B_x' = \gamma\left(B_x + \frac{v}{c^2} E_y \right)$$

Similarly,

$$B_y' = \frac{\partial A_x'}{\partial z'} - \frac{\partial A_z'}{\partial x'} = \gamma \frac{\partial A_x}{\partial z} + \frac{\gamma v}{c^2}\frac{\partial A_x'}{\partial t} - \frac{\partial A_z'}{\partial x}$$

$$= \gamma \frac{\partial A_x}{\partial z} + \frac{\gamma v}{c^2}\frac{\partial A_x}{\partial t} - \frac{\partial}{\partial x}\left[\gamma A_z - \frac{\gamma v}{c^2} V_c\right]$$

$$= \gamma \frac{\partial A_x}{\partial z} + \frac{\gamma v}{c^2}\frac{\partial A_x}{\partial t} - \gamma \frac{\partial A_z}{\partial x} + \frac{\gamma v}{c^2}\frac{\partial V_c}{\partial x} = \gamma\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) - \frac{\gamma v}{c^2}\left(-\frac{\partial V_c}{\partial x} - \frac{\partial A_x}{\partial t}\right)$$

Therefore

$$B_y' = \gamma\left(B_y - \frac{v}{c^2} E_x \right)$$

Finally,

$$B_z' = \frac{\partial A_y'}{\partial x'} - \frac{\partial A_x'}{\partial y'} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_z$$

In reality $\vec{E}$ and $\vec{B}$ do *NOT* transform as *separate* 4-vector fields, but combine *together* to transform not as a 4-vector but as elements of a *4-D second rank tensor*:

4-*vectors* transform as

$$m_k' = \sum_{j=1}^{4} a_{kj} m_j \ ; \quad k = 1,2,3,4$$

whereas 4-D *second* rank tensors transform according to:

$$F_{jk}' = \sum_{\ell=1}^{4} \sum_{m=1}^{4} a_{j\ell} a_{km} F_{\ell m} \ ; \quad j,k = 1,2,3,4$$

So there will (in general) be 16 elements in the 4-D second rank tensor $\tilde{F}$ (cf. 4 in the 4-D vector or first rank tensor $\vec{M}$).

Following the above remarks, suppose we try to take the equivalent of 4-D "curl" of the 4-vector potential $\vec{\Phi}$ (which gives us the *electromagnetic field tensor* $\tilde{F}$); i.e. take

$$\tilde{F} = \vec{\Box} \times \vec{\Phi} \tag{62}$$

by which we mean that the elements or components of $\tilde{F}$ are given by (in analogy with the 3-D case)

$$F_{jk} = \frac{\partial \Phi_k}{\partial x_j} - \frac{\partial \Phi_j}{\partial x_k} \; ; \quad j,k = 1,2,3,4 \tag{63}$$

(this certainly "mixes" the derivatives as the 3-D forms require); in general it has 16 "elements" or components. Notice however that

$$F_{kj} = \frac{\partial \Phi_j}{\partial x_k} - \frac{\partial \Phi_k}{\partial x_j} = -F_{jk}$$

so that the matrix of the elements of this tensor is antisymmetric (the field tensor is antisymmetric). Thus the matrix F which consists of the elements of $\tilde{F}$ has the form:

$$F = \begin{bmatrix} 0 & F_{12} & F_{13} & F_{14} \\ -F_{12} & 0 & F_{23} & F_{24} \\ -F_{13} & -F_{23} & 0 & F_{34} \\ -F_{14} & -F_{24} & -F_{34} & 0 \end{bmatrix}$$

so that there are only *6* independent elements. They are

$$F_{12} = \frac{\partial \Phi_2}{\partial x_1} - \frac{\partial \Phi_1}{\partial x_2} \equiv \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} = B_3$$

$$F_{13} = \frac{\partial \Phi_3}{\partial x_1} - \frac{\partial \Phi_1}{\partial x_3} \equiv \frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3} = -B_2$$

$$F_{23} = \frac{\partial \Phi_3}{\partial x_2} - \frac{\partial \Phi_2}{\partial x_3} \equiv \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} = B_1$$

$$F_{14} = \frac{\partial \Phi_4}{\partial x_1} - \frac{\partial \Phi_1}{\partial x_4} \equiv \frac{i}{c}\frac{\partial V_c}{\partial x_1} - \frac{1}{ic}\frac{\partial A_1}{\partial t} = -\frac{i}{c}\left(-\frac{\partial V_c}{\partial x} - \frac{\partial A_x}{\partial t}\right) = -\frac{i}{c}E_x = -\frac{i}{c}E_1$$

$$F_{24} = \frac{\partial \Phi_4}{\partial x_2} - \frac{\partial \Phi_2}{\partial x_4} = \frac{i}{c}\frac{\partial V_c}{\partial x_2} - \frac{1}{ic}\frac{\partial A_2}{\partial t} = -\frac{i}{c}\left(-\frac{\partial V_c}{\partial y} - \frac{\partial A_y}{\partial t}\right) = -\frac{iE_y}{c} = -\frac{iE_2}{c}$$

$$F_{34} = \frac{\partial \Phi_4}{\partial x_3} - \frac{\partial \Phi_3}{\partial x_4} = \frac{i}{c}\frac{\partial V_c}{\partial x_3} - \frac{1}{ic}\frac{\partial A_3}{\partial t} = -\frac{i}{c}\left(-\frac{\partial V_c}{\partial z} - \frac{\partial A_z}{\partial t}\right) = -\frac{iE_z}{c} = -\frac{iE_3}{c}$$

So the matrix of the tensor $\tilde{F}$ becomes

$$F = \begin{bmatrix} 0 & B_3 & -B_2 & -i\frac{E_1}{c} \\ -B_3 & 0 & B_1 & -i\frac{E_2}{c} \\ B_2 & -B_1 & 0 & -i\frac{E_3}{c} \\ i\frac{E_1}{c} & i\frac{E_2}{c} & i\frac{E_3}{c} & 0 \end{bmatrix} \tag{64}$$

Now recall the non-relativistic form of Maxwell's equations in *free space*:

$$\vec{\nabla} . \vec{B} = 0 \tag{65}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0} \tag{66}$$

and

$$\vec{\nabla} . \vec{E} = \frac{\rho_f}{\varepsilon_0} \tag{67}$$

$$\vec{\nabla} \times \vec{B} - \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}_f \tag{68}$$

We can re-express (65) and (66) in component form as:

$$\frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3} = 0$$

or

$$\frac{\partial F_{23}}{\partial x_1} + \frac{\partial F_{31}}{\partial x_2} + \frac{\partial F_{12}}{\partial x_3} = 0 \qquad (69)$$

while:

$$\frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3} + \frac{\partial B_1}{\partial t} = 0 \quad \text{becomes}$$

$$ic\frac{\partial F_{34}}{\partial x_2} + ic\frac{\partial F_{42}}{\partial x_3} + ic\frac{\partial F_{23}}{\partial x_4} = 0 \qquad (70)$$

$$\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1} + \frac{\partial B_2}{\partial t} = 0 \quad \text{becomes}$$

$$ic\frac{\partial F_{14}}{\partial x_3} + ic\frac{\partial F_{43}}{\partial x_1} + ic\frac{\partial F_{31}}{\partial x_4} = 0,$$

or, multiplying both sides of the equation by -1:

$$ic\frac{\partial F_{41}}{\partial x_3} + ic\frac{\partial F_{34}}{\partial x_1} + ic\frac{\partial F_{13}}{\partial x_4} = 0 \qquad (71)$$

Finally

$$\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2} + \frac{\partial B_3}{\partial t} = 0 \quad \text{can be written as}$$

$$ic\frac{\partial F_{24}}{\partial x_1} + ic\frac{\partial F_{41}}{\partial x_2} + ic\frac{\partial F_{12}}{\partial x_4} = 0 \qquad (72)$$

Thus the two homogeneous Maxwell's Equations (65) and (66) can be written in a compact form in terms of derivatives of components of the field tensor $\tilde{F}$:

$$\frac{\partial F_{jk}}{\partial x_\ell} + \frac{\partial F_{k\ell}}{\partial x_j} + \frac{\partial F_{\ell j}}{\partial x_k} = 0 \;;\quad (j \neq k \neq \ell) \tag{73}$$

with $(j\,k\,\ell) = (1\,2\,3)$ yielding (65) and $(j\,k\,\ell) = (1\,2\,4),\ (1\,3\,4),\ (2\,3\,4)$ yielding (66).

In a similar way the component forms of (67) and (68) yield

$$\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3} = \frac{\rho_f}{\varepsilon_0} = ic\frac{\partial F_{14}}{\partial x_1} + ic\frac{\partial F_{24}}{\partial x_2} + ic\frac{\partial F_{34}}{\partial x_3}$$

and with

$$\frac{\rho_f}{\varepsilon_0} = \mu_0 c^2 \rho_f = -ic\,\mu_0(ic\,\rho_f),$$

then

$$\frac{\partial F_{14}}{\partial x_1} + \frac{\partial F_{24}}{\partial x_2} + \frac{\partial F_{34}}{\partial x_3} = -\mu_0(ic\,\rho_f) = -\mu_0 J_4 \tag{74}$$

On the other hand

$$\frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3} - \frac{1}{c^2}\frac{\partial E_1}{\partial t} = \mu_0 J_{f1} = -\frac{\partial F_{21}}{\partial x_2} - \frac{\partial F_{31}}{\partial x_3} - \frac{\partial F_{41}}{\partial x_4}$$

So:

$$\frac{\partial F_{21}}{\partial x_2} + \frac{\partial F_{31}}{\partial x_3} + \frac{\partial F_{41}}{\partial x_4} = -\mu_0 J_1 \tag{75}$$

Also:

$$\frac{\partial B_1}{\partial x_3} - \frac{\partial B_3}{\partial x_1} - \frac{1}{c^2}\frac{\partial E_2}{\partial t} = \mu_0 J_{f2} = -\frac{\partial F_{32}}{\partial x_3} - \frac{\partial F_{12}}{\partial x_1} - \frac{\partial F_{42}}{\partial x_4}$$

therefore

$$\frac{\partial F_{12}}{\partial x_1} + \frac{\partial F_{32}}{\partial x_3} + \frac{\partial F_{42}}{\partial x_4} = -\mu_0 J_2 \tag{76}$$

Finally:

$$\frac{\partial B_2}{\partial x_1} - \frac{\partial B_1}{\partial x_2} - \frac{1}{c^2}\frac{\partial E_3}{\partial t} = \mu_0 J_{f3} = -\frac{\partial F_{13}}{\partial x_1} - \frac{\partial F_{23}}{\partial x_2} - \frac{\partial F_{43}}{\partial x_4}$$

therefore

$$\frac{\partial F_{13}}{\partial x_1} + \frac{\partial F_{23}}{\partial x_2} + \frac{\partial F_{43}}{\partial x_4} = -\mu_0 J_3 \tag{77}$$

So that the two inhomogeneous Maxwell's Equations (67) and (68) contract to:

$$\sum_{j=1}^{4} \frac{\partial F_{jk}}{\partial x_j} = -\mu_0 J_k \quad : \quad k = 1,2,3,4 \tag{78}$$

[Recall that the diagonal elements of $\tilde{F}$ are zero, and

$$J = (\vec{J}, ic\rho) \rightarrow (\vec{J}_f, ic\rho_f)$$

in free space.]

10. Transformation Properties of the $\vec{E}$ and $\vec{B}$ Fields

It was asserted previously that these fields combine together to transform as elements F_{jk} of a 4-D second rank tensor – the electromagnetic field tensor $\tilde{F}$. In order to verify this it will be necessary to use the transformation equations for the 4-vector gradient

$$\vec{\square} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\right) = \left(\vec{\nabla}, \frac{1}{ic}\frac{\partial}{\partial t}\right)$$

and the 4-vector potential

$$\vec{\Phi} = \left(\vec{A}, \frac{iV_c}{c}\right) = (\Phi_1, \Phi_2, \Phi_3, \Phi_4)$$

Both are 4-vectors and hence transform as:

$$\frac{\partial}{\partial x_j'} = \sum_{\ell=1}^{4} a_{j\ell}\frac{\partial}{\partial x_\ell} \qquad \text{(Lorentz Transformation)}$$

and

$$\Phi_k^{\prime} = \sum_{m=1}^{4} a_{km} \Phi_m$$

So:

$$F_{jk}^{\prime} = \frac{\partial \Phi_k^{\prime}}{\partial x_j^{\prime}} - \frac{\partial \Phi_j^{\prime}}{\partial x_k^{\prime}}$$

by definition becomes

$$F_{jk}^{\prime} = \frac{\partial}{\partial x_j^{\prime}} \left(\sum_{m=1}^{4} a_{km} \Phi_m \right) - \frac{\partial}{\partial x_k^{\prime}} \left(\sum_{n=1}^{4} a_{jn} \Phi_n \right)$$

$$= \sum_{m=1}^{4} a_{km} \frac{\partial \Phi_m}{\partial x_j^{\prime}} - \sum_{n=1}^{4} a_{jn} \frac{\partial \Phi_n}{\partial x_k^{\prime}}$$

Using the chain rule:

$$F_{jk}^{\prime} = \sum_{m=1}^{4} a_{km} \left(\sum_{i=1}^{4} \frac{\partial \Phi_m}{\partial x_i} \frac{\partial x_i}{\partial x_j^{\prime}} \right) - \sum_{n=1}^{4} a_{jn} \left(\sum_{p=1}^{4} \frac{\partial \Phi_n}{\partial x_p} \frac{\partial x_p}{\partial x_k^{\prime}} \right)$$

$$= \sum_{m=1}^{4} \sum_{i=1}^{4} a_{km} a_{ji} \frac{\partial \Phi_m}{\partial x_i} - \sum_{n=1}^{4} \sum_{p=1}^{4} a_{jn} a_{kp} \frac{\partial \Phi_n}{\partial x_p}$$

$$= \sum_{m=1}^{4} \sum_{i=1}^{4} a_{km} a_{ji} \frac{\partial \Phi_m}{\partial x_i} - \sum_{i=1}^{4} \sum_{m=1}^{4} a_{ji} a_{km} \frac{\partial \Phi_i}{\partial x_m}$$

$$= \sum_{m=1}^{4} \sum_{i=1}^{4} a_{ji} a_{km} \frac{\partial \Phi_m}{\partial x_i} - \sum_{m=1}^{4} \sum_{i=1}^{4} a_{ji} a_{km} \frac{\partial \Phi_i}{\partial x_m}$$

$$= \sum_{m=1}^{4} \sum_{i=1}^{4} a_{ji} a_{km} \left(\frac{\partial \Phi_m}{\partial x_i} - \frac{\partial \Phi_i}{\partial x_m} \right)$$

But by definition:

$$F_{im} = \frac{\partial \Phi_m}{\partial x_i} - \frac{\partial \Phi_i}{\partial x_m};$$

therefore

$$F_{jk}^{/} = \sum_{m=1}^{4} \sum_{i=1}^{4} a_{ji} a_{km} F_{im} = \sum_{i=1}^{4} \sum_{m=1}^{4} a_{ji} a_{km} F_{im} \tag{79}$$

which was the form asserted previously.

Equations (79) can be re-expressed in matrix form

$$F_{jk}^{/} = \sum_{i=1}^{4} \sum_{m=1}^{4} a_{ji} F_{im} a_{km} = \sum_{i=1}^{4} \sum_{m=1}^{4} a_{ji} F_{im} a_{mk}^{T}$$

where T means transpose. Therefore

$$F^{/} = A F A^{T}$$

If we perform this matrix multiplication, we should obtain again:

$$\boxed{\begin{aligned} E_1^{/} &= \gamma(E_1 - vB_2) & B_1^{/} &= \gamma\left(B_1 + \frac{vE_2}{c^2}\right) \\ E_2^{/} &= \gamma(E_2 + vB_1) & B_2^{/} &= \gamma\left(B_2 - \frac{vE_1}{c^2}\right) \\ E_3^{/} &= E_3 & B_3^{/} &= B_3 \end{aligned}} \tag{80}$$

which can be generalized (recall $\vec{v} = v\hat{k}$ here) to the case where $K^{/}$ moves in an arbitrary direction at constant velocity $\vec{v}$ with respect to system K, to yield

$$\begin{aligned} \vec{E}_{\parallel}^{/} &= \vec{E}_{\parallel} & \vec{E}_{\perp}^{/} &= \gamma(\vec{E}_{\perp} + \vec{v} \times \vec{B}_{\perp}) \\ \vec{B}_{\parallel}^{/} &= \vec{B}_{\parallel} & \vec{B}_{\perp}^{/} &= \gamma\left(\vec{B}_{\perp} - \frac{1}{c^2}\vec{v} \times \vec{E}_{\perp}\right) \end{aligned} \tag{81}$$

(here $\parallel$ and $\perp$ mean parallel and perpendicular to $\vec{v}$).

Notice that we can *always* choose a $\hat{k}$ direction to be along $\vec{v}$, so

$$\vec{E}_\perp = \vec{E}_1 + \vec{E}_2 \; ; \quad \vec{B}_\perp = \vec{B}_1 + \vec{B}_2$$

$$\vec{E}_\parallel = \vec{E}_3 \; ; \quad \vec{B}_\parallel = \vec{B}_3$$

Therefore

$$\vec{E}_\parallel^{\,\prime} = \vec{E}_\parallel \quad \text{and} \quad \vec{B}_\parallel^{\,\prime} = \vec{B}_\parallel$$

while

$$\vec{E}_\perp^{\,\prime} = \gamma(E_1 - vB_2)\hat{i} + \gamma(E_2 + vB_1)\hat{j}$$

$$= \gamma(\vec{E}_1 + \vec{E}_2 + v(B_1\hat{j} - B_2\hat{i}))$$

But

$$v(B_1\hat{j} - B_2\hat{i}) = v\hat{k} \times (B_1\hat{i} + B_2\hat{j}) = \vec{v} \times \vec{B}_\perp$$

Therefore

$$\vec{E}_\perp^{\,\prime} = \gamma(\vec{E}_\perp + \vec{v} \times \vec{B}_\perp)$$

Similarly

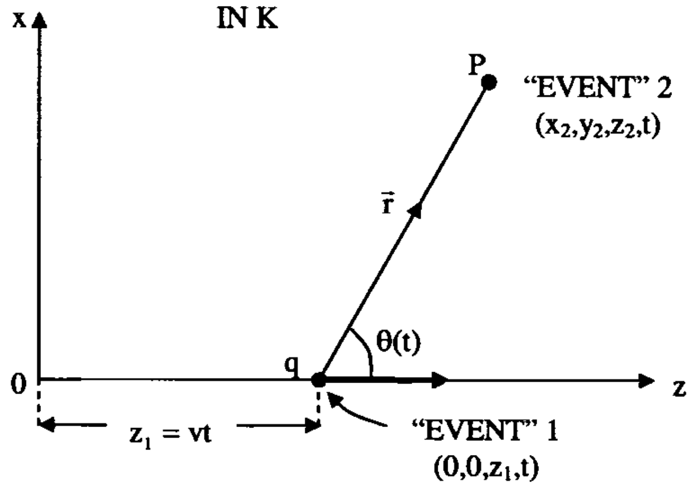
$$\vec{B}_\perp^{\,\prime} = \gamma\left(B_1 + \frac{vE_2}{c^2}\right)\hat{i} + \gamma\left(B_2 - \frac{vE_1}{c^2}\right)\hat{j}$$

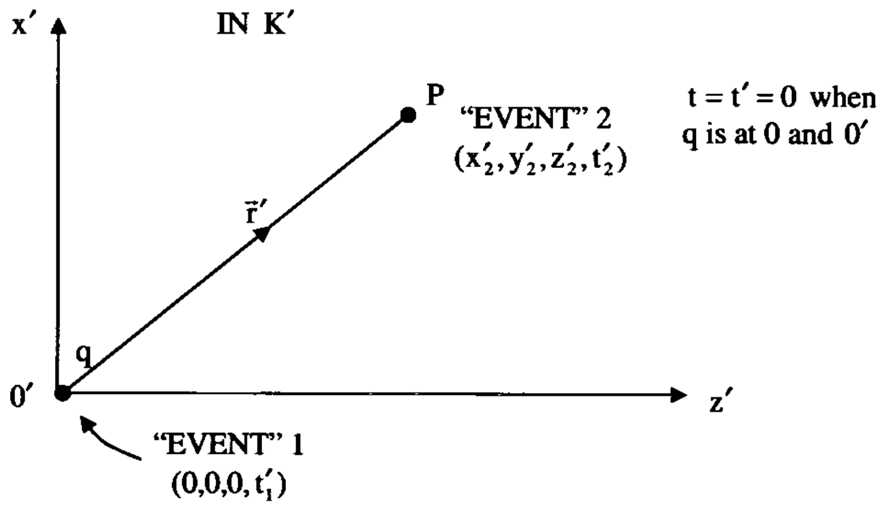
$$= \gamma\left(\vec{B}_1 + \vec{B}_2 - \frac{1}{c^2}v(E_1\hat{j} - E_2\hat{i})\right)$$

$$= \gamma\left(\vec{B}_\perp - \frac{1}{c^2}\vec{v} \times \vec{E}_\perp\right)$$

11. Fields Produced by a Point Charge in Uniform Motion

Consider the case of a point charge q moving at constant velocity $\vec{v} = v\hat{k}$ (i.e. along the z-axis of system K). Imagine trying to measure *simultaneously* the location of the charge q in K and the fields $\vec{E}$ and $\vec{B}$ produced by q in K; does the electric field $\vec{E}$ in K produced by a moving q still obey Coulomb's law based on distances measured in K, etc.?





In K', q is stationary and so we can write down simply its $\vec{E}'$ and $\vec{B}'$ fields ("event" 2 in K') along with its location ("event" 1 in K').

In K'

EVENT 1: $(x_1', y_1', z_1', t_1') = (0, 0, 0, t_1')$

which can be expressed in terms of the unprimed coordinates by the usual Lorentz Transformation

$$\left(0, 0, \gamma(z_1 - vt) = 0 \ , \quad \gamma\left(t - \frac{z_1 v}{c^2}\right)\right)$$

EVENT 2: located at $(x_2', y_2', z_2', t_2') = \left(x_2, y_2, \gamma(z_2 - vt), \gamma\left(t - \frac{z_2 v}{c^2}\right)\right)$

with fields

$$\vec{E}' = \frac{q\hat{r}'}{4\pi\varepsilon_0 (r')^2} \ ; \quad \vec{B}' = 0$$

In component form:

$$E_x' = \frac{q(x_2' - 0)}{4\pi\varepsilon_0 (r')^3} = \frac{q x_2'}{4\pi\varepsilon_0 (r')^3}$$

$$E_y' = \frac{q(y_2' - 0)}{4\pi\varepsilon_0 (r')^3} = \frac{q y_2'}{4\pi\varepsilon_0 (r')^3}$$

$$E_z' = \frac{q(z_2' - 0)}{4\pi\varepsilon_0 (r')^3} = \frac{q z_2'}{4\pi\varepsilon_0 (r')^3}$$

In K

The fields in K can be found using the inverse of equations (80) ($v \rightarrow -v$) in terms of those written down in $K^{/}$

$$E_x = \gamma(E_x^{/} + vB_y^{/}) = \gamma E_x^{/} = \frac{\gamma q x_2^{/}}{4\pi\varepsilon_0 (r^{/})^3}$$

$$B_x = \gamma\left(B_x^{/} - \frac{vE_y^{/}}{c^2}\right) = -\frac{\gamma v E_y^{/}}{c^2} = -\frac{vE_y}{c^2} = \frac{1}{c^2}(\vec{v}\times\vec{E})_x$$

$$E_y = \gamma(E_y^{/} - vB_x^{/}) = \gamma E_y^{/} = \frac{\gamma q y_2^{/}}{4\pi\varepsilon_0 (r^{/})^3}$$

$$B_y = \gamma\left(B_y^{/} + \frac{vE_x^{/}}{c^2}\right) = \frac{\gamma v E_x^{/}}{c^2} = \frac{vE_x}{c^2} = \frac{1}{c^2}(\vec{v}\times\vec{E})_y$$

$$E_z = E_z^{/} = \frac{q z_2^{/}}{4\pi\varepsilon_0 (r^{/})^3}$$

$$B_z = B_z^{/} = 0 = \frac{1}{c^2}(\vec{v}\times\vec{E})_z$$

Noting $x_2^{/} = x_2,\ y_2^{/} = y_2$ and $z_2^{/} = \gamma(z_2 - vt)$ then

$$(r^{/})^3 = [(x_2^{/})^2 + (y_2^{/})^2 + (z_2^{/})^2]^{3/2} = [(x_2^2 + y_2^2 + \gamma^2(z_2 - vt)^2]^{3/2}$$

so

$$\vec{E} = \hat{i}E_x + \hat{j}E_y + \hat{k}E_z = \frac{\gamma q[\hat{i}x_2 + \hat{j}y_2 + \hat{k}(z_2 - vt)]}{4\pi\varepsilon_0[x_2^2 + y_2^2 + \gamma^2(z_2 - vt)^2]^{3/2}}$$

However in K,

$$\vec{r}_p = \hat{i}x_2 + \hat{j}y_2 + \hat{k}z_2 \; ; \quad \vec{r}_q = \hat{k}vt$$

Therefore

$$\vec{E} = \frac{\gamma q(\vec{r}_p - \vec{r}_q)}{4\pi\varepsilon_0[x_2^2 + y_2^2 + \gamma^2(z_2 - vt)^2]^{3/2}}$$

However

$$z_2 - vt = z_2 - z_1 = r\cos\theta$$

while

$$x_2^2 + y_2^2 = r^2 - (z_2 - z_1)^2 = r^2 - r^2\cos^2\theta = r^2\sin^2\theta$$

so

$$x_2^2 + y_2^2 + \gamma^2(z_2 - vt)^2 = r^2\sin^2\theta + \gamma^2 r^2\cos^2\theta$$

$$= r^2\left(\sin^2\theta + \left(1 - \frac{v^2}{c^2}\right)^{-1}\cos^2\theta\right) = \frac{r^2}{\left(1 - \frac{v^2}{c^2}\right)}\left(\sin^2\theta - \frac{v^2}{c^2}\sin^2\theta + \cos^2\theta\right)$$

$$= \gamma^2 r^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right) = \gamma^2 r^2(1 - \beta^2\sin^2\theta)$$

Thus in K:

$$\vec{r} = \vec{r}_p - \vec{r}_q$$

and

$$\vec{E} = \frac{\gamma q(\vec{r}_p - \vec{r}_q)}{4\pi\varepsilon_0\gamma^3|\vec{r}_p - \vec{r}_q|^3(1 - \beta^2\sin^2\theta)^{3/2}}$$

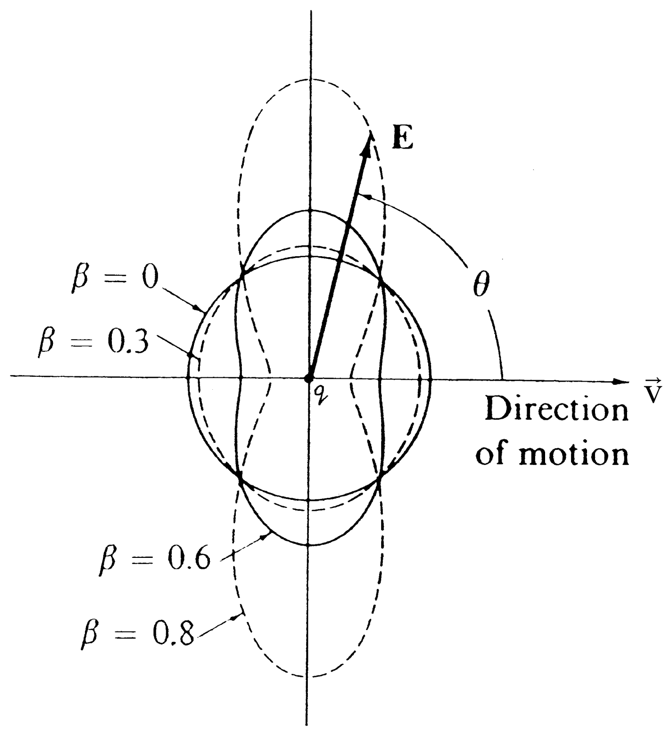
Therefore

$$\vec{E} = \frac{q(1 - \beta^2)(\vec{r}_p - \vec{r}_q)}{4\pi\varepsilon_0|\vec{r}_p - \vec{r}_q|^3(1 - \beta^2\sin^2\theta)^{3/2}}$$

and

$$\vec{B} = \hat{i}B_x + \hat{j}B_y + \hat{k}B_z = \frac{1}{c^2}(\vec{v} \times \vec{E})$$

Plots of $\vec{E}$ and $\vec{B}$ on the surface of a sphere of (fixed) radius $|\vec{r}_p - \vec{r}_q|$ as a function of angle θ are shown below.



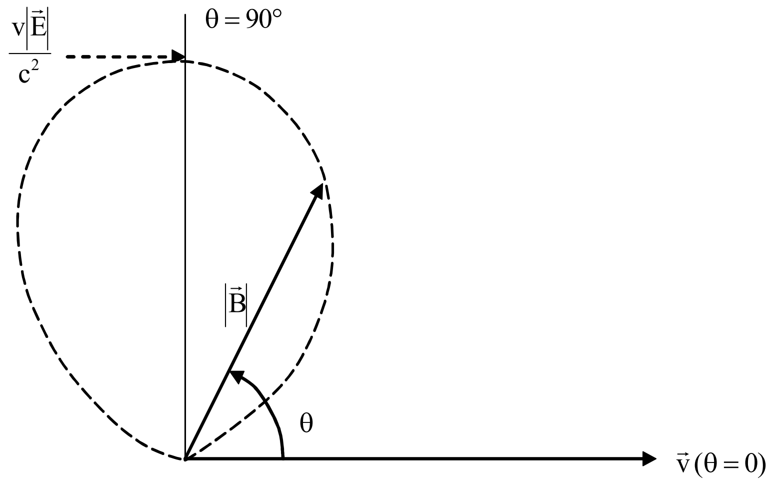
Magnitude of E at a fixed distance $r = |\vec{r}_p - \vec{r}_q|$ as a function of angle θ for various values of $\beta = \frac{v}{c}$.

$$\vec{B} = \frac{1}{c^2}\vec{v} \times \vec{E}$$

Therefore

$$|\vec{B}| = \frac{1}{c}\beta|\vec{E}|\sin\theta$$

so "multiply" each curve for $\vec{E}$ previous by $\frac{\beta}{c}\sin\theta$ to get:



12. Infinite Linear System of Charges in Uniform Motion

Consider an infinite line of charge (linear density λ) moving with uniform velocity $\vec{v} = v\hat{k}$ along the z-axis of system K.

Basically one wants to measure the location of all elements of charge in this line (the elements are usually used in calculations of the field) along with the $\vec{E}$ and $\vec{B}$ fields produced at some field point P, simultaneously.

Since the line is infinite, we can locate the field point P on the x-axis in K without loss of generality.

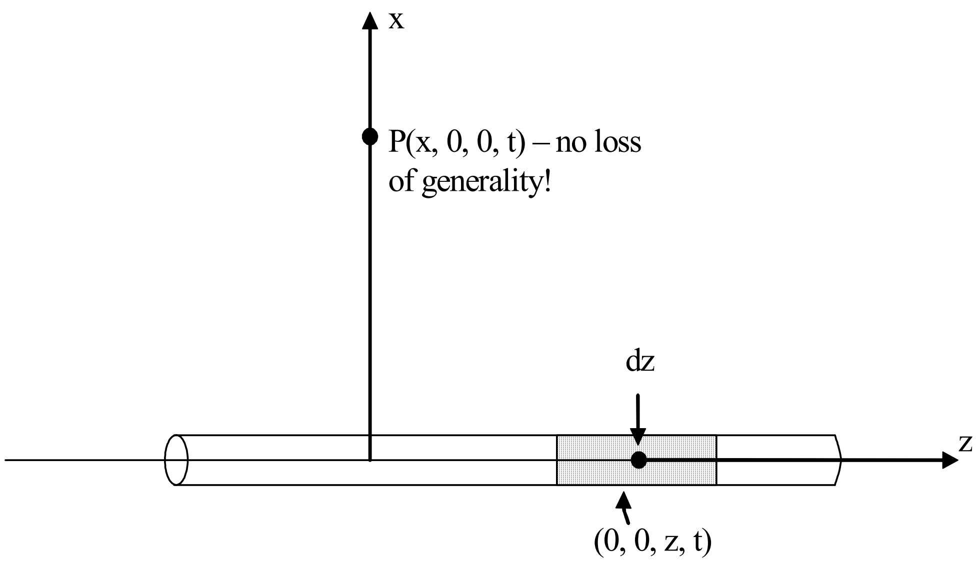
In K', the events which determined the locations of all the charge elements in this infinite line and the fields at P are, of course, *not* simultaneous. However according to observers in K' the line of charge is *at rest*, and thus the fields in K' are *static* and given by:

$$\vec{E}'(x', 0, 0, t') = \frac{\lambda'}{2\pi\varepsilon_0 x'}\hat{i}$$

The field points radially outwards i.e. along $\hat{i}$ for points on the x' axis, λ' is the linear charge density measured in K'. (See page 69 for the electric field produced by an infinite, stationary line of charge.)

$\vec{B}' = 0$ everywhere at all times (*no* current).

In K:



Using the transformation equations

$$E_x \text{ (at P)} = \gamma E_x^{/} = \frac{\gamma\lambda^{/}}{2\pi\varepsilon_0 x^{/}} = \frac{\gamma\lambda^{/}}{2\pi\varepsilon_0 x}$$

However we must relate $\lambda^{/}$ to λ; notice that the element dz carries charge dq given by

$$dq = \lambda dz \;;$$

the corresponding element $dz^{/}$ in $K^{/}$ carries charge

$$dq^{/} = \lambda^{/} dz^{/}.$$

Since charge is *invariant*:

$$dq^{/} = dq$$

Therefore

$$\lambda^{/} dz^{/} = \lambda dz$$

However the PROPER length of the element is $dz^{/}$ (measured by observers at *rest* with respect to it). Therefore

$$dz = \frac{1}{\gamma} dz^{/} = \sqrt{1 - \frac{v^2}{c^2}}\, dz^{/}$$

Therefore

$$\lambda' = \frac{\lambda}{\gamma} \quad : \quad \gamma\lambda' = \lambda$$

Using this, we can find the fields' components at point P:

$$E_x = \gamma(E_x' + vB_y') = \gamma E_x' = \frac{\gamma\lambda'}{2\pi\varepsilon_0 x'} = \frac{\lambda}{2\pi\varepsilon_0 x}$$

$$B_x = \gamma\left(B_x' - \frac{vE_y'}{c^2}\right) = 0 \text{ as } B_x' = 0 \text{ and } E_y' = 0 \text{ at P}$$

Similarly,

$$E_y = \gamma(E_y' - vB_x') = 0$$

$$B_y = \gamma\left(B_y' + \frac{vE_x'}{c^2}\right) = \frac{\gamma v E_x'}{c^2} = \frac{v\lambda}{2\pi\varepsilon_0 c^2 x} = \frac{\mu_0 v\lambda}{2\pi x}$$

$$E_z = E_z' = 0$$

$$B_z = B_z' = 0$$

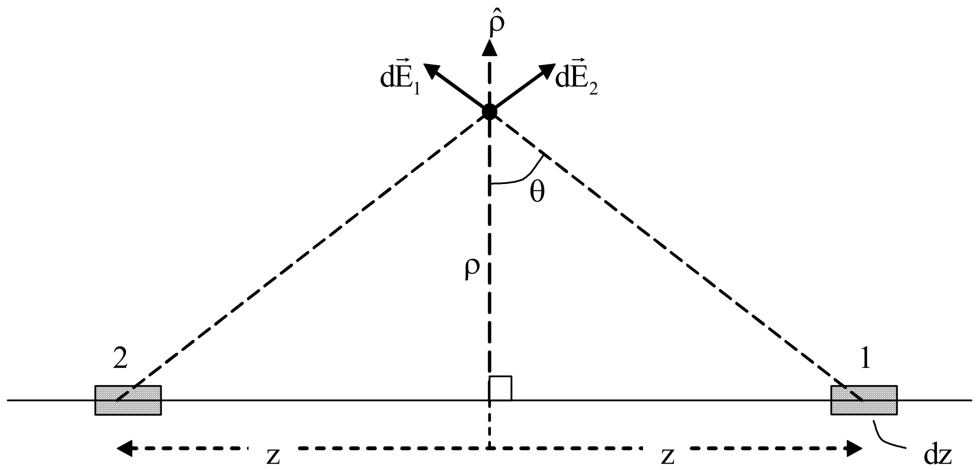
Combining these and generalizing the position of field point P, we get:

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 x}\hat{i} \qquad \rightarrow \qquad \frac{\lambda}{2\pi\varepsilon_0 \rho}\hat{\rho}$$

$$\vec{B} = \frac{v\lambda}{2\pi\varepsilon_0 c^2 x}\hat{j} \qquad \rightarrow \qquad \frac{\mu_0 I}{2\pi\rho}\hat{\varphi}$$

where $\rho, \hat{\rho} \equiv$ cylindrical radial coordinates; $I = v\lambda$ in K and $\varepsilon_0 c^2 = \frac{1}{\mu_0}$; notice $\vec{B}$ is time-independent because I(v) is uniform (constant).

Infinite, stationary line of charge



By symmetry – field is radially outwards from line

$$d\vec{E}_T = d\vec{E}_1 + d\vec{E}_2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{\lambda\, dz}{\rho^2 + z^2}\right) 2\cos\theta\, \hat{\rho}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{2\lambda\, dz\, \rho}{(\rho^2 + z^2)^{3/2}}\, \hat{\rho}$$

Therefore

$$|\vec{E}_T| = \frac{\lambda\rho}{2\pi\varepsilon_0}\int_0^\infty \frac{dz}{(\rho^2 + z^2)^{3/2}} = \frac{\lambda\rho}{2\pi\varepsilon_0}\left[\frac{z}{\rho^2(\rho^2 + z^2)^{1/2}}\right]_0^\infty$$

$$\frac{z}{\rho^2(\rho^2 + z^2)^{1/2}} \sim \frac{1}{\rho^2}\left(1 - \frac{1}{2}\frac{\rho^2}{z^2} \ldots\right) \quad \text{for } z >> \rho$$

$$\rightarrow \frac{1}{\rho^2} \quad \text{when } z \rightarrow \infty$$

Therefore

$$|\vec{E}_T| = \frac{\lambda}{2\pi\varepsilon_0\rho}$$