Physics 7590 Homework # 5

1) a) $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{bmatrix}$, where

$\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. Thus, $\beta = \tanh\theta$;

and hence $1-\beta^2 = 1-\tanh^2\theta = 1 - \frac{\sinh^2\theta}{\cosh^2\theta} = \frac{\cosh^2\theta - \sinh^2\theta}{\cosh^2\theta}$

$= \frac{1}{\cosh^2\theta}$. Hence

$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh\theta$ and $\beta\gamma = (\tanh\theta)(\cosh\theta) = \sinh\theta$.

Thus, $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\theta & i\sinh\theta \\ 0 & 0 & -i\sinh\theta & \cosh\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(i\theta) & \sin(i\theta) \\ 0 & 0 & -\sin(i\theta) & \cos(i\theta) \end{bmatrix}$

b) $u_3' = \frac{u_3 - v}{1 - \frac{u_3 v}{c^2}}$;

$u_3'/c = \frac{u_3/c - v/c}{1 - \frac{u_3}{c}\frac{v}{c}}$;

$\tanh\varphi' = \frac{\tanh\varphi - \tanh\theta}{1 - \tanh\varphi \tanh\theta} = \tanh(\varphi - \theta)$. Therefore,

$\boxed{\varphi' = \varphi - \theta}$.

PHYS 7590 Homework # 5

2. a) Moving clock runs slow by a factor $\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{5}{3}$.

Since 18 years elapsed on the moving clock, it follows that $\left(\frac{5}{3}\right) 18 = 30$ years elapsed on the stationary (Earth) clock. So the brother will be $21 + 30 = 51$ years old.

b) By Earth clock, it took 15 years to get to star X, at a speed of $v = \frac{4}{5}c$. So the distance from Earth to X is $d = \left(\frac{4}{5}c\right)(15 \text{ years}) = 12c$ years (or 12 light years)

c) $z = 12c$ years, $t = 15$ years

d) $z' = 0$, $t' = 9$ years [She got on at the origin in K' and she rode along with K'; so she'll still be at the origin. If you doubt these values, use the Lorentz transformations with the values of z and t from part c).]

e) K'' is moving with velocity $-v\hat{k}$ relative to K. So

$$z'' = \gamma(z + vt) = \frac{5}{3}\left[12c + \left(\frac{4}{5}c\right)15\right] = 40c \text{ years}$$

$$t'' = \gamma\left(t + \frac{v}{c^2}z\right) = \frac{5}{3}\left[15 + \frac{4c}{5}\frac{1}{c^2}12c\right] = 41 \text{ years}.$$

f) Since $t'' - t' = 41 - 9 = 32$ years, the traveling twin would have to set her clock ahead 32 years; so her clock will now read $18 + 32 = 50$ years at her arrival. Note that this is $\frac{5}{3}(30)$ years – precisely what she would calculate if the stay-at-home twin had been the traveler, for 30 years of his own time!

g) (i) According to the traveling twin, the one on earth is the one traveling and therefore the stay-at-home twin's clock will run slower by a factor of $\gamma = \frac{5}{3}$. So just before the jump $t = \frac{t'}{\gamma} = \frac{3}{5}(9) = 5.4$ years

so the brother will be 26.4 years old.

(ii) $t = \frac{t''}{\gamma} = \frac{3}{5}(41) = 24.6$ years so the brother will be 45.6 years old.

h) It will take another 5.4 years $[\frac{3}{5}(9 \text{ years})]$ of Earth time for the return [see part g) (i)]; so when she gets back, she will say her twin's age is $45.6 + 5.4 = 51$ years – which is what we found in part a). But note that to make it work from the traveler's point of view, you must take into account the jump in perceived age of stay-at-home when she changes coordinates from K' to K''.

5) In the lab frame (K):

$E = E_1 + E_2 + E_3 + \cdots$ and $P_z = P_1 + P_2 + P_3 + \cdots$

In the center of momentum (or CM) frame K':

$P_z' = \gamma_{(v)}\left(P_z - \frac{v}{c^2}E\right) = 0$. Thus, $P_z - \frac{v}{c^2}E = 0$ and hence

$$v = \frac{c^2 P_z}{E} = \frac{P_1 + P_2 + P_3 + \cdots}{E_1 + E_2 + E_3 + \cdots} c^2 .$$

3. $z(t) = \sqrt{b^2 + (ct)^2}$; $x = y = 0$

a) $d\tau = \sqrt{1 - u^2/c^2}\, dt$, where

$$\vec{u} = \hat{e}_z \frac{dz}{dt} = \hat{e}_z \frac{2c^2 t}{2\sqrt{b^2+c^2t^2}} = \hat{e}_z \frac{c^2 t}{\sqrt{b^2+c^2t^2}}.$$ Thus,

$$u^2/c^2 = \frac{c^2t^2}{b^2+c^2t^2}$$; and hence

$$1 - u^2/c^2 = \frac{b^2}{b^2+c^2t^2}.$$ It follows that

$$d\tau = \frac{b}{\sqrt{b^2+c^2t^2}}\, dt$$; and hence

$$\tau = b \int \frac{dt}{\sqrt{b^2+c^2t^2}}.$$

Let $t = \frac{b}{c}\tan\xi$. Then $dt = \frac{b}{c}\sec^2\xi\, d\xi$ and

$b^2 + c^2t^2 = b^2(1+\tan^2\xi) = b^2\sec^2\xi$. Thus,
(*)

$$\tau = b \int \frac{b/c \sec^2\xi\, d\xi}{b\sec\xi} = \frac{b}{c}\int \sec\xi\, d\xi$$

$$= \frac{b}{c}\ln(\sec\xi + \tan\xi) + K$$ constant of integration.

But $\tan\xi = \frac{ct}{b}$; and from (*): $\sec\xi = \frac{\sqrt{b^2+c^2t^2}}{b}$.

Therefore,

$$\tau = \frac{b}{c} \ln\left[\frac{1}{b}\left(ct + \sqrt{b^2 + c^2t^2}\right)\right] + K.$$

But at $t=0$, $\tau = \frac{b}{c} \ln 1 + K = K \underset{=}{\neq} 0$ (given). So

~~t~~ $K = 0$ and hence

$$\boxed{\tau = \frac{b}{c} \ln\left[\frac{1}{b}\left(ct + \sqrt{b^2 + c^2t^2}\right)\right]} \quad (**)$$

b) From (**), we get:

$$b e^{c\tau/b} = \sqrt{b^2 + c^2t^2} + ct = z + \sqrt{z^2 - b^2}.$$

$$\sqrt{z^2 - b^2} = b e^{c\tau/b} - z$$

$$\cancel{z^2} - b^2 = b^2 e^{2c\tau/b} + \cancel{z^2} - 2bz\, e^{c\tau/b}$$

$$z = \frac{b^2\left(e^{2c\tau/b} + 1\right)}{2b\, e^{c\tau/b}} = b\, \frac{e^{c\tau/b} + e^{-c\tau/b}}{2}$$

$$\boxed{z = b \cosh(c\tau/b)}$$

$$u = \frac{c^2 t}{\sqrt{b^2 + c^2t^2}} = c\, \frac{ct}{\sqrt{b^2 + c^2t^2}} = c\, \frac{\sqrt{z^2 - b^2}}{z}. \quad \text{But}$$

$$\sqrt{z^2 - b^2} = \sqrt{b^2\left[\cosh^2(c\tau/b) - 1\right]} = \sqrt{b^2 \sinh^2(c\tau/b)}$$

$$= b \sinh(c\tau/b). \quad \text{Therefore,}$$

$$\boxed{u = c\, \frac{b \sinh(c\tau/b)}{b \cosh(c\tau/b)} = c \tanh(c\tau/b)}.$$

4. We have (see notes):

$$u_3' = \frac{u_3 - v}{1 - \frac{v u_3}{c^2}} \qquad \text{and} \qquad t' = \gamma(v)\left[t - \frac{v}{c^2} z\right].$$

$$a_3' = \frac{du_3'}{dt'}, \quad \text{where}$$

$$du_3' = \frac{du_3}{1 - \frac{v u_3}{c^2}} - \frac{u_3 - v}{\left(1 - \frac{v u_3}{c^2}\right)^2}\left[-\frac{v}{c^2} du_3\right]$$

$$= \frac{du_3}{\left(1 - \frac{v u_3}{c^2}\right)^2}\left[1 - \frac{v u_3}{c^2} + \frac{v}{c^2}(u_3 - v)\right]$$

$$= \frac{du_3}{\left(1 - \frac{v u_3}{c^2}\right)^2}\left[1 - \frac{v^2}{c^2}\right] = \frac{du_3}{[\gamma(v)]^2\left(1 - \frac{v u_3}{c^2}\right)^2};$$

and

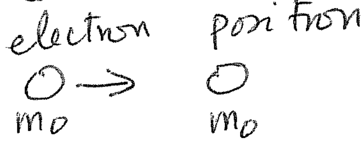
$$dt' = \gamma(v)\left[dt - \frac{v}{c^2} dz\right] = \gamma(v)\left[dt - \frac{v}{c^2}\frac{dz}{dt} dt\right]$$

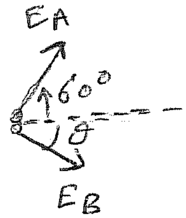
$$= \gamma(v)\left[1 - \frac{v}{c^2} u_3\right] dt$$ ~~where~~ where we have used: $u_3 = \frac{dz}{dt}$.

It follows that:

$$a_3' = \frac{du_3'}{dt'} = \frac{du_3/dt}{[\gamma(v)]^3\left(1 - \frac{v}{c^2} u_3\right)^3} = \frac{a_3}{[\gamma(v)]^3\left[1 - \frac{v}{c^2} u_3\right]^3}.$$

6. Let m_0 denote the rest mass of the electron/positron.
We use conservation of the 4-vector momentum (i.e. conservation of relativistic 3-D momentum and conservation of energy.)





Conservation of energy:

$$\sqrt{P_e^2 c^2 + m_0^2 c^4} + m_0 c^2 = E_A + E_B \qquad (1)$$

Conservation of momentum:

$$P_e = \frac{E_A}{c}\cos 60^\circ + \frac{E_B}{c}\cos\theta \qquad (2) \quad \text{and}$$

$$0 = \frac{E_A}{c}\sin 60^\circ - \frac{E_B}{c}\sin\theta \qquad (3).$$

From (2), we get: $E_B \cos\theta = P_e c - \frac{1}{2} E_A \qquad (4)$

from (3), we get: $E_B \sin\theta = \frac{\sqrt{3}}{2} E_A \qquad (5)$

Squaring (4) and (5), and adding the resulting equations yield:

$$E_B^2 = P_e^2 c^2 - E_A P_e c + E_A^2 \qquad (6).$$

On the other hand, using (1), we get

$$E_B^2 = \left[\sqrt{P_e^2 c^2 + m_0^2 c^4} + m_0 c^2 - E_A\right]^2$$
$$= P_e^2 c^2 + m_0^2 c^4 + 2\sqrt{P_e^2 c^2 + m_0^2 c^4}\,(m_0 c^2 - E_A) + m_0^2 c^4 - 2 E_A m_0 c^2 + E_A^2, \text{ or}$$

$$E_B^2 = P_e^2 c^4 + 2 m_o^2 c^4 + 2 m_o c^2 \sqrt{P_e^2 c^2 + m_o^2 c^4} - 2 E_A \sqrt{P_e^2 c^2 + m_o^2 c^4}$$
$$- 2 E_A m_o c^2 + E_A^2 \qquad (7)$$

Comparing (6) and (7), we get:

$$- E_A P_e c = 2 m_o^2 c^4 + 2 m_o c^2 \sqrt{P_e^2 c^2 + m_o^2 c^4} - 2 E_A \sqrt{P_e^2 c^2 + m_o^2 c^4}$$
$$- 2 E_A m_o c^2$$, which we can solve for E_A:

$$E_A = \frac{m_o^2 c^4 + m_o c^2 \sqrt{P_e^2 c^2 + m_o^2 c^4}}{m_o c^2 + \sqrt{P_e^2 c^2 + m_o^2 c^4} - P_e c/2}$$

$$= \frac{\left(m_o^2 c^4 + m_o c^2 \sqrt{P_e^2 c^2 + m_o^2 c^4}\right) \cdot \left(m_o c^2 - \sqrt{P_e^2 c^2 + m_o^2 c^4} - P_e c/2\right)}{\left(m_o c^2 + \sqrt{P_e^2 c^2 + m_o^2 c^4} - P_e c/2\right) \cdot \left(m_o c^2 - \sqrt{P_e^2 c^2 + m_o^2 c^4} - P_e c/2\right)}$$

$$= \frac{m_o c^2 \left(m_o c^2 + \sqrt{P_e^2 c^2 + m_o^2 c^4}\right) \cdot \left(m_o c^2 - \sqrt{P_e^2 c^2 + m_o^2 c^4} - P_e c/2\right)}{m_o^2 c^4 - m_o P_e c^3 + P_e^2 c^2/4 - P_e^2 c^2 - m_o^2 c^4}$$

$$= m_o c^2 \frac{\cancel{m_o^2 c^4} - P_e^2 c^2 - \cancel{m_o^2 c^4} - \frac{1}{2} P_e m_o c^3 - \frac{1}{2} P_e c \sqrt{P_e^2 c^2 + m_o^2 c^4}}{- m_o P_e c^3 - \frac{3}{4} P_e^2 c^2}$$

$$= m_o c^2 \frac{- P_e c^2/2 \left[m_o c + 2 P_e + \sqrt{P_e^2 + m_o^2 c^2}\right]}{- P_e c^2 \left[m_o c + \frac{3}{4} P_e\right]}$$

$$\boxed{E_A = \frac{m_o c^2}{2} \, \frac{m_o c + 2 P_e + \sqrt{P_e^2 + m_o^2 c^2}}{m_o c + \frac{3}{4} P_e}}$$

7) Initial momentum = momentum of particle 1 whose energy is twice its rest energy:

$$E_1^2 - p^2c^2 = m_0^2 c^4$$

$$(2m_0c^2)^2 - p^2c^2 = m_0^2 c^4 \rightarrow p^2c^2 = 3m_0^2c^4$$

$$\rightarrow p = \sqrt{3}\, m_0 c$$

Initial (total) energy $= E_1 + E_2$

$$E_T = 2m_0c^2 + m_0c^2 = 3m_0c^2 .$$

Both initial momentum and initial energy are conserved. so final energy is $3m_0c^2$ and final momentum is $\sqrt{3}\, m_0 c$.

Let M_0 be the rest mass of the resulting composite particle. Then

$$E^2 - p^2c^2 = M_0^2 c^4$$

$$9m_0^2c^4 - 3m_0^2c^4 = M_0^2c^4 \rightarrow \boxed{M_0 = \sqrt{6}\, m_0} \quad (\simeq 2.45 m_0).$$

In this process some kinetic energy was converted into mass (rest energy), so $M_0 > 2m_0$.

To find the velocity of the composite particle:

$$E = \gamma M_0 c^2$$

$$3m_0c^2 = \gamma\sqrt{6}\, m_0 c^2$$

$$\frac{1}{\gamma^2} = 1 - \frac{u^2}{c^2} = \frac{6}{9} = \frac{2}{3} \rightarrow \frac{u^2}{c^2} = \frac{1}{3} \rightarrow \boxed{u = \frac{c}{\sqrt{3}}}$$

8) First we calculate the pion's energy:

$$E^2 = p^2c^2 + m_0^2 c^4$$
$$= \frac{9}{16} m_0^2 c^4 + m_0^2 c^4 = \frac{25}{16} m_0^2 c^4$$

$$\rightarrow E = \frac{5}{4} m_0 c^2$$

Conservation of energy $\Rightarrow \frac{5}{4} m_0 c^2 = E_A + E_B$ (1)
where γ_A is the photon emitted in the direction of the initial pion and γ_B the photon emitted in the opposite direction.

Conservation of momentum $\Rightarrow \frac{3}{4} m_0 c = \frac{E_A}{c} - \frac{E_B}{c}$

so $\frac{3}{4} m_0 c^2 = E_A - E_B$ (2)

Adding (1) and (2), we get $\boxed{E_A = m_0 c^2}$

Substracting (2) from (1), we get $\boxed{E_B = \frac{1}{4} m_0 c^2}$

9) $T_0 = 0.81 \times 10^{-10}$ sec (lifetime of the particle in its rest frame)

$\rightarrow T = \frac{T_0}{\sqrt{1 - u^2/c^2}}$ is the particle's lifetime in a frame in which the particle is moving at speed u.

Distance ℓ that the particle travels in lifetime is
$\ell = uT$, from which we get $u = \frac{\ell}{T}$

$\therefore T^2 = \frac{T_0^2}{1 - u^2/c^2} = \frac{T_0^2}{1 - \frac{\ell^2}{c^2 T^2}}$. Solving for T, we get

$$T = \left(T_0^2 + \frac{\ell^2}{c^2}\right)^{1/2}.$$

Kinetic energy of the particle is:

$$K = m_0 c^2 \left[\frac{1}{\sqrt{1 - u^2/c^2}} - 1\right]$$

$$= m_0 c^2 \left[\frac{T}{T_0} - 1\right]$$

$$= m_0 c^2 \left[\left(1 + \frac{\ell^2}{c^2 T_0^2}\right)^{1/2} - 1\right]$$

$$= 1,190 \text{ Mev} \left[\left(1 + \frac{10^{-6}}{(0.81 \times 10^{-10} \times 3 \times 10^{8})^2}\right)^{1/2} - 1\right]$$

$$= 1.007 \text{ MeV}.$$