PHYS 3496 — Sample Midterm Exam

Problem 1: Find the steady-state temperature $T(\rho, \phi)$ inside a circular plate of radius a if the upper semicircular boundary is held at $100°$ and the lower one at $-100°$.

Hints:

1. This is a two-dimensional problem; use Laplace's Equation in polar coordinates:

$$\frac{\partial^2 T}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial T}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 T}{\partial \phi^2} = 0$$

with the boundary condition on the circle $\rho = a$:

$$T(a, \phi) = \begin{cases} -100° & \text{if } -\pi < \phi < 0 \\ 100° & \text{if } 0 < \phi < \pi. \end{cases}$$

2. Two linearly independent solutions of the second order, linear ODE

$$x^2 y'' + xy' - n^2 y = 0$$

are $y = x^n$ and $y = x^{-n}$.

Problem 2: An electric dipole of moment $\mathbf{p}$ is located at the origin. The dipole creates an electric potential at $\mathbf{r}$ given by

$$\psi(\mathbf{r}) = \frac{\mathbf{p} \cdot \mathbf{r}}{4\pi\epsilon_0 r^3} = \frac{\mathbf{p} \cdot \hat{\mathbf{e}}_r}{4\pi\epsilon_0 r^2}.$$

Show that the electric field $\mathbf{E} = -\nabla\psi$ at $\mathbf{r}$ is given by

$$\mathbf{E} = \frac{3(\mathbf{p} \cdot \hat{\mathbf{e}}_r)\hat{\mathbf{e}}_r - \mathbf{p}}{4\pi\epsilon_0 r^3}.$$

Hints: 1. Without loss of generality, we may choose the z-axis to be the axis of the dipole so that $\mathbf{p} = p_0\hat{\mathbf{e}}_z$. Write both $\psi(\mathbf{r})$ and ∇ in spherical polar coordinates then compute $\mathbf{E} = -\nabla\psi$.

2. You may use:

$$\hat{\mathbf{e}}_z = \cos\theta\,\hat{\mathbf{e}}_r - \sin\theta\,\hat{\mathbf{e}}_\theta$$

Problem 3: Consider the Chebyshev ODE:

$$(1-x^2)y'' - xy' + n^2 y = 0; \qquad\qquad -1 \le x \le 1.$$

(a) Let $w(x) := \frac{1}{p_0(x)} e^{\int \frac{p_1(x)}{p_0(x)}\, dx}$, where $p_0(x) = 1 - x^2$ and $p_1(x) = -x$.

Show that $w(x) = 1/\sqrt{1-x^2}$.

(b) Show that multiplying the ODE by $w(x) = 1/\sqrt{1-x^2}$ puts it into a self-adjoint form.

(c) Show that the Hermitian operator boundary condition

$$\left[v^* \bar{p}_0(x) u' - (v^*)' \bar{p}_0(x) u\right]_{-1}^{1} = 0$$

holds for polynomial solutions u and v of the Chebyshev ODE. Here

$$\bar{p}_0(x) = w(x) p_0(x) = \frac{1}{\sqrt{1-x^2}}(1-x^2) = \sqrt{1-x^2}.$$

(d) The Chebyshev polynomials $T_0(x), T_1(x), T_2(x), \cdots$ are solutions of the Chebyshev ODE that correspond to $n = 0, 1, 2, \cdots$, respectively. Using the results of parts (b) and (c) above, write down the orthogonality condition for the Chebyshev polynomials.

PHYS 3496

Sample Midterm
~~[illegible]~~
Solutions

1. $$\frac{\partial^2 T}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial T}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 T}{\partial \varphi^2} = 0$$

Let $T(\rho,\varphi) = R(\rho)\,\Psi(\varphi)$. Then

$$\Psi(\varphi)\frac{d^2 R(\rho)}{d\rho^2} + \frac{\Psi(\varphi)}{\rho}\frac{dR(\rho)}{d\rho} + \frac{R(\rho)}{\rho^2}\frac{d^2\Psi}{d\varphi^2} =$$

Dividing the last equation by $R(\rho)\,\Psi(\varphi)$, we get:

$$\frac{1}{R}\frac{d^2 R}{d\rho^2} + \frac{1}{\rho R}\frac{dR}{d\rho} + \frac{1}{\rho^2 \Psi}\frac{d^2\Psi}{d\varphi^2} = 0$$

or

$$\underbrace{\frac{\rho^2}{R}\frac{d^2 R}{d\rho^2} + \frac{\rho}{R}\frac{dR}{d\rho}}_{\text{a function of } \rho \text{ only}} = \underbrace{-\frac{1}{\Psi}\frac{d^2\Psi}{d\varphi^2}}_{\text{a function of } \varphi \text{ only}}$$

The last identity can happen only if both sides are equal to the same constan say α. Thus

$$\frac{\rho^2}{R}\frac{d^2R}{d\rho^2}+\frac{\rho}{R}\frac{dR}{d\rho}=-\frac{1}{\Psi}\frac{d^2\Psi}{d\varphi^2}=\alpha$$

If we rotate the circular plate by any integer multiple of 2π ($\varphi \rightarrow \varphi + 2n\pi$), we obtain the same boundary value problem

$\Rightarrow$ $\Psi(\varphi)$ must be a periodic function of φ, with period 2π. Thus, $\alpha = n^2$ with n an integer.

$$-\frac{1}{\Psi}\frac{d^2\Psi}{d\varphi^2}=n^2 \rightarrow \Psi_n(\varphi)=\begin{Bmatrix}\cos n\varphi \\ \sin n\varphi\end{Bmatrix}$$

From the boundary condition $T(a,\varphi) = f(\varphi) = \begin{cases} -100 & \text{if } -\pi<\varphi<0 \\ 100 & \text{if } 0<\varphi<\pi \end{cases}$, we infer that

$\Psi_n(\varphi)$ must be an odd function of φ

$\Rightarrow \Psi_n(\varphi) \sim \sin n\varphi$

ρ equation:

$$\frac{\rho^2}{R}\frac{d^2R}{d\rho^2} + \frac{\rho}{R}\frac{dR}{d\rho} = n^2 \rightarrow$$

$$\rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} - n^2R = 0$$

$$\rightarrow R(\rho) = \begin{Bmatrix} \rho^n \\ \cancel{\rho^{-n}} \end{Bmatrix}$$

because $\rho^{-n} \xrightarrow[\rho \to 0]{} \infty$

$$\rightarrow R_n(\rho) \sim \rho^n$$

For each $n = 1, 2, 3, ---$, we obtain

the eigensolution: $T_n(\rho, \varphi) = R_n(\rho)\, \Psi_n(\varphi)$
$= \rho^n \sin n\varphi$.

To match the boundary condition at $\rho = a$,

we take a linear combination of all possible eigensolutions:

$$T(\rho, \varphi) = \sum_{n=1}^{\infty} b_n \rho^n \sin n\varphi \qquad (*)$$

$$T(a, \varphi) = f(\varphi) = \sum_{n=1}^{\infty} b_n a^n \sin n\varphi$$

which is a Fourier sine series of the odd function $f(\varphi)$, with period 2π, and

Fourier coefficients

$$b_n a^n = \frac{2}{\pi}\int_0^{\pi} f(\varphi)\sin n\varphi \, d\varphi = \frac{2}{\pi}\int_0^{\pi} 100 \sin n\varphi \, d\varphi$$

$$= \frac{200}{\pi}\int_0^{\pi} \sin n\varphi \, d\varphi$$

$$= \frac{200}{\pi}\left(-\frac{\cos n\varphi}{n}\right)\Bigg|_{\varphi=0}^{\pi}$$

$$= \frac{200}{n\pi}(1-\cos n\pi)$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{400}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

$$\rightarrow b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{400}{n\pi}\frac{1}{a^n} & \text{if } n \text{ is odd.} \end{cases}$$

Substituting into (*), we get

$$T(\rho,\varphi) = \frac{400}{\pi}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n}\left(\frac{\rho}{a}\right)^n \sin n\varphi$$

Sample Midterm, Problem 2

$$\psi(\vec{r}) = \frac{\vec{P}\cdot\hat{e}_r}{4\pi\varepsilon_0 r^2} = P_0 \frac{\hat{e}_z\cdot\hat{e}_r}{4\pi\varepsilon_0 r^2} = \frac{P_0}{4\pi\varepsilon_0} \frac{\cos\theta}{r^2}$$

$$\vec{E} = -\vec{\nabla}\psi(\vec{r}) = -\left[\frac{\partial\psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\cancelto{0}{\frac{\partial\psi}{\partial\varphi}}\hat{e}_\varphi\right]$$

$$= -\frac{P_0}{4\pi\varepsilon_0}\left[-\frac{2\cos\theta}{r^3}\hat{e}_r - \frac{\sin\theta}{r^3}\hat{e}_\theta\right]$$

$$= \frac{2P_0\cos\theta\,\hat{e}_r + P_0\sin\theta\,\hat{e}_\theta}{4\pi\varepsilon_0 r^3}$$

$$= \frac{3P_0\cos\theta\,\hat{e}_r - P_0\cos\theta\,\hat{e}_r + P_0\sin\theta\,\hat{e}_\theta}{4\pi\varepsilon_0 r^3}$$

$$= \frac{3(\vec{P}\cdot\hat{e}_r)\hat{e}_r - P_0\overbrace{(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta)}^{\hat{e}_z}}{4\pi\varepsilon_0 r^3}$$

$$= \frac{3(\vec{P}\cdot\hat{e}_r)\hat{e}_r - \vec{P}}{4\pi\varepsilon_0 r^3}$$

$$(1-x^2)\,y'' - x y' + n^2 y = 0 \quad -1 \le x \le 1 \qquad (*)$$

a) $$w(x) = \frac{1}{1-x^2}\, e^{\int \frac{-x}{1-x^2}dx} = \frac{1}{1-x^2}\, e^{\frac{1}{2}\ln|1-x^2|}$$

$$= \frac{1}{1-x^2}\, e^{\frac{1}{2}\ln(1-x^2)} \quad \text{since } 1-x^2 \ge 0$$

$$= \frac{1}{1-x^2}\, e^{\ln\sqrt{1-x^2}} = \frac{1}{1-x^2}\sqrt{1-x^2} = \frac{1}{\sqrt{1-x^2}}$$

b) Multiplying the ODE by $w(x) = 1/\sqrt{1-x^2}$, we get:

$$\underbrace{\sqrt{1-x^2}}_{\bar{P}_0(x)}\, y'' \underbrace{- \frac{x}{\sqrt{1-x^2}}}_{\bar{P}_1(x)}\, y' + \frac{n^2}{\sqrt{1-x^2}}\, y = 0 \qquad (**)$$

$$\frac{d}{dx}\bar{P}_0(x) = \frac{d}{dx}\left(\sqrt{1-x^2}\right) = \frac{d}{dx}(1-x^2)^{1/2} = \frac{1}{2}(1-x^2)^{-1/2}(-2x)$$
(chain Rule)

$$= \frac{-x}{\sqrt{1-x^2}} = \bar{P}_1(x).$$ Hence the ODE in $(**)$ is self-adjoint.

(c) For u, v polynomial solutions of $(*)$ [and hence $(**)$], $v^* u'$ and $(v^*)' u$ are also polynomials and hence finite at $+1$ and -1. It follows that

$$\left[v^* \bar{P}_0(x)\, u' - (v^*)' \bar{P}_0(x)\, u\right]_{-1}^{+1} = \left[\sqrt{1-x^2}\left(v^* u' - (v^*)' u\right)\right]_{-1}^{+1} = 0$$

since $\sqrt{1-x^2} = 0$ and $|v^* u' - (v^*)' u|$ is finite at ± 1.

(d) $$\int_{-1}^{+1} T_m^*(x)\, T_n(x)\, w(x)\, dx = 0 \text{ if } m \ne n; \text{ that is,}$$

$$\int_{-1}^{+1} T_m(x)\, T_n(x) \frac{1}{\sqrt{1-x^2}}\, dx = 0 \text{ if } m \ne n, \text{ since}$$

$$\int_{-1}^{+1} T_m(x)\, T_n(x) \frac{1}{\sqrt{1-x^2}}\, dx = 0 \text{ if } m \neq n, \text{ since}$$

$T_m(x)$ is real.