PHYS 3496
Sample Final Exam

Problem 1: Find the steady-state temperature distribution in a solid cylinder of height 10 and radius 1 if the top and curved surface of the cylinder are held at 0° and the base is held at 100°.

Hint: Laplace's Equation in cylindrical coordinates (ρ, ϕ, z) is:

$$\nabla^2 u = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0;$$

but because of the symmetry of this problem with respect to the z-axis (the axis of the cylinder), u is independent of ϕ. Thus, $u = u(\rho, z)$; and Laplace's Equation becomes:

$$\nabla^2 u = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) + \frac{\partial^2 u}{\partial z^2} = 0.$$

Problem 2: If the magnetic induction is given, in cylindrical coordinates, by $\mathbf{B} = B_\phi(\rho)\hat{\mathbf{e}}_\phi$, show that

$$(\mathbf{B} \cdot \nabla)\mathbf{B} = -\frac{B_\phi^2}{\rho}\hat{\mathbf{e}}_\rho.$$

Problem 3: The quantum mechanical orbital angular momentum operator is defined as $\mathbf{L} = -i(\mathbf{r} \times \nabla)$.

(a) Show that

$$\mathbf{L} = i\left(\hat{\mathbf{e}}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial \phi} - \hat{\mathbf{e}}_\phi \frac{\partial}{\partial \theta}\right).$$

(b) Resolving $\hat{\mathbf{e}}_\theta$ and $\hat{\mathbf{e}}_\phi$ into Cartesian components, determine L_x, L_y, and L_z in terms of θ, ϕ, and their derivatives.

<u>Problem 4</u>: Consider Laguerre's ODE:

$$xy'' + (1-x)y' + py = 0. \tag{1}$$

(a) Show that multiplying the ODE by $w(x) = e^{-x}$ puts it into a self-adjoint form.

(b) Show that the Hermitian operator boundary condition

$$\left[v^* x e^{-x} u' - (v^*)' x e^{-x} u\right]_0^\infty = 0$$

holds for polynomial solutions u and v of Equation (1).

(c) The Laguerre polynomials $L_0(x), L_1(x), L_2(x), \cdots$ are solutions of the Laguerre ODE that correspond to $p = 0, 1, 2, \cdots$, respectively. Using the results of parts (a) and (b) above, write down the orthogonality condition for the Laguerre polynomials.

(d) Let $\psi = e^{-x/2} y$. Then by substituting $y = e^{x/2}\psi$ into Equation (1), show that ψ satisfies the ODE

$$x\psi'' + \psi' + \left(\frac{1}{2} - \frac{x}{4} + p\right)\psi = 0 \tag{2}$$

and show that the ODE in Equation (2) is self-adjoint.

(e) For each $m = 0, 1, 2, \cdots$, let $\psi_m(x) = e^{-x/2} L_m(x)$. Then ψ_m is a solution of Equation (2) for $p = m$. Show that the Hermitian operator boundary condition for Equation (2):

$$\left[v^* x u' - (v^*)' x u\right]_0^\infty = 0$$

holds for $u = \psi_m$ and $v = \psi_n$.

(f) Using the results of parts (d) and (e), write down the orthogonality condition for the ψ_m's and show that this is equivalent to the orthogonality condition for the L_m's in part (c).

<u>Remark</u>: All of the results above show that multiplying the whole ODE by $w(x) = e^{-x}$ or replacing the dependent variable y by $\psi = \sqrt{w(x)}y = e^{-x/2}y$ lead to the same results, and hence the two approaches are equivalent.

<u>Problem 5</u>: Develop the Taylor expansion of

$$f(z) = \frac{1}{(z+1)(z+2)}$$

about the point $z = 0$. Specify the exact range of validity of your expansion.

<u>Problem 6</u>: Evaluate the integral

$$I = \int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}.$$

<u>Hint</u>: Make the change of variable $z = e^{i\theta}$ to rewrite the integral as a contour integral around the unit circle $|z| = 1$.

<u>Problem 7</u> (10 marks): Evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{x^2}{x^4 + 1} dx.$$

<u>Hint</u>: In computing the residues, L'Hôpital's rule may be useful.

<u>Problem 8</u>: Evaluate the integral

$$I = \int_0^{\infty} \frac{x^p}{x^2 + 1} dx, \ 0 < p < 1.$$

<u>Hint</u>: Consider the contour integral

$$\oint \frac{z^p}{z^2 + 1} dz$$

for a suitable contour that avoids the branch point 0 and the branch cut taken along the positive x-axis.