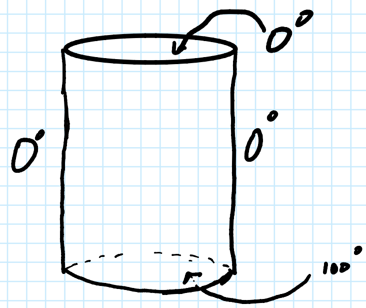
1.



$$\nabla^2 u = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) + \frac{\partial^2 u}{\partial z^2} = 0 \qquad (i)$$

let $u(\rho, z) = R(\rho) Z(z)$. Then, substituting into (i), we get:

$$\frac{Z(z)}{\rho}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + R(\rho)\frac{d^2 Z}{dz^2} = 0.$$ Multiplying by $\frac{1}{RZ}$, we get:

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0$$

$$\Rightarrow \frac{1}{\rho R}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) = -\frac{1}{Z}\frac{d^2 Z}{dz^2} = -K^2 \qquad (ii)$$

$$(ii) \Rightarrow \frac{d^2 Z}{dz^2} - K^2 Z = 0 \longrightarrow Z(z) = \begin{Bmatrix} e^{-Kz} \\ e^{Kz} \end{Bmatrix}$$

To make $Z(10) = 0$, we use

$\boxed{Z(z) = \sinh K(10 - z)}$, which is a linear combination of e^{Kz} and e^{-Kz}.

Next we try to find $R(\rho)$: from (ii), we get

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) = -K^2 \longrightarrow$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + K^2 R = 0 \Rightarrow$$

$\rho \quad d\rho \qquad d\rho\,)$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + K^2 R = 0.$$ Multiplying by ρ^2, we get:

$$\rho^2 \frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} + K^2\rho^2 R = 0.$$ This is a modified Bessel Equation, whose solutions are $J_0(K\rho)$ and $N_0(K\rho)$. But $N_0(K\rho)$ is infinite at $\rho = 0$; so we leave that solution out.

Thus, $\boxed{R(\rho) = J_0(K\rho)}$.

From $u(1, z) = 0$ (boundary condition on the curved surface of the cylinder), we get:

$R(1) = 0$ and hence $J_0(K) = 0$. Thus,

$K = K_m$, a zero of J_0.

Thus, for each $m = 1, 2, 3, \ldots$, we obtain one eigenvalue K_m, the m^{th} zero of J_0 and the corresponding eigensolution:

$$u_m(\rho, z) = R_m(\rho)\, Z_m(z)$$
$$= J_0(K_m\rho)\sinh K_m(10 - z).$$

To match the boundary condition at the bottom of the cylinder, we write $u(\rho, z) = \sum_{m=1}^{\infty} b_m J_0(K_m\rho)\sinh K_m(10 - z)$

$$u(\rho, 0) = 100 = \sum_{m=1}^{\infty} b_m J_0(K_m\rho)\sinh(10K_m) \quad \text{(iii)}$$

Multiplying both sides by $\rho J_0(K_\mu\rho)$ and integrating from 0 to 1 $\rightarrow$

$$100\int_0^1 \rho J_0(K_\mu\rho)\,d\rho = \sum_{m=1}^{\infty} b_m \sinh(10K_m)\int_0^1 \rho J_0(K_m\rho) J_0(K_\mu\rho)\,d\rho$$

$$= b_\mu \sinh(10K_\mu)\,\frac{J_1^2(K_\mu)}{2} \quad \text{(iv)}.$$

But from $\int x J_0(x)\,dx = x J_1(x)$, with $x = K_\mu\rho$, we get:

But from $\int x J_0(x)\,dx = x J_1(x)$, with $x = \kappa_\mu \rho$, we get:

$$\kappa_\mu^2 \int \rho J_0(\kappa_\mu \rho)\,d\rho = \kappa_\mu \rho J_1(\kappa_\mu \rho) \text{ ; and hence}$$

$$\int_0^1 \rho J_0(\kappa_\mu \rho)\,d\rho = \frac{1}{\kappa_\mu} \rho J_1(\kappa_\mu \rho)\Big|_{\rho=0}^{1} = \frac{J_1(\kappa_\mu)}{\kappa_\mu}.$$

$$(iv) \longrightarrow \frac{100\, J_1(\kappa_\mu)}{\kappa_\mu} = b_\mu \sinh(10\,\kappa_\mu)\, \frac{J_1^2(\kappa_\mu)}{2}$$

$$\longrightarrow b_\mu = \frac{200}{\kappa_\mu \sinh(10\kappa_\mu)\, J_1(\kappa_\mu)} \quad \text{or}$$

$$b_m = \frac{200}{\kappa_m \sinh(10\kappa_m)\, J_1(\kappa_m)}, \text{ for } m \geq 1. \text{ Thus,}$$

$$u(\rho, z) = \sum_{m=1}^{\infty} \frac{200}{\kappa_m \sinh(10\kappa_m)\, J_1(\kappa_m)} J_0(\kappa_m \rho) \sinh \kappa_m (10-z)$$

where κ_m is the m^{th} zero of J_0.

2. $\vec{B} = B_\varphi(\rho)\,\hat{e}_\varphi$

$$(\vec{B}\cdot\vec{\nabla})\vec{B} = \left[B_\varphi\,\hat{e}_\varphi\cdot\left(\hat{e}_\rho\frac{\partial}{\partial\rho} + \frac{\hat{e}_\varphi}{\rho}\frac{\partial}{\partial\varphi} + \hat{e}_z\frac{\partial}{\partial z}\right)\right]\left(B_\varphi(\rho)\,\hat{e}_\varphi\right)$$

$$= \frac{B_\varphi}{\rho}\frac{\partial}{\partial\varphi}\left(B_\varphi(\rho)\,\hat{e}_\varphi\right)$$

$$= \frac{B_\varphi^2}{\rho}\frac{\partial\hat{e}_\varphi}{\partial\varphi}$$

$\hat{e}_x$

But $\hat{e}_\varphi = -\sin\varphi\,\hat{e}_x + \cos\varphi\,\hat{e}_y$ and

$\hat{e}_\rho = \cos\varphi\,\hat{e}_x + \sin\varphi\,\hat{e}_y$. Hence

$\dfrac{\partial\hat{e}_\varphi}{\partial\varphi} = -\hat{e}_\rho$. Thus,

$$(\vec{B}\cdot\vec{\nabla})\vec{B} = -\frac{B_\varphi^2}{\rho}\hat{e}_\rho$$

3. $\vec{L} = -i\,(\vec{r}\times\vec{\nabla})$

(a) $$\vec{L} = -i\, r\,\hat{e}_r \times \left[\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_\varphi \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\right]$$

$$= -i\left[\hat{e}_\varphi \frac{\partial}{\partial\theta} - \hat{e}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right]$$

$$= i\left[\hat{e}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} - \hat{e}_\varphi \frac{\partial}{\partial\theta}\right]$$

(b) $$\hat{e}_\theta = \hat{e}_x \cos\theta\cos\varphi + \hat{e}_y \cos\theta\sin\varphi - \hat{e}_z \sin\theta$$

$$\hat{e}_\varphi = -\hat{e}_x \sin\varphi + \hat{e}_y \cos\varphi$$

$$\rightarrow \vec{L} = i\left[(\hat{e}_x\cos\theta\cos\varphi + \hat{e}_y\cos\theta\sin\varphi - \hat{e}_z\sin\theta)\cdot \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} + (\hat{e}_x\sin\varphi - \hat{e}_y\cos\varphi)\frac{\partial}{\partial\theta}\right]$$

$$= \hat{e}_x \overbrace{\left[i\sin\varphi\frac{\partial}{\partial\theta} + i\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right]}^{L_x}$$

$$+ \hat{e}_y \overbrace{\left[-i\cos\varphi\frac{\partial}{\partial\theta} + i\cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right]}^{L_y}$$

$$+ \hat{e}_z \underbrace{\left[-i\frac{\partial}{\partial\varphi}\right]}_{L_z}$$

4. $x y'' + (1-x) y' + p y = 0 \qquad (1)$

(a) $$\underbrace{x e^{-x}}_{\bar{P}_0(x)} y'' + \underbrace{(1-x)e^{-x}}_{\bar{P}_1(x)} y' + p e^{-x} y = 0 \qquad (*)$$

$$\frac{d}{dx}\bar{P}_0(x) = \frac{d}{dx}\left[x e^{-x}\right] = e^{-x} - x e^{-x} = (1-x)e^{-x}$$

4. $xy'' + (1-x)y' + py = 0$ (1)

(a) $\underbrace{xe^{-x}}_{\bar{P}_0(x)} y'' + \underbrace{(1-x)e^{-x}}_{\bar{P}_1(x)} y' + pe^{-x} y = 0$ (*)

$\frac{d}{dx}\bar{P}_0(x) = \frac{d}{dx}[xe^{-x}] = e^{-x} - xe^{-x} = (1-x)e^{-x}$

$= \bar{P}_1(x)$. Hence the ODE in (*) is self-adjoint.

(b) for u, v polynomial solutions (1),

$v^* u'$ and $(v^*)' u$ are polynomials

Thus $xe^{-x} v^* u'$ and $xe^{-x}(v^*)'u$ both vanish at $x=0$ and at ∞ since $\lim_{x\to\infty} e^{-x}(\text{polynomial}) = 0$

(c) $\int_0^\infty L_m^*(x) L_n(x) w(x)\, dx = 0$ if $m \neq n$

or $\int_0^\infty L_m(x) L_n(x) e^{-x}\, dx = 0$ if $m \neq n$, since $L_m(x)$ is real.

(d) Let $\psi = e^{-x/2} y$. Then $y = e^{x/2}\psi$. Thus,

$y' = \frac{1}{2}e^{x/2}\psi + e^{x/2}\psi'$ and

$y'' = \frac{1}{4}e^{x/2}\psi + e^{x/2}\psi' + e^{x/2}\psi''$

Substituting into (1), we get

$$x\left[\cancel{e^{x/2}}\left(\tfrac{1}{4}y + y' + y''\right)\right]$$
$$+(1-x)\,\cancel{e^{x/2}}\left(\tfrac{1}{2}y + y'\right) + p\,\cancel{e^{x/2}}\,y = 0$$

$$\Rightarrow \underbrace{x}_{P_0}\, y'' + \underbrace{1}_{P_1}\, y' + \left(\tfrac{1}{2} - \tfrac{1}{4}x + p\right) y = 0 \quad (**)$$

$\frac{d}{dx}P_0 = \frac{d}{dx}x = 1 = P_1$; hence the ODE in (**)

is self-adjoint.

(e) Since $u = y_m = e^{-x/2} L_m$ and
$v = y_n = e^{-x/2} L_n$, it follows

that $v^* u'$ and $(v^*)' u$ are both of the form

$e^{-x} \cdot$ (polynomial) and hence

$x v^* u'$ and $x (v^*)' u$ are both of the form

$x e^{-x} \cdot$ [polynomial] and they both vanish

at 0 and ∞ as in part (b)

(f) $\int_0^\infty y_m(x)\, y_n(x)\, dx = 0$ if $m \neq n$
[the weight function is 1 in this case]

This is equivalent to
$\int_0^\infty L_m(x) e^{-x/2} L_n(x) e^{-x/2}\, dx = 0$ if $m \neq n$ or

$$\int_0^\infty L_m(x)\, L_n(x)\, e^{-x}\, dx = 0 \text{ if } m \neq n$$

which is the orthogonality condition for the Laguerre polynomials (part (c)).

5. $$f(z) = \frac{1}{(z+1)(z+2)} = \frac{1}{z+1} - \frac{1}{z+2}$$

$$\frac{1}{z+1} = \frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n z^n, \quad |z|<1$$

$$\frac{1}{z+2} = \frac{1}{2}\,\frac{1}{1+\frac{z}{2}} = \frac{1}{2}\left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 + \cdots\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{n+1}} z^n, \quad |z/2|<1 \text{ or } |z|<2$$

Thus, $$f(z) = \frac{1}{z+1} - \frac{1}{z+2}$$

$$= \sum_{n=0}^{\infty} (-1)^n z^n - \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{n+1}} z^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \left[1 - \frac{1}{2^{n+1}}\right] z^n$$

$$= \sum_{n=0} \cdots \frac{\cdots}{2^{n+1}} \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2^{n+1}-1}{2^{n+1}} z^n$$

$$= \left[\frac{1}{2} - \frac{3}{4} z + \frac{7}{8} z^2 - \frac{15}{16} z^3 + \cdots \right]$$

The radius of convergence is the distance from $z=0$ to the nearest singular point, $z=-1$. So $R=1$. The series converges for $|z|<1$.

6. $$\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$$

Let $z = e^{i\theta}$. Then $dz = i e^{i\theta} d\theta = i z \, d\theta$

and hence $d\theta = \frac{dz}{iz} = -i \frac{dz}{z}$.

Also $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}(z+z^{-1})$.

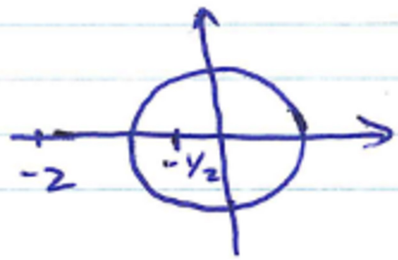
Thus,
$$I = \int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} = -i \oint_C \frac{dz}{z[5+2(z+z^{-1})]}$$

where C is the unit circle $|z|=1$, run counterclockwise.

$$I = -i \oint_C \frac{dz}{2z^2+5z+2} = -i \oint_C \frac{dz}{(2z+1)(z+2)}$$

$$= -i \left[2\pi i \cdot \operatorname{Res}\left(\frac{1}{(2z+1)(z+2)}, z=-\frac{1}{2} \right) \right]$$

$$= -i\left[2\pi i \cdot \mathrm{Res}\left(\frac{1}{(2z+1)(z+2)}, z=-\frac{1}{2}\right)\right]$$

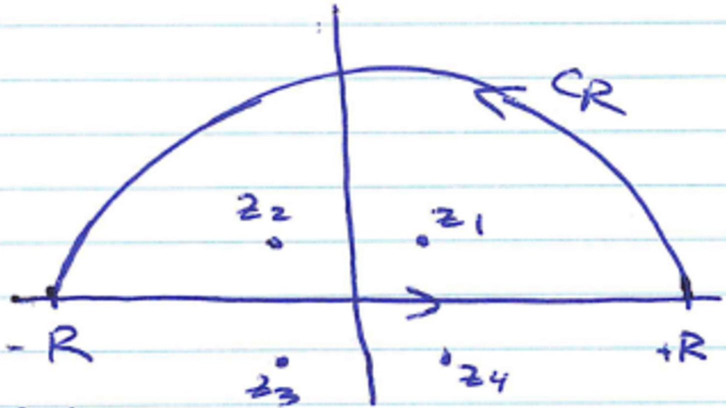


$$I = 2\pi \lim_{z \to -1/2} (z + 1/2)\frac{1}{(2z+1)(z+2)}$$

$$= 2\pi \lim_{z \to -1/2} \frac{1}{2(z+2)}$$

$$= 2\pi \frac{1}{2(3/2)} = \frac{2\pi}{3}$$

7. $I = \int_{-\infty}^{+\infty} \frac{x^2}{x^4+1}\, dx$



Let $R > 1$ and let C be the contour shown above. Then

$$\oint_C \frac{z^2}{z^4+1}\, dz = \int_{-R}^{+R} \frac{x^2}{x^4+1}\, dx + \int_{C_R} \frac{z^2}{z^4+1}\, dz$$

$= 2\pi i \sum$ (Residues of $\frac{z^2}{z^4+1}$ in the upper half-plane).

The poles of $\frac{z^2}{z^4+1}$ are the roots of $z^4+1=0$ i.e. the fourth roots of $-1 = e^{i\pi}$.

$z_1 = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$, $\quad z_2 = e^{i(\pi/4 + 2\pi/4)} = e^{i3\pi/4}$

$= \frac{-1+i}{\sqrt{2}}$

$z_3 = e^{i(\pi/4 + 4\pi/4)} = -z_1$; and $z_4 = -z_2$

$z_3 = e^{i(\pi/4 + 4\pi/4)} = -z_1$; and $z_4 = -z_2$

~~Thus~~ $\cancel{I = 2\pi i \left[\mathrm{Res} \left(z^2 \right.\right.}$

Letting $R \longrightarrow \infty$, $\displaystyle\int_{C_R} \frac{z^2}{z^4+1}\, dz \longrightarrow 0$ since

$$\lim_{|z| \to \infty} \left| z \cdot \frac{z^2}{z^4+1} \right| = 0; \text{ and}$$

$$\int_{-R}^{+R} \frac{x^2}{x^4+1}\, dx \longrightarrow \int_{-\infty}^{+\infty} \frac{x^2}{x^4+1}\, dx = I.$$

Thus, $I = 2\pi i \left[\mathrm{Res}\left(\frac{z^2}{z^4+1}, z_1 \right) + \mathrm{Res}\left(\frac{z^2}{z^4+1}, z_2 \right) \right]$

$$= 2\pi i \left[\lim_{z \to z_1} \frac{(z-z_1) z^2}{z^4+1} + \lim_{z \to z_2} \frac{(z-z_2) z^2}{z^4+1} \right]$$

$$= 2\pi i \left[\lim_{z \to z_1} \frac{z^2 + 2z(z-z_1)}{4z^3} + \lim_{z \to z_2} \frac{z^2 + 2z(z-z_2)}{4z^3} \right]$$

$$= 2\pi i \left[\frac{1}{4z_1} + \frac{1}{4z_2} \right]$$

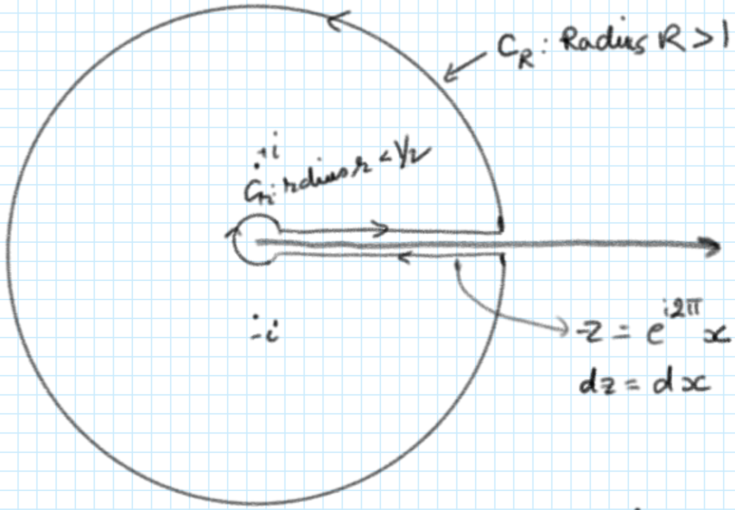
$$I = \frac{\pi i}{2} \left[e^{-i\pi/4} + e^{-i3\pi/4} \right]$$

$$= -\frac{\pi}{2i} \left[e^{-i\pi/4} - e^{i\pi/4} \right]$$

$$= \pi \, \frac{e^{i\pi/4} - e^{-i\pi/4}}{2i} = \pi \sin \pi/4 = \frac{\pi}{\sqrt{2}}.$$

8. $I = \int^{\infty} x^p \, dx \qquad 0 < p < 1.$

8. $I = \int_0^\infty \frac{x^p}{x^2+1}\,dx \qquad 0 < p < 1.$



$$\oint_C \frac{z^p}{z^2+1}\,dz = \int_r^R \frac{x^p}{x^2+1}\,dx + \int_{C_R} \frac{z^p}{z^2+1}\,dz + \int_R^r \frac{(xe^{i2\pi})^p}{x^2+1}\,dx$$

$$+ \int_{C_r} \frac{z^p}{z^2+1}\,dz$$

$$= 2\pi i\left[\operatorname{Res}\left(\frac{z^p}{z^2+1}, i\right) + \operatorname{Res}\left(\frac{z^p}{z^2+1}, -i\right)\right].$$

$$\operatorname{Res}\left(\frac{z^p}{z^2+1}, i\right) = \lim_{z \to i = e^{i\pi/2}} \frac{(z-i)\,z^p}{z^2+1} = \frac{e^{ip\pi/2}}{2i}$$

$$\operatorname{Res}\left(\frac{z^p}{z^2+1}, -i\right) = \lim_{z \to -i = e^{i3\pi/2}} \frac{(z+i)\,z^p}{z^2+1} = \frac{e^{i3p\pi/2}}{-2i}.$$

Thus, $\left(1 - e^{i2\pi p}\right) \int_r^R \frac{x^p}{x^2+1}\,dx + \int_{C_R} \frac{z^p}{z^2+1}\,dz + \int_{C_r} \frac{z^p}{z^2+1}\,dz$

$$= 2\pi i\left[\frac{e^{ip\pi/2}}{2i} - \frac{e^{i3p\pi/2}}{2i}\right] = \pi\left[e^{ip\pi/2} - e^{i3p\pi/2}\right].$$

Now let $r \to 0$ and $R \to \infty$.

Since $\lim_{|z| \to \infty} \left| z\,\frac{z^p}{z^2+1} \right| = 0 \quad (p<1)$, $\int_{C_R} \frac{z^p}{z^2+1}\,dz \xrightarrow[R\to\infty]{} 0$

Also, $\left| \int_{C_r} \frac{z^p}{z^2+1}\,dz \right| < 2\left| \int_{C_r} z^p\,dz \right| = 2\left| \int_0^{2\pi} r^p e^{ip\theta}\, i r e^{i\theta}\,d\theta \right|$

$$\leq 2\int_0^{2\pi} r^{p+1}\,d\theta = 2\pi\, r^{p+1} \xrightarrow[r\to 0]{} 0$$

Thus, $\int \frac{z^p}{\ldots}\,dz \longrightarrow 0.$

$$\leq 2 \int_0 r \, d\theta = 2\pi r \xrightarrow[r \to 0]{} 0$$

Thus, $\displaystyle\int_{C_r} \frac{z^p}{z^2+1}\,dz \xrightarrow[r \to 0]{} 0$.

Moreover, $\displaystyle\int_r^R \frac{x^p}{x^2+1}\,dx \xrightarrow[r \to 0]{R \to \infty} \int_0^\infty \frac{x^p}{x^2+1}\,dx = I$. Thus,

$$\left(1 - e^{i2p\pi}\right) I = \pi \left(e^{ip\pi/2} - e^{i3p\pi/2}\right).$$

Dividing both sides by $\left(-2i\, e^{ip\pi}\right)$, we get:

$$\left(\frac{e^{ip\pi} - e^{-ip\pi}}{2i}\right) I = \pi \left(\frac{e^{ip\pi/2} - e^{-ip\pi/2}}{2i}\right).$$ Hence

$$I = \pi \frac{\sin(p\pi/2)}{\sin(p\pi)} = \pi \frac{\sin(p\pi/2)}{2\sin(p\pi/2)\cos(p\pi/2)}$$

$$\boxed{I = \frac{\pi}{2\cos(p\pi/2)}}$$