Review of Legendre functions

$$(1-x^2)\,y'' - 2x\,y' + \ell(\ell+1)\,y = 0 \leftarrow \text{Legendre Equation}$$

Recall: we looked for solutions of the form $y = \sum_{n=0}^{\infty} a_n x^n$

$$\rightarrow y = a_0\left[1 - \frac{\ell(\ell+1)}{2!}x^2 + \frac{\ell(\ell+1)(\ell-2)(\ell+3)}{4!}x^4 - + \cdots\right]$$

$$+ a_1\left[x - \frac{(\ell-1)(\ell+2)}{3!}x^3 + \frac{(\ell-1)(\ell+2)(\ell-3)(\ell+4)}{5!}x^5 - + \cdots\right]$$

To get solutions that converge for all x in $[-1,+1]$ $\rightarrow \begin{pmatrix} x=\cos\theta \\ 0\le\theta\le\pi \end{pmatrix}$

one of the two series solutions had to terminate. This happens if $\ell = 0, 1, 2, \ldots$

$\rightarrow$ Legendre polynomial of degree ℓ: $P_\ell(x)$.

$P_\ell(1) = 1$ for all $\ell = 0, 1, 2, \ldots$

$P_\ell(x)$ is an odd function of x if ℓ is odd $(\ell = 1, 3, 5, \ldots)$

$P_\ell(x)$ is an even function of x if ℓ is even $(\ell = 0, 2, 4, \ldots)$

e.g.
$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$
$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$
$$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$$
$$\vdots$$

Orthogonality condition

$$\int_{-1}^{+1} P_\ell(x)\, P_m(x)\, dx = \begin{cases} 0 & \text{if } \ell \neq m \\ \frac{2}{2\ell+1} & \text{if } \ell = m \end{cases}$$

$$= \frac{2}{2\ell+1}\, \underline{\delta_{\ell m}}$$

$$f(x) = \sum_{\ell=0}^{\infty} c_\ell\, P_\ell(x) \quad \text{Legendre series}$$

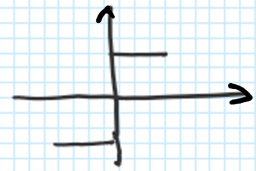
$$\int_{-1}^{+1} f(x)\, P_m(x)\, dx = \sum_{\ell=0}^{\infty} c_\ell \underbrace{\int_{-1}^{+1} P_\ell(x)\, P_m(x)\, dx}_{\frac{2}{2\ell+1}\delta_{\ell m}}$$

$$= c_m \frac{2}{2m+1}$$

$$\longrightarrow \boxed{c_m = \frac{2m+1}{2} \int_{-1}^{+1} f(x)\, P_m(x)\, dx}$$

Example : $f(x) = \begin{cases} 1 & 0 < x < 1 & \left(0 < \theta < \frac{\pi}{2}\right) \\ -1 & -1 < x < 0 & \left(\frac{\pi}{2} < \theta < \pi\right) \end{cases}$

$$f(x) = \sum_{\ell=0}^{\infty} c_\ell P_\ell(x)$$



$$c_\ell = \frac{2\ell+1}{2} \int_{-1}^{+1} \underline{f(x)}\ \underline{P_\ell(x)}\, dx$$

$$= \begin{cases} 0 & \text{if } \ell \text{ is even} \\ (2\ell+1) \int_0^1 f(x) P_\ell(x)\, dx \end{cases}$$ (since $f(x)$ is an odd function of x)

$$c_\ell = (2\ell+1) \int_0^1 P_\ell(x)\, dx \quad \text{for } \ell = 1, 3, 5, \ldots$$

$$c_1 = 3 \int_0^1 x\, dx = \frac{3}{2}$$

$$c_3 = 7 \int_0^1 P_3(x)\, dx = -\frac{7}{8}$$

$$c_5 = 11 \int_0^1 P_5(x)\, dx = \frac{11}{16}$$

$\vdots$

$$f(x) = c_1 P_1(x) + c_3 P_3(x) + c_5 P_5(x) + \cdots$$

$$= \frac{3}{2} P_1(x) - \frac{7}{8} P_3(x) + \frac{11}{16} P_5(x) + \cdots$$

Associated Legendre Equation:

$$(1-x^2) y'' - 2x y' + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right] y = 0 \qquad (m^2 \le \ell^2)$$

$\rightarrow$ solutions : $P_\ell^m(x)$ $\leftarrow$ Associated Legendre Functions

To get solutions that are valid for $-1 \le x \le +1$, we must have:

$\ell = 0, 1, 2, 3, \ldots$

$m = -\ell, -\ell+1, \ldots, -1, 0, 1, \ldots, \ell-1, \ell$

$$P_\ell^0(x) \equiv P_\ell(x).$$