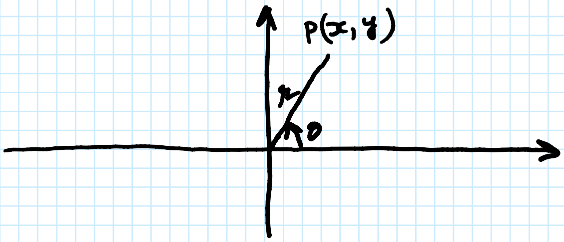
Review of Complex Numbers (Section 1.8 in Arfken's book or Chapter 2 in Boas' book.)

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$$



$$z = x + iy \xleftrightarrow[\text{correspondence}]{\text{one-to-one}} P(x,y) \text{ in the plane}$$

Cartesian representation

$$z = x + iy$$

$\Downarrow$

more convenient if we are adding or subtracting

e.g. $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

Polar representation

$$z = r e^{i\theta}$$

$\Downarrow$

more convenient if we are multiplying or dividing

e.g. $z_1 \cdot z_2 = (r_1 e^{i\theta_1}) \cdot (r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$$

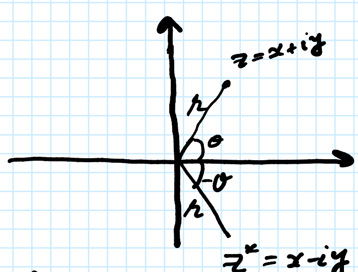
$$z = x + iy = r\cos\theta + i r\sin\theta = r\underbrace{(\cos\theta + i\sin\theta)}_{e^{i\theta}} = r e^{i\theta}$$

<u>complex conjugate</u>:

$$z = x + iy$$
$$z^* = x - iy$$

$\Downarrow$

$$z \cdot z^* = (x + iy)(x - iy) = x^2 + y^2 = r^2 = |z|^2$$



$$z = r e^{i\theta}$$
$$z^* = r e^{-i\theta}$$

$\Downarrow$

$$z z^* = r^2 = |z|^2$$

If $z \neq 0$

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{x - iy}{x^2 + y^2} = \frac{z^*}{|z|^2}$$

$$\frac{1}{z} = \frac{z^*}{|z|^2}$$

$$= \frac{x - iy}{x^2+y^2} = \frac{\bar{z}}{|z|^2} \leftarrow$$

Elementary Functions:

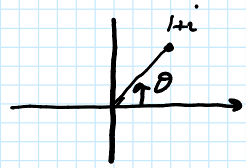
- **Power**: for n in $\mathbb{N}$, $z^n = (re^{i\theta})^n = r^n e^{in\theta}$

Example: Compute $(1+i)^{100}$

It is convenient to write $1+i$ in polar form

$r = \sqrt{x^2+y^2} = \sqrt{1+1} = \sqrt{2}$

$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1} 1 \rightarrow \theta = \begin{cases} \pi/4 \\ \text{or} \\ 5\pi/4 \times \end{cases}$



But since x, y are both positive, $z = 1+i$ is in the first quadrant

$\rightarrow 0 < \theta < \pi/2 \rightarrow \theta = \pi/4$.

So $1+i = \sqrt{2}\, e^{i\pi/4}$

$\rightarrow (1+i)^{100} = 2^{50} e^{i\,100\pi/4} = 2^{50} e^{i25\pi} = 2^{50} \cancel{e^{i24\pi}}^{\,1} e^{i\pi}$

$= 2^{50} e^{i\pi} = -2^{50}$ $\quad$ [$e^{i\pi} = \cos\pi + i\sin\pi = -1$

$\Downarrow$

$\boxed{e^{i\pi} + 1 = 0}$ Euler's identity relating five fundamental mathematical constants:

$0, 1, i, e$ and π.]

- $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all z.

$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$ " " " z.

$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ " " " z.

$\rightarrow e^{iz} = \cos z + i \sin z$ and $e^{-iz} = \cos z - i \sin z$

$\rightarrow \cos z = \frac{e^{iz} + e^{-iz}}{2}$ and $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$

$2 \qquad\qquad 2i$

In particular, $e^{i\theta} = \cos\theta + i\sin\theta$, for any real θ : Euler's formula.

Also
$$e^{z_1}\cdot e^{z_2} = e^{z_1+z_2}$$
$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}, \text{ etc..}$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$
$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$
$$\cos^2 z + \sin^2 z = 1$$
etc ---

- Hyperbolic Trigonometric functions:

$$\cosh z = \frac{e^z + e^{-z}}{2}, \qquad \sinh z = \frac{e^z - e^{-z}}{2}$$

$\rightarrow$ $\cosh z = \cos(iz)$; $\sinh z = -i\sin(iz)$

$\rightarrow$ $\cosh^2 z - \sinh^2 z = 1$, etc --

- Roots : Let $w = \rho e^{i\varphi}$ be an n^{th} root of $z = r e^{i\theta}$. Then

$$w^n = z$$

$\rightarrow$ $\rho^n e^{in\varphi} = r e^{i\theta}$

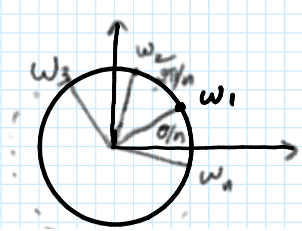
$\rightarrow$ $\begin{cases} \rho^n = r \\ \text{and} \\ n\varphi = \theta + 2k\pi : k \text{ integer} \end{cases}$

$\rightarrow$ $\begin{cases} \rho = r^{1/n} \\ \varphi = \dfrac{\theta}{n} + \dfrac{2k\pi}{n} \end{cases}$ $\rightarrow$ enough to take $k = 0, 1, --, n-1$

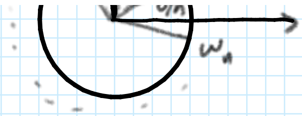
$\rightarrow$ n distinct n^{th} roots, equally spaced on the circle of radius $r^{1/n}$ centered at the origin:

$k=0 \longrightarrow w_1 = r^{1/n} e^{i\theta/n}$

$k=1 \longrightarrow w_2 = r^{1/n} e^{i\left(\frac{\theta}{n} + \frac{2\pi}{n}\right)}$



$k=1 \rightarrow w_2 = r^{1/n} e^{i(\frac{\theta}{n}+\frac{2\pi}{n})}$



$\vdots$

$k=n-1 \rightarrow w_n = r^{1/n} e^{i[\frac{\theta}{n}+\frac{2(n-1)\pi}{n}]}$

Note that for $k=n$ we go back to w_1 as the angle will be $\frac{\theta}{n}+2\pi$

$\rightarrow$ so no new root!

So every complex number has exactly n n^{th} roots

$\rightarrow$ the root function is multi-valued. Note that

$$w_1 + w_2 + \cdots + w_n = 0$$

- $\ln z$: As in $\mathbb{R}$, $\ln z$ is defined as the inverse function of the exponential; that is,

$$w = \ln z \iff z = e^w$$

$\ln z$ is defined for all $z \neq 0$.

- From $e^{w_1} \cdot e^{w_2} = e^{w_1+w_2}$, we get

$$\ln(e^{w_1} \cdot e^{w_2}) = w_1 + w_2, \text{ or}$$

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2$$

In particular, $\ln(r e^{i\theta}) = \ln(r e^{i(\theta+2k\pi)})$

$$= \underbrace{\ln r}_{\text{Real logarithm}} + i(\theta+2k\pi) : k \text{ integer}$$

$\rightarrow$ $\ln z$ is multivalued

<u>Example</u>: $\ln(-1) = \ln(e^{i\pi}) = \ln 1 + i(\pi+2k\pi) = i(\pi+2k\pi)$

$\rightarrow \ln(-1) = \pm i\pi, \pm i3\pi, \ldots$

- Complex power:

In $\mathbb{R}$: $\ln a^b = b\ln a \iff a^b = e^{b\ln a}$, $a>0$

In $\mathbb{C}$: Define, similarly, $a^b = e^{b\ln a}$, $a \neq 0$

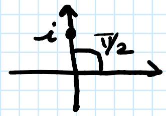
(in $\mathbb{C}$, $\ln a$ is defined for any $a \neq 0$.)

(in $\mathbb{C}$, $\ln a$ is defined for any $a \neq 0$.)

$\rightarrow$ a^b is multivalued.

__Example:__ $i^{2i} = e^{2i \ln i}$

$$i = 1\, e^{i\pi/2}$$



$$\rightarrow \ln i = \ln 1 + i\left(\pi/2 + 2k\pi\right) = i\left(\pi/2 + 2k\pi\right) = i\,(4k+1)\,\pi/2 \,, \quad k: \text{integer}$$

$$\rightarrow i^{2i} = e^{2i \ln i} = e^{2i \cdot i\,(4k+1)\pi/2} = e^{-(4k+1)\pi} \;:\; k = 0, \pm 1, \pm 2, \dots$$

$$= e^{(-4k-1)\pi} = e^{(4n-1)\pi} \;:\; n = -k \,;\; n = 0, \pm 1, \pm 2, \dots$$

$$\rightarrow i^{2i} = e^{-\pi},\ e^{3\pi},\ e^{-5\pi},\ e^{7\pi},\ e^{-9\pi},\ \dots$$