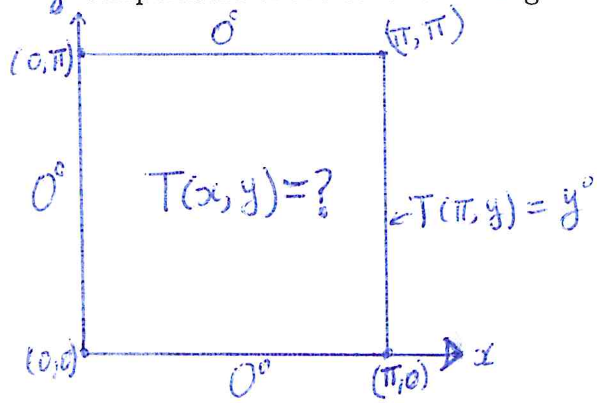
PHYS 3496 Midterm Exam
Tuesday, October 22, 2024

Instructions: Please read the following instructions before you start working on your problems.

- Write your name and student number on the provided examination booklet.
- All course materials must remain closed during the exam but you are allowed to have one eight and a half by eleven inch sheet of formulae and equations.
- **No calculators allowed** (they are not needed!)
- **Write all necessary steps to get full credit**

Good Luck!

Problem 1 (12 marks): Find the steady-state temperature distribution $T(x, y)$ inside the square metal plate given by $0 \leq x \leq \pi$ and $0 \leq y \leq \pi$ if the temperature of the right edge is $T(\pi, y) = y$ for $0 < y < \pi$ and the temperature of the other three edges is $0°$.



Hints:

1. The temperature in the plate satisfies the two-dimensional Laplace's equation

$$\nabla^2 T(x, y) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

2.

$$\int y \sin(ay)dy = \frac{-y\cos(ay)}{a} + \frac{\sin(ay)}{a^2} + C$$

Write all necessary steps, including separation of variables and applying all boundary conditions, to get full credit

Problem 2 (6 marks): The quantum mechanical orbital angular momentum operator is defined as $\mathbf{L} = -i(\mathbf{r} \times \nabla)$.

(a) Show that

$$\mathbf{L} = i\left(\hat{\mathbf{e}}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - \hat{\mathbf{e}}_\phi \frac{\partial}{\partial\theta}\right).$$

(b) Resolving $\hat{\mathbf{e}}_\theta$ and $\hat{\mathbf{e}}_\phi$ into Cartesian components

$$\begin{aligned} \hat{\mathbf{e}}_\theta &= \hat{\mathbf{e}}_x \cos\theta\cos\phi + \hat{\mathbf{e}}_y \cos\theta\sin\phi - \hat{\mathbf{e}}_z \sin\theta \\ \hat{\mathbf{e}}_\phi &= -\hat{\mathbf{e}}_x \sin\phi + \hat{\mathbf{e}}_y \cos\phi, \end{aligned}$$

determine L_x, L_y, and L_z in terms of θ, ϕ, and their derivatives.

<u>Problem 3</u> (7 marks): Consider the Hermite ODE:

$$y'' - 2xy' + 2\alpha y = 0; \qquad -\infty < x < \infty. \tag{1}$$

The Hermite polynomials $H_0(x), H_1(x), H_2(x), \cdots$ are real polynomial solutions of the Hermite ODE that correspond to $\alpha = 0, 1, 2, \cdots$, respectively.

(a) Let $\psi = e^{-x^2/2}y$. Then by substituting $y = e^{x^2/2}\psi$ into Equation (1), show that ψ satisfies the ODE

$$\psi'' + \left(2\alpha + 1 - x^2\right)\psi = 0 \tag{2}$$

and show that the ODE in Equation (2) is self-adjoint.

(b) For each $m = 0, 1, 2, \cdots$, let $\psi_m(x) = e^{-x^2/2}H_m(x)$. Then ψ_m is a solution of Equation (2) for $\alpha = m$. Show that the Hermitian operator boundary condition for Equation (2):

$$\left[v^*u' - (v^*)'\,u\right]_{-\infty}^{\infty} = 0$$

holds for $u = \psi_m$ and $v = \psi_n$.

(c) Using the results of parts (a) and (b), write down the orthogonality condition for the ψ_m's and use that to write the orthogonality condition for the Hermite polynomials H_m's.

1.

$$D^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad (*)$$

We look for solutions of the form $T(x,y) = X(x)\,Y(y)$.
Substituting into $(*)$, we get:

$$Y \frac{d^2 X}{dx^2} + X \frac{d^2 Y}{dy^2} = 0$$

$$\xrightarrow{\cdot \frac{1}{XY}} \quad \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0 \quad \text{or} \quad \frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2}$$

The left-hand side is a function of x only and the right-hand side is a function of y only. Since this has to hold for all x, y, which vary independently, this can happen only if both sides are equal to the same constant, say α. Thus,

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \alpha$$

From the boundary conditions, we must have

$$X(0) = 0 \quad \text{and} \quad Y(0) = Y(\pi) = 0.$$

Since we need $Y(y)$ to be zero at two values of y, we need Y to be a sine function (not exponentials)

$\rightarrow \quad \alpha = +k^2 > 0.$ Hence

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = k^2 \qquad (**)$$

$$\frac{d^2X}{dx^2} - k^2 X = 0 \longrightarrow X(x) = \begin{Bmatrix} e^{kx} \\ e^{-kx} \end{Bmatrix}$$

Boundary condition on the left edge $\rightarrow X(0) = 0$

$$\rightarrow X(x) = \sinh(kx) = \frac{e^{kx} - e^{-kx}}{2}$$

$\rightarrow \quad X(x) = \sinh(kx) = \dfrac{e^{kx} - e^{-kx}}{2}$

$$\frac{d^2 Y}{dy^2} + k^2 y = 0 \quad (\text{from } (**)) \longrightarrow Y(y) = \left\{ \begin{matrix} \cancel{\cos(ky)} \\ \sin(ky) \end{matrix} \right\}$$

Since $Y(0) = 0$ (B.C. at the lower edge of the plate), $\cos(ky)$ is not an acceptable solution. Thus, $Y(y) = \sin(ky)$.

Since $Y(\pi) = 0$ (B.C. at the top edge of the plate), we get $\sin(k\pi) = 0 \rightarrow k$ is an integer; $k = n = 1, 2, 3, \ldots$ (Eigenvalues)

For each $n = 1, 2, 3, \ldots$, we obtain the eigensolution:

$$T_n(x, y) = X_n(x)\, Y_n(y)$$

$= \sinh(nx) \sin(ny)$, which is a solution of $(*)$ and satisfies 3 of the 4 boundary conditions

$[T(x,0) = T(x,\pi) = 0$ for $0 < x < \pi$

$T(0,y) = 0$ for $0 < y < \pi]$.

To match the fourth boundary condition ($T(\pi, y) = y$ for $0 < y < \pi$), we form the linear combination of all eigensolutions:

$$T(x,y) = \sum_{n=1}^{\infty} B_n T_n(x,y)$$

$$\rightarrow \quad T(x,y) = \sum_{n=1}^{\infty} B_n \sinh(nx) \sin(ny) \quad (***)$$

$T(\pi, y) = y \longrightarrow$

$$y = \sum_{n=1}^{\infty} \underbrace{\left(B_n \sinh(n\pi)\right)}_{b_n} \sin(ny):$$

a Fourier sine series with period 2π and coefficients $b_n = B_n \sinh(n\pi)$

$b_n = \dfrac{2}{\;} \displaystyle\int^{\pi} y \sin(ny)\, dy$

$$b_n = \frac{2}{\pi} \int_0^{\pi} y \sin(ny)\, dy$$

$$= \frac{2}{\pi} \left[-\frac{y \cos(ny)}{n} + \frac{\sin(ny)}{n^2} \right]_{y=0}^{\pi}$$

$$= \frac{2}{\pi} \left[-\frac{\pi \cos(n\pi)}{n} \right] = -\frac{2\cos(n\pi)}{n} = \frac{2(-1)^{n+1}}{n}$$

$$\rightarrow B_n = \frac{b_n}{\sinh(n\pi)} = \frac{2(-1)^{n+1}}{n \sinh(n\pi)}$$

Substituting into $(***)$, we obtain the unique solution of our BVP:

$$\boxed{T(x,y) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \sinh(n\pi)} \sinh(nx) \sin(ny)}.$$

2. a $\vec{L} = -i\,(\vec{r} \times \vec{\nabla})$

(a) $$\vec{L} = -i\, r\,\hat{e}_r \times \left[\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_\varphi \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\right]$$

$$= -i\left[\hat{e}_\varphi \frac{\partial}{\partial \theta} - \hat{e}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial \varphi}\right]$$

$$= i\left[\hat{e}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial \varphi} - \hat{e}_\varphi \frac{\partial}{\partial \theta}\right]$$

(b) $$\hat{e}_\theta = \hat{e}_x \cos\theta\cos\varphi + \hat{e}_y \cos\theta\sin\varphi - \hat{e}_z \sin\theta$$

$$\hat{e}_\varphi = -\hat{e}_x \sin\varphi + \hat{e}_y \cos\varphi$$

$$\rightarrow \vec{L} = i\left[(\hat{e}_x \cos\theta\cos\varphi + \hat{e}_y \cos\theta\sin\varphi - \hat{e}_z \sin\theta)\cdot \frac{1}{\sin\theta}\frac{\partial}{\partial \varphi} + (\hat{e}_x \sin\varphi - \hat{e}_y \cos\varphi)\frac{\partial}{\partial \theta}\right]$$

$$= \hat{e}_x \overbrace{\left[i \sin\varphi \frac{\partial}{\partial \theta} + i \cot\theta \cos\varphi \frac{\partial}{\partial \varphi}\right]}^{L_x}$$

$$+ \hat{e}_y \overbrace{\left[-i\cos\varphi \frac{\partial}{\partial \theta} + i\cot\theta \sin\varphi \frac{\partial}{\partial \varphi}\right]}^{L_y}$$

$$+ \hat{e}_z \underbrace{\left[-i \frac{\partial}{\partial \varphi}\right]}_{L_z}$$

3. $y'' - 2xy' + 2\alpha y = 0$ (1) ; $-\infty < x < +\infty$

a) let $\psi(x) = e^{-x^2/2} y$. Then $y = e^{x^2/2}\psi(x)$. Thus,

$$y' = e^{x^2/2}\psi'(x) + x e^{x^2/2}\psi(x) \text{ and}$$

$$y'' = e^{x^2/2}\psi''(x) + 2x e^{x^2/2}\psi'(x) + (x^2+1)e^{x^2/2}\psi(x).$$

Substituting into (1), we get:

$$e^{x^2/2}\left[\psi'' + 2x\psi' + (x^2+1)\psi\right] - 2x e^{x^2/2}\left[\psi' + x\psi\right] + 2\alpha e^{x^2/2}\psi = 0$$

$$\Rightarrow \psi'' + \cancel{2x\psi'} + (x^2+1)\psi - 2x(\cancel{\psi'} + x\psi) + 2\alpha\psi = 0$$

$$\Rightarrow \psi'' + (2\alpha + 1 - x^2)\psi = 0 \quad (2)$$

Here $P_0(x) = 1$ and $P_1(x) = 0$

Since $P_0'(x) = 0 = P_1(x)$, it follows that the ODE in (2) is self-adjoint.

b) Since $u = \psi_m = e^{-x^2/2} H_m$ and

$v = \psi_n = e^{-x^2/2} H_n$, where H_m and H_n are Hermite polynomials (of degrees m and n, respectively), it follows that

$v^* u'$ and $(v^*)' u$ are both of the form $e^{-x^2}\cdot$ (polynomial), and hence they both vanish at $\pm\infty$ since $\lim_{x\to\pm\infty} e^{-x^2}\cdot$ (any polynomial) $= 0$

Thus, $\left[v^* u' - (v^*)' u\right]_{-\infty}^{+\infty} = 0$.

c) It follows from parts a) and b) that the ODE in (2) is Hermitian

Thus, eigensolutions (ψ_m and ψ_n) corresponding to different eigenvalues $m \neq n$ are orthogonal on $(-\infty, +\infty)$. Thus,

$$\int_{-\infty}^{+\infty} \psi_m^*(x)\,\psi_n(x)\,dx = 0 \text{ if } m \neq n, \text{ or}$$

$$\boxed{\int_{-\infty}^{+\infty} \psi_m(x)\,\psi_n(x)\,dx = 0 \text{ if } m \neq n}$$ (since $\psi_m(x)$ is real).

Replacing $\psi_m(x)$ with $e^{-x^2/2}H_m(x)$ and $\psi_n(x)$ with $e^{-x^2/2}H_n(x)$, we

Replacing $\psi_m(x)$ with $e^{-x^2/2} H_m(x)$ and $\psi_n(x)$ with $e^{-x^2/2} H_n(x)$, we get the orthogonality condition for the Hermite polynomials on $(-\infty,+\infty)$:

$$\int_{-\infty}^{+\infty} \left[e^{-x^2/2} H_m(x)\right] \cdot \left[e^{-x^2/2} H_n(x)\right] dx = 0 \text{ if } m \neq n, \text{ or}$$

$$\boxed{\int_{-\infty}^{+\infty} e^{-x^2} H_m(x)\, H_n(x)\, dx = 0 \quad \text{if } m \neq n}$$