11.8.4 $\quad I = \int_0^{2\pi} \dfrac{\cos 3\theta}{5 - 4\cos\theta}\, d\theta$

Let $z = e^{i\theta}$. Then $d\theta = -i \dfrac{dz}{z}$

$\cos\theta = \dfrac{z + z^{-1}}{2}$ and $\cos 3\theta = \dfrac{e^{i3\theta} + e^{-i3\theta}}{2} = \dfrac{z^3 + z^{-3}}{2}$

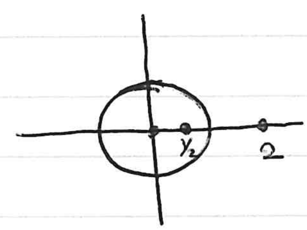
Thus, $I = -\dfrac{i}{2} \oint_C \dfrac{z^3 + z^{-3}}{5 - 2(z + z^{-1})} \dfrac{dz}{z}$

$\uparrow$ unit circle

$$= -\frac{i}{2} \oint_C \frac{z^3 + z^{-3}}{5z - 2z^2 - 2}\, dz$$

$$= \frac{i}{2} \oint_C \frac{z^6 + 1}{z^3 [2z^2 - 5z + 2]}\, dz$$

$$= \frac{i}{2} \oint_C \frac{z^6 + 1}{z^3 (2z - 1)(z - 2)}\, dz$$



$= \dfrac{i}{2}(2\pi i)\Big[$ Residue of $\dfrac{z^6 + 1}{z^3 (2z-1)(z-2)}$ at $1/2$

$+$ Residue at 0. $\Big]$

Residue at $1/2$ $= \lim_{z \to 1/2} (z - 1/2) \dfrac{z^6 + 1}{z^3 (2z-1)(z-2)}$

$= \dfrac{1}{2} \lim_{z \to 1/2} \dfrac{z^6 + 1}{z^3 (z - 2)} = \dfrac{1}{2} \dfrac{(1/2)^6 + 1}{(\frac{1}{2})^3 (\frac{1}{2} - 2)} = \dfrac{1}{2} \dfrac{1 + 64}{8(-3/2)}$

$= -\dfrac{65}{24}$

$$\text{Residue at } \tfrac{1}{2} = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} \frac{z^6+1}{(2z-1)(z-2)}$$

$$= \frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2} \frac{1}{(2z-1)(z-2)}$$

[the z^6 term gives 0 after ~~differenti~~ differentiating twice and putting $z=0$.]

$$= \frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2} \left[\frac{1/3}{z-2} - \frac{2/3}{2z-1} \right]$$

$$= \frac{1}{2} \lim_{z \to 0} \left[\frac{2/3}{(z-2)^3} - \frac{16/3}{(2z-1)^3} \right]$$

$$= \frac{1}{2} \left[\frac{2/3}{(-8)} - \frac{16/3}{(-1)} \right]$$

$$= \frac{1}{2} \left[-\frac{1}{12} + \frac{16}{3} \right]$$

$$= \frac{1}{2} \, \frac{-1+64}{12} = \frac{63}{24} \qquad \left(= \frac{21}{8} \right)$$

$$I = \frac{i}{2} \, 2\pi i \left[\frac{-65}{24} + \frac{63}{24} \right]$$

$$= -\pi \left[\frac{-2}{24} \right] = \frac{\pi}{12}.$$

11.8.5 $\quad I = \int_0^{\pi} \cos^{2n}\theta \, d\theta = ?$

First note that $\displaystyle\int_{\pi}^{2\pi} \cos^{2n}\theta \, d\theta = \int_0^{\pi} \cos^{2n}(t+\pi)\, dt$

$(\theta = t + \pi)$

$$= \int_0^{\pi} [\cos(t+\pi)]^{2n}\, dt$$

$$= \int_0^{\pi} [-\cos t]^{2n}\, dt = \int_0^{\pi} \cos^{2n} t \, dt = I$$

Thus, $\displaystyle I = \frac{1}{2}\left[\int_0^{\pi} \cos^{2n}\theta\, d\theta + \int_{\pi}^{2\pi} \cos^{2n}\theta \, d\theta\right]$

$$= \frac{1}{2}\int_0^{2\pi} \cos^{2n}\theta \, d\theta$$

Let $z = e^{i\theta}$, then $\cos\theta = \frac{1}{2}(z + z^{-1})$ and $d\theta = -i\,\frac{dz}{z}$

Thus, $\displaystyle I = \frac{1}{2} \oint_C \left[\frac{1}{2}(z+z^{-1})\right]^{2n}\left(-i\,\frac{dz}{z}\right)$,

$C \rightarrow$ unit circle $|z| = 1$

$$= \frac{-i}{2^{2n+1}} \oint_C \frac{(z^2+1)^{2n}}{z^{2n+1}}\, dz$$

$$= \frac{-i}{2^{2n+1}}\, 2\pi i \left(\text{Residue of } \frac{(z^2+1)^{2n}}{z^{2n+1}} \text{ at } z = 0\right)$$

$$= \frac{\pi}{2^{2n}} \cdot \left(\text{coefficient of } z^{2n} \text{ in } (z^2+1)^{2n}\right)$$

$$= \frac{\pi}{2^{2n}}\binom{2n}{n} = \frac{\pi}{2^{2n}}\,\frac{(2n)!}{n!\,n!}$$

$$= \pi\, \frac{(2n)!}{(2^n n!)\,(2^n n!)} = \pi\,\frac{(2n)!}{(2n)!!\,(2n)!!} = \frac{\pi\,(2n-1)!!}{(2n)!!}$$

11.8.9 $I=\int_{-\infty}^{+\infty}\frac{\sin^2 x}{x^2}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{1-\cos 2x}{x^2}\,dx$

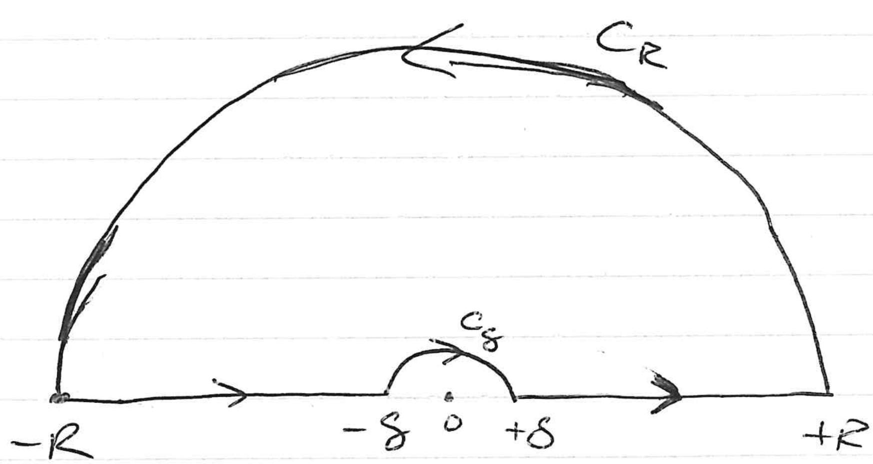
Note that for any $\delta>0$ in $\mathbb{R}$,

$$\int_{-\infty}^{-\delta}\frac{\sin 2x}{x^2}\,dx + \int_{\delta}^{\infty}\frac{\sin 2x}{x^2}\,dx = \int_{\infty}^{\delta}\frac{\sin 2u}{u^2}\,du + \int_{\delta}^{\infty}\frac{\sin 2x}{x^2}\,dx \quad (u=-x)$$

$$= -\int_{\delta}^{\infty}\frac{\sin 2u}{u^2}\,du + \int_{\delta}^{\infty}\frac{\sin 2x}{x^2}\,dx = 0$$

Hence $\int_{-\infty}^{+\infty}\frac{\sin 2x}{x^2}\,dx = \lim_{\delta\to 0^+}\left[\int_{-\infty}^{-\delta}\frac{\sin 2x}{x^2}\,dx + \int_{\delta}^{\infty}\frac{\sin 2x}{x^2}\,dx\right] = 0$

Thus, $I = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{1-(\cos 2x + i\sin 2x)}{x^2}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{1-e^{i2x}}{x^2}\,dx$



$$\oint_C \frac{1-e^{i2z}}{z^2}\,dz = \int_{-R}^{-\delta}\frac{1-e^{i2x}}{x^2}\,dx + \int_{C_\delta}\frac{1-e^{i2z}}{z^2}\,dz + \int_{\delta}^{R}\frac{1-e^{i2x}}{x^2}\,dx + \int_{C_R}\frac{1-e^{i2z}}{z^2}\,dz$$

$= 0$ (no ~~poles~~ singularities of $\frac{1-e^{i2z}}{z^2}$ inside C)

As $R\to\infty$, $\delta\to 0$, $\int_{C_R}\frac{1-e^{i2z}}{z^2}\,dz \longrightarrow 0$

$\cancel{\oint_{C_R}\frac{1}{z^2}dz \text{ is}}$ $\frac{1}{z^2}$ is of the form in Eqn 11.95 and $\frac{e^{i2z}}{z^2}$ of the form in Eqn. 11.100

$$\int_{-R}^{-\delta} \frac{1-e^{i2x}}{x^2}\,dx + \int_{\delta}^{R} \frac{1-e^{i2x}}{x^2}\,dx \longrightarrow ⨍_{-\infty}^{+\infty} \frac{1-e^{i2x}}{x^2}\,dx$$

$$\int_{C_\delta} \frac{1-e^{i2z}}{z^2}\,dz \longrightarrow -\pi i \ \mathrm{Res}\left(\frac{1-e^{i2z}}{z^2}, 0\right)$$

Thus, $$⨍_{-\infty}^{+\infty} \frac{1-e^{i2x}}{x^2}\,dx = \pi i \ \mathrm{Res}\left(\frac{1-e^{i2z}}{z^2}, 0\right)$$

$$\frac{1-e^{i2z}}{z^2} = \frac{1-[1+i2z - 2z^2 + \cdots]}{z^2} = -\frac{2i}{z} + 2 + \cdots$$

$$\rightarrow \mathrm{Res}\left(\frac{1-e^{i2z}}{z^2}, 0\right) = -2i.$$

Thus, $$⨍_{-\infty}^{+\infty} \frac{1-e^{i2x}}{x^2}\,dx = \pi i\,(-2i) = 2\pi$$

and hence $$I = \int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{1}{2} ⨍_{-\infty}^{+\infty} \frac{1-e^{i2x}}{x^2}\,dx = \pi$$

11.8.16 $$\int_{-\infty}^{+\infty} \frac{x^2}{1+x^4}\,dx$$

This is of the form in Eqn. 11.95. Thus,

$$\int_{-\infty}^{+\infty} \frac{x^2}{1+x^4}\,dx = 2\pi i \sum\left(\text{Residues of } \frac{z^2}{1+z^4} \text{ in upper half-plane.}\right)$$

The poles of $\frac{z^2}{1+z^4}$ are the ~~roots of z^4~~ roots ~~zeros~~ of $1+z^4=0$, i.e. the four fourth roots $-1 = e^{i\pi}$.

$$z_1 = e^{i\pi/4} = (1+i)/\sqrt{2} \qquad z_2 = e^{i\left(\frac{\pi}{4} + \frac{2\pi}{4}\right)} = e^{i3\pi/4} = \frac{-1+i}{\sqrt{2}}$$

$$z_3 = e^{i\left(\pi/4 + \frac{4\pi}{4}\right)} = -z_1 \quad \text{and} \quad z_4 = e^{i\left(\pi/4 + \frac{6\pi}{4}\right)} = -z_2 .$$

Only z_1 and z_2 are in the upper half-plane.

Residue of $\frac{z^2}{1+z^4}$ at $z_1 = \lim\limits_{z \to z_1} \frac{(z-z_1)\, z^2}{1+z^4}$

$$= \lim_{z \to z_1} \frac{z^2 + (z-z_1)2z}{4z^3} = \frac{z_1^2}{4z_1^3} = \frac{1}{4z_1} = \frac{1}{4} e^{-i\pi/4}$$

Similarly, Residue at $z_2 = \frac{1}{4z_2} = \frac{1}{4} e^{-i3\pi/4}$

Thus, $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2}{1+x^4}\, dx = 2\pi i \, \frac{1}{4} \left[e^{-i\pi/4} + e^{-i3\pi/4} \right]$

$$= \frac{\pi i}{2} \left[\frac{1-i}{\sqrt{2}} + \frac{-1-i}{\sqrt{2}} \right]$$

$$= \frac{\pi i}{2} \left[\frac{-2i}{\sqrt{2}} \right] = \frac{\pi}{\sqrt{2}} = \frac{\pi\sqrt{2}}{2}$$

[redacted]

[redacted]

11.8.11 Note that $\displaystyle\int_{-\infty}^{+\infty} \frac{\sin \omega t}{\omega^2}\, d\omega$

$$= \lim_{\delta \to 0^+} \left[\int_{-\infty}^{-\delta} \frac{\sin \omega t}{\omega^2}\, d\omega + \int_{\delta}^{\infty} \frac{\sin \omega t}{\omega^2}\, d\omega \right]$$

$$= \lim_{\delta \to 0^+} \left[\int_{\infty}^{\delta} \frac{\sin u t}{u^2}\, du + \int_{\delta}^{\infty} \frac{\sin \omega t}{\omega^2}\, d\omega \right]$$

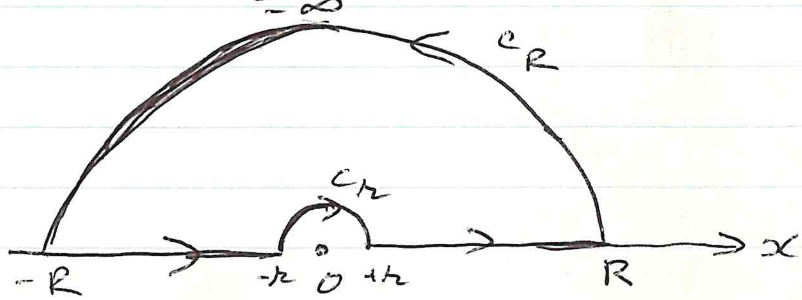
$(u = -\omega)$

$= 0$. Thus,

$$I = \int_{-\infty}^{+\infty} \frac{1 - \cos \omega t}{\omega^2}\, d\omega = \int_{-\infty}^{+\infty} \frac{1 - \cos \omega t}{\omega^2}\, d\omega$$

$$= \int_{-\infty}^{+\infty} \frac{1 - (\cos \omega t + i \sin \omega t)}{\omega^2}\, d\omega$$

$$= \int_{-\infty}^{+\infty} \frac{1 - e^{i\omega t}}{\omega^2}\, d\omega = \int_{-\infty}^{+\infty} \frac{1 - e^{ixt}}{x^2}\, dx$$



$$\oint_C \frac{1-e^{izt}}{z^2}\,dz = \int_{-R}^{-r} \frac{1-e^{ixt}}{x^2}\,dx + \int_{C_r} \frac{1-e^{izt}}{z^2}\,dz$$

$$+ \int_r^R \frac{1-e^{ixt}}{x^2}\,dx + \int_{C_R} \frac{1-e^{izt}}{z^2}\,dz$$

$$= 0 \quad \left(\text{no singularities of } \frac{1-e^{izt}}{z^2} \text{ inside } C.\right)$$

Note $z=0$ is a simple pole of $\frac{1-e^{izt}}{z^2}$. Hence

$$\int_{C_r} \frac{1-e^{izt}}{z^2}\,dz = -\pi i \operatorname{Res}\left(\frac{1-e^{izt}}{z^2}, 0\right).$$

Also $$\int_{C_R} \frac{1-e^{izt}}{z^2}\,dz = \int_{C_R} \frac{1}{z^2}\,dz - \int_{C_R} \frac{e^{izt}}{z^2}\,dz$$

$\xrightarrow[R \to \infty]{}$ 0 by equations 11.96 and 11.102, respectively.

[t is presumably time, so $t > 0$].

Let $r \to 0$, $R \to \infty$:

$$\int_{-R}^{-r} \frac{1-e^{ixt}}{x^2}\,dx + \int_r^R \frac{1-e^{ixt}}{x^2}\,dx \longrightarrow \int_{-\infty}^{+\infty} \frac{1-e^{ixt}}{x^2}\,dx = I$$

$$\int_{C_r} \frac{1-e^{izt}}{z^2}\,dz \longrightarrow -\pi i \operatorname{Res}\left(\frac{1-e^{izt}}{z^2}, 0\right)$$

$$\int_{C_R} \frac{1-e^{izt}}{z^2}\,dz \longrightarrow 0. \quad \text{Thus,}$$

$$I - \pi i \operatorname{Res}\left(\frac{1-e^{izt}}{z^2}, 0\right) = 0$$

$$I = \pi i \operatorname{Res}\left(\frac{1-e^{izt}}{z^2}, 0\right)$$

$$= \pi i \lim_{z \to 0} z \cdot \frac{1-e^{izt}}{z^2}$$

$$= \pi i \lim_{z \to 0} \frac{1-e^{izt}}{z}$$

$$= \pi i \lim_{z \to 0} \frac{-ite^{izt}}{1}$$

$$= \pi i(-it) = \pi t.$$

It follows that $\displaystyle\int_{-\infty}^{+\infty} \frac{2(1-\cos\omega t)}{\omega^2}\, d\omega = 2I = 2\pi t.$

11.8.12 b)

$$I = \int_{-\infty}^{+\infty} \frac{x \sin x}{x^2+a^2}\, dx = -i \int_{-\infty}^{+\infty} \frac{x\, i\sin x}{x^2+a^2}\, dx$$

$$= -i\left[\int_{-\infty}^{+\infty} \frac{x\cos x}{x^2+a^2}\, dx + \int_{-\infty}^{+\infty} \frac{x\, i\sin x}{x^2+a^2}\, dx\right]$$

($\displaystyle\int_{-\infty}^{+\infty} \frac{x\cos x}{x^2+a^2}\, dx = 0$ since $\dfrac{x\cos x}{x^2+a^2}$ is an odd function.)

Thus, $\displaystyle I = -i \int_{-\infty}^{+\infty} \frac{x e^{ix}}{x^2+a^2}\, dx$

This is of the form in Equation 11.100, with $\displaystyle\lim_{|z| \to \infty} \frac{z}{z^2+a^2} = 0$

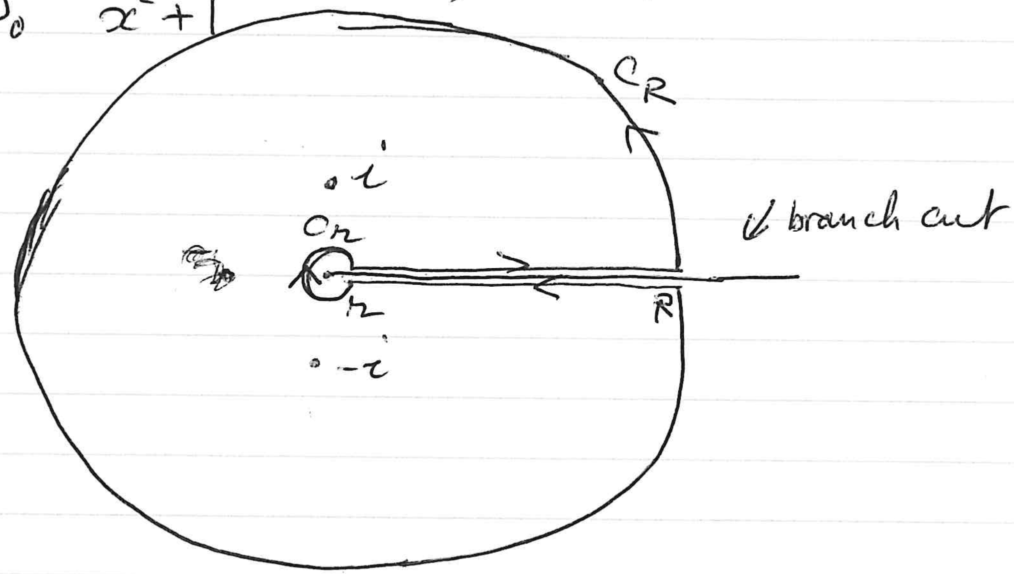
Thus, $I = -i\, 2\pi i \ \mathrm{Res}\left(\dfrac{z e^{iz}}{z^2+a^2}, ia\right)$ (Eq. 11.103)

$$= 2\pi \lim_{z \to ia} \frac{(z-ia) z e^{iz}}{z^2+a^2}$$

$$= 2\pi \lim_{z \to ia} \frac{z e^{iz}}{z+ia}$$

$$= 2\pi \frac{ia e^{-a}}{2ia} = \pi e^{-a}$$

11.8.17 $I = \displaystyle\int_0^\infty \frac{x^p \ln x}{x^2+1} dx$, $0 < p < 1$



$$\oint_C \frac{z^p \ln z}{z^2+1} dz = \int_r^R \frac{x^p \ln x}{x^2+1} dx + \int_{C_R} \frac{z^p \ln z}{z^2+1} dz$$

$$+ \int_R^r \frac{x^p e^{i2\pi p}[\ln x + 2\pi i]}{x^2+1} dx + \int_{C_r} \frac{z^p \ln z}{z^2+1} dz$$

$$= 2\pi i \left[\text{Residue of } \frac{z^p \ln z}{z^2+1} \text{ at } i + \text{Residue at } -i\right]$$

Since $\displaystyle\lim_{|z| \to \infty} [z f(z)] = \lim_{|z| \to \infty} \frac{z^{p+1} \ln z}{z^2+1} = 0$ $(p+1 < 2)$

we have that $\lim_{R \to \infty} \int_{C_R} \frac{z^p \ln z}{z^2+1} dz = 0$

Also, on C_r $\left| \frac{z^p \ln z}{z^2+1} \right| \sim |r^p \ln r| \xrightarrow[r \to 0]{} 0$ since $p>0$

Hence $\int_{C_r} \frac{z^p \ln z}{z^2+1} dz \xrightarrow[r \to 0]{} 0$

So setting $r \to 0$ and $R \to \infty$, we obtain

$$\int_0^\infty \frac{x^p \ln x}{x^2+1} dx - e^{i2\pi p} \int_0^\infty \frac{x^p \ln x}{x^2+1} dx$$

$$- 2\pi i e^{i2\pi p} \int_0^\infty \frac{x^p}{x^2+1} dx = 2\pi i \left[\operatorname{Res}\left(\frac{z^p \ln z}{z^2+1}, i\right) + \operatorname{Res}\left(\frac{z^p \ln z}{z^2+1}, -i\right) \right]$$

But $\int_0^\infty \frac{x^p}{x^2+1} dx = \frac{\pi}{2 \cos p\pi/2}$ (Example 11.8.8),

$$\operatorname{Res}\left(\frac{z^p \ln z}{z^2+1}, i\right) = \lim_{z \to i = e^{i\pi/2}} \frac{(z-i) z^p \ln z}{z^2+1}$$

$$= \lim_{z \to i} \frac{z^p \ln z}{z+i} = \frac{e^{ip\pi/2} \, i\pi/2}{2i} = \frac{\pi}{4} e^{ip\pi/2},$$

and $\operatorname{Res}\left(\frac{z^p \ln z}{z^2+1}, -i\right) = \lim_{z \to -i = e^{i3\pi/2}} \frac{(z+i) z^p \ln z}{z^2+1}$

$$= \lim_{z \to -i} \frac{z^p \ln z}{z-i} = \frac{e^{i3p\pi/2} [i 3\pi/2]}{-2i} = -\frac{3\pi}{4} e^{i3p\pi/2}$$

Thus, $(1 - e^{i2p\pi}) I - 2\pi i e^{i2\pi p} \frac{\pi}{2\cos(p\pi/2)}$

$$= 2\pi i \left[\frac{\pi}{4} e^{ip\pi/2} - \frac{3\pi}{4} e^{i3p\pi/2} \right]$$

Dividing by $\left[-2i\, e^{ip\pi}\right]$, we get

$$(\sin p\pi)\, I + \frac{\pi^2 e^{ip\pi}}{2\cos(p\pi/2)} = \pi\left[\frac{3\pi}{4} e^{ip\pi/2} - \frac{\pi}{4} e^{-ip\pi/2}\right]$$

$$2I \sin(p\pi/2)\cos(p\pi/2) = \frac{\pi^2}{4}\left[3e^{ip\pi/2} - e^{-ip\pi/2} - \frac{2e^{ip\pi}}{\cos(p\pi/2)}\right]$$

Now, $3e^{ip\pi/2} - e^{-ip\pi/2} - \dfrac{2e^{ip\pi}}{\cos(p\pi/2)}$

$$= 3\cos(p\pi/2) + 3i\sin(p\pi/2) - \cos(p\pi/2) + i\sin(p\pi/2)$$

$$- \frac{2\cos p\pi}{\cos(p\pi/2)} - \frac{2i\sin p\pi}{\cos(p\pi/2)}$$

$$= 2\cos(p\pi/2) - \frac{2\cos p\pi}{\cos(p\pi/2)}$$

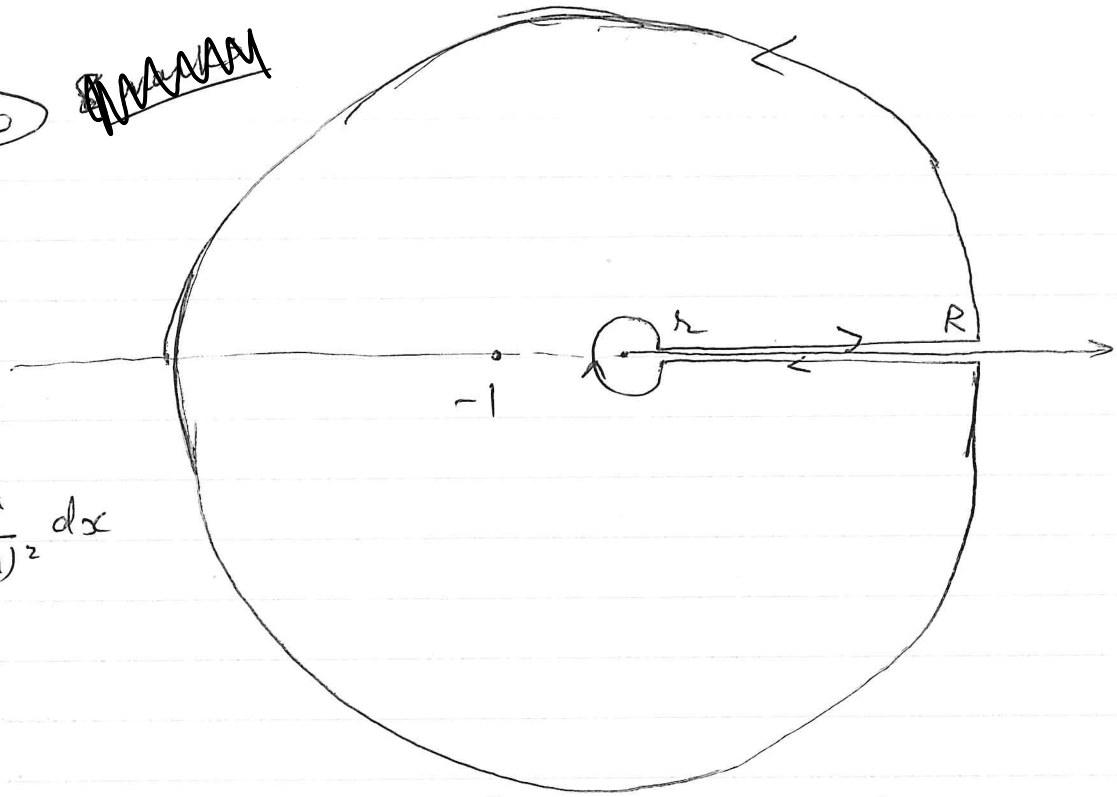
$$= 2\cos(p\pi/2) - \frac{2(\cos^2(p\pi/2) - \sin^2(p\pi/2))}{\cos(p\pi/2)}$$

$$= \frac{2\sin^2(p\pi/2)}{\cos(p\pi/2)}, \quad \text{Thus,}$$

$$2I\sin(p\pi/2)\cos(p\pi/2) = \frac{\pi^2}{4}\,\frac{2\sin^2(p\pi/2)}{\cos(p\pi/2)}$$

$$\rightarrow I = \frac{\pi^2}{4}\,\frac{\sin(p\pi/2)}{\cos^2(p\pi/2)}.$$

11.8.20



Let

$$I = \int_0^\infty \frac{x^a}{(x+1)^2}\,dx$$

$$\oint_C \frac{z^a}{(z+1)^2}\,dz = \int_r^R \frac{x^a}{(x+1)^2}\,dx + \int_{C_R} \frac{z^a}{(z+1)^2}\,dz$$

$$+ \int_R^r \frac{x^a e^{i2\pi a}}{(1+x)^2}\,dx + \int_{C_r} \frac{z^a}{(z+1)^2}\,dz$$

$$= 2\pi i \operatorname{Res}\left(\frac{z^a}{(1+z)^2}, -1\right)$$

$\lim_{|z|\to\infty} \frac{z\, z^a}{(z+1)^2} = 0$ since $a<1$; hence

$$\lim_{R\to\infty} \int_{C_R} \frac{z^a}{(z+1)^2}\,dz = 0$$

Also $\left|\int_{C_r} \frac{z^a\,dz}{(z+1)^2}\right| \sim r^{a+1} \xrightarrow[r\to 0]{} 0$ since $a>-1$ and hence $a+1>0$.

So, by letting $r\to 0$ and $R\to\infty$, we obtain

$$\left(1-e^{i2\pi a}\right)\int_0^\infty \frac{x^a}{(1+x)^2}\,dx = 2\pi i \; \mathrm{Res}\left(\frac{z^a}{(z+1)^2},\ -1=e^{i\pi}\right)$$

$$\left(1-e^{i2\pi a}\right) I = 2\pi i \lim_{z\to -1=e^{i\pi}} \frac{d}{dz}\left[(z+1)^2\frac{z^a}{(z+1)^2}\right]$$

$$= 2\pi i \lim_{z\to e^{i\pi}} \left(a z^{a-1}\right)$$

$$= 2\pi i \; a \, e^{i\pi(a-1)}$$

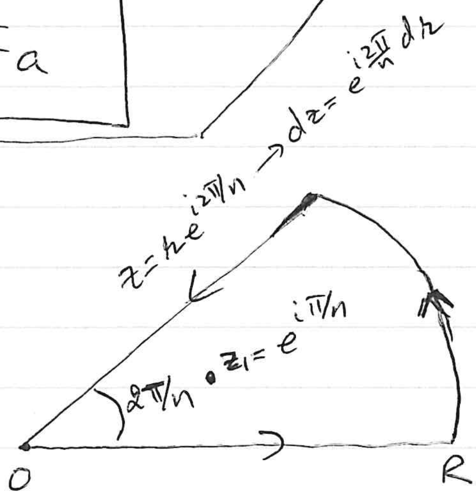
$$\left(1-e^{i2\pi a}\right) I = -2\pi i \, a \, e^{i\pi a}$$

Dividing both sides by $\left(-2i\,e^{i\pi a}\right)$, we get

$$\frac{e^{i\pi a}-e^{-i\pi a}}{2i}\, I = \pi a$$

$$\longrightarrow \boxed{I = \frac{\pi a}{\sin \pi a}}$$

11.8.22



$$\oint_C \frac{dz}{1+z^n} = \int_0^R \frac{dx}{1+x^n} + \int_{C_R} \frac{dz}{1+z^n} + \int_R^0 \frac{e^{i2\pi/n}\,dr}{1+r^n}$$

$$= 2\pi i \; \mathrm{Res}\left(\frac{1}{1+z^n},\ z_1 = e^{i\pi/n}\right)$$

The poles of $\frac{1}{1+z^n}$ are the n n^{th} roots of $-1 = e^{i\pi}$

$$z_1 = e^{i\pi/n}$$
$$z_2 = e^{i3\pi/n}$$
$$\vdots$$
$$z_n = e^{i(2n-1)\pi/n} = e^{-i\pi/n}$$

$\to$ only z_1 lies inside C.

Since $\lim_{|z|\to\infty} \frac{z}{1+z^n} = 0$, $\lim_{R\to\infty} \int_{C_R} \frac{dz}{1+z^n} = 0$

Thus, taking $R \to \infty$, we get:

$$\left(1-e^{i2\pi/n}\right)\int_0^\infty \frac{dx}{1+x^n} = 2\pi i \operatorname{Res}\left(\frac{1}{1+z^n}, e^{i\pi/n}\right)$$

$$= 2\pi i \lim_{z\to e^{i\pi/n}} \frac{z-e^{i\pi/n}}{1+z^n}$$

$$\left(1-e^{i2\pi/n}\right) I = 2\pi i \lim_{z\to e^{i\pi/n}} \frac{1}{n z^{n-1}}$$

$$= 2\pi i \frac{1}{n e^{i(n-1)\pi/n}}$$

$$= -2\pi i \frac{1}{n e^{-i\pi/n}}$$

$$= -2i e^{i\pi/n} \frac{\pi}{n}$$

$$\to \frac{e^{i\pi/n} - e^{-i\pi/n}}{2i} I = \pi/n \to I = \int_0^\infty \frac{dx}{1+x^n} = \frac{\pi/n}{\sin(\pi/n)}$$

$\sin(\pi/n)$