PHYS 3496
[redacted]

8.3.1 a, b, c)

$$(1-x^2)\,y'' - 2x\,y' + n(n+1)\,y = 0$$

Let $y = \sum_{j=0}^{\infty} a_j\, x^{j+s}$, $a_0 \neq 0$. Then

$$y' = \sum_{j=0}^{\infty} (j+s)\, a_j\, x^{j+s-1} \quad \text{and}$$

$$y'' = \sum_{j=0}^{\infty} (j+s)(j+s-1)\, a_j\, x^{j+s-2}.$$

substituting y' and y'' into ODE, we get:

$$(1-x^2)\sum_{j=0}^{\infty} (j+s)(j+s-1)\, a_j\, x^{j+s-2}$$

$$-\,2x\sum_{j=0}^{\infty} (j+s)\, a_j\, x^{j+s-1} + n(n+1)\sum_{j=0}^{\infty} a_j\, x^{j+s} = 0$$

$$\rightarrow \sum_{j=0}^{\infty} (j+s)(j+s-1)\, a_j\, x^{j+s-2}$$

$$-\sum_{j=0}^{\infty} \left[(j+s)(j+s-1) + 2(j+s) - n(n+1)\right] a_j\, x^{j+s} = 0$$

$$\rightarrow \sum_{j=-2}^{\infty} (j+s+2)(j+s+1)\, a_{j+2}\, x^{j+s}$$

$$-\sum_{j=0}^{\infty} \big[\underbrace{(j+s)(j+s+1) - n(n+1)}_{(j+s-n)(j+s+n+1)}\big]\, a_j\, x^{j+s} = 0$$

$$\rightarrow \cancel{a_0}\, s(s-1)\, a_0\, x^{s-2} + (s+1)s\, a_1\, x^{s-1}$$

$$+ \sum_{j=0}^{\infty} \left[(j+s+2)(j+s+1)\, a_{j+2} - (j+s-n)(j+s+n+1)\, a_j\right] x^{j+s} = 0$$

$s(s-1)\, a_0 = 0 \rightarrow \boxed{s(s-1)=0}$ (Indicial Equation), since $a_0 \neq 0$.

$\rightarrow s = 0$ or $s = 1$.

$s = 0$: both a_0 and a_1 are arbitrary; and

$$\sum_{j=0}^{\infty} \left[(j+2)(j+1)\, a_{j+2} - (j-n)(j+n+1)\, a_j\right] x^j = 0$$

$\rightarrow (j+2)(j+1)\, a_{j+2} - (j-n)(j+n+1)\, a_j = 0$ for all $j \geq 0$

$$\rightarrow a_{j+2} = \frac{(j-n)(j+n+1)}{(j+2)(j+1)}\, a_j \quad (*)$$

So $a_2 = -\dfrac{n(n+1)}{2!}\, a_0$

$$a_4 = \frac{(2-n)(n+3)}{(4)(3)}\, a_2 = \frac{(n-2)n(n+1)(n+3)}{4!}\, a_0$$

$$a_6 = \frac{(4-n)(n+5)}{(6)(5)}\, a_4 = -\frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}\, a_0$$

$\vdots$

$$a_3 = \frac{(1-n)(n+2)}{(3)(2)}\, a_1 = -\frac{(n-1)(n+2)}{3!}\, a_1$$

$$a_5 = \frac{(3-n)(n+4)}{(5)(4)}\, a_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}\, a_1$$

$\vdots$

$$\rightarrow y = a_0 \Big[1 - \frac{n(n+1)}{2!} x^2 + \frac{(n-2)\, n\, (n+1)(n+3)}{4!} x^4 - \frac{(n-4)(n-2)\, n\, (n+1)(n+3)(n+5)}{6!} x^6 + \cdots \Big]$$

$$+ a_1 \Big[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!} x^7 + \cdots \Big]$$

$$= y_{even} + y_{odd}$$

Since y contains two arbitrary constants (a_0 and a_1), $y = y_{even} + y_{odd}$ is the general solution of the linear, second order ODE; and hence we don't need to consider the case of $s=1$. [Note that $s=1 \longrightarrow y_{odd}$, nothing new!]

(d) If the series in y_{even} or y_{odd} continue to infinity then, inspecting both series, we see that, for very large l ($l \gg n$), we get

at $x = \pm 1$: $$|a_l x^l| \sim |a_0| \frac{(l-1)!}{l!} \text{ or } |a_1| \frac{(l-1)!}{l!}$$

$$= \{ |a_0| \text{ or } |a_1| \} \cdot \frac{1}{l}$$

Moreover, for both series, we have at $x = \pm 1$;

$$\frac{a_{j+2}\, x^{j+2}}{a_j\, x^j} = \frac{a_{j+2}}{a_j} x^2 = \frac{a_{j+2}}{a_j}$$

$$= \frac{(j-n)(j+n+1)}{(j+2)(j+1)} \sim 1 \quad \text{for } j \gg n$$

→ no alternating sign

Hence if the series in y_{even} continues to infinity

then $y_{even}(\pm 1) \sim a_0 \left[1 - \frac{n(n+1)}{2!} + \cdots + \frac{1}{2l} + \frac{1}{2l+2} + \cdots \right]$

and $y_{odd}(\pm 1) \sim \pm a_1 \left[1 - \frac{(n-1)(n+2)}{3!} + \cdots + \frac{1}{2l+1} + \frac{1}{2l+3} + \cdots \right]$

and they both diverge if they don't terminate

e) for n an integer ≥ 0 ; $n = 0, 1, 2, \ldots$

one (but not the other) of the series terminate

$n = 0 \rightarrow y_{even} = a_0$ [polynomial of deg.

$n = 1 \rightarrow y_{odd} = a_1 x$ [" of deg.

$n=2 \rightarrow y_{even} = a_0\left[1-3x^2\right]$ (polynomial of degree 2)

$n=3 \rightarrow y_{odd} = a_1\left[x-\frac{5}{3}x^3\right]$ (polynomial of deg 3)

etc...

Note that one of the series terminates also for $n =$ negative integer; but we don't get anything new:

$$n=-1 \Longleftrightarrow n=0$$
$$n=-2 \Longleftrightarrow n=1$$
$$\vdots$$
$$n=-(\ell+1) \Longleftrightarrow n=\ell \qquad \text{since, for } n=-(\ell+1):$$

$$n(n+1) = (-\ell-1)(-\ell) = \ell(\ell+1).$$

(8.3.2) $\quad y''-2xy'+2\alpha y=0 \qquad -\infty<x<+\infty$

Multiplying with $w(x) = \frac{1}{P_0}e^{\int \frac{P_1}{P_0}dx} = e^{\int -2x\,dx} = e^{-x^2}$;

we get $\quad e^{-x^2}y''-2xe^{-x^2}y'+2\alpha e^{-x^2}y=0 \quad (*)$

which is self-adjoint since

$$\frac{d}{dx}\left[e^{-x^2}\right] = -2xe^{-x^2}, \text{ and hence}$$

$$(*) \rightarrow \frac{d}{dx}\left[e^{-x^2}y'\right]+2\alpha e^{-x^2}y=0.$$

We obtain a Hermitian eigenvalue problem if we include $w(x) = e^{-x^2}$ in the definition of the inner product, that is, if we define

$$\langle v, u \rangle = \int_{-\infty}^{+\infty} v^*(x)\, w(x)\, u(x)\, dx$$

$$= \int_{-\infty}^{+\infty} v^*(x)\, u(x)\, e^{-x^2}\, dx$$

and if the Sturm-Liouville boundary condition

$$\left[v^* e^{-x^2} u' - (v^*)' e^{-x^2} u \right]_{x=-\infty}^{+\infty} = 0 \quad \text{is satisfied} \quad (**)$$

As shown in 8.3.3 below, ~~As shown in below,~~ $(**)$ holds if $\alpha = 0, 1, 2, \ldots$ (eigenvalues) so the solutions are polynomial and hence $e^{-x^2} \cdot (\text{polynomial}) \xrightarrow[x \to \pm\infty]{} 0$.

8.3.3 $y'' - 2xy' + 2\alpha y = 0$

a) Let $y = \sum_{j=0}^{\infty} a_j x^{j+s}$, $\underline{a_0 \neq 0}$. Then

$$y' = \sum_{j=0}^{\infty} (j+s) a_j x^{j+s-1} \text{ and}$$

$$y'' = \sum_{j=0}^{\infty} (j+s)(j+s-1) a_j x^{j+s-2}. \quad \text{Thus,}$$

$$\sum_{j=0}^{\infty} (j+s)(j+s-1) a_j x^{j+s-2} - \sum_{j=0}^{\infty} 2(j+s) a_j x^{j+s}$$

$$+ \sum_{j=0}^{\infty} 2\alpha a_j x^{j+s} = 0$$

$$\Rightarrow \sum_{j=-2}^{\infty} (j+s+2)(j+s+1) a_{j+2} x^{j+s}$$

$$- \sum_{j=0}^{\infty} 2(j+s-\alpha) a_j x^{j+s} = 0$$

$$\Rightarrow s(s-1) a_0 x^{s-2} + (s+1) s a_1 x^{s-1}$$

$$+ \sum_{j=0}^{\infty} \left[(j+s+2)(j+s+1) a_{j+2} - 2(j+s-\alpha) a_j \right] x^{j+s} = \quad (*)$$

$$\Rightarrow s(s-1) a_0 = 0$$

$$\Rightarrow s(s-1) = 0 \text{ since } a_0 \neq 0$$

$$\Rightarrow s = 0 \text{ or } s = 1$$

for $s=0$, $(*) \longrightarrow$ a_0 and a_1 are arbitrary

and

$$\sum_{j=0}^{\infty} \left[(j+2)(j+1) a_{j+2} - 2(j-\alpha) a_j \right] x^j = 0$$

$$\rightarrow \boxed{a_{j+2} = \frac{2(j-\alpha)}{(j+2)(j+1)} a_j} \quad \text{for all } j \geq 0 \quad (**)$$

$$j=0 \rightarrow a_2 = \frac{2(-\alpha)}{2!} a_0$$

$$j=2 \rightarrow a_4 = \frac{2(2-\alpha)}{(4)(3)} a_2 = 2^2 \frac{(-\alpha)(2-\alpha)}{4!} a_0$$

$\vdots$

$$j=2k-2 \rightarrow a_{2k} = 2^k \frac{(-\alpha)(2-\alpha)\cdots(2k-2-\alpha)}{(2k)!} a_0$$

$$j=1 \rightarrow a_3 = \frac{2(1-\alpha)}{3!} a_1$$

$$j=3 \rightarrow a_5 = \frac{2(3-\alpha)}{(5)(4)} a_3 = 2^2 \frac{(1-\alpha)(3-\alpha)}{5!} a_1$$

$\vdots$

$$j=2k-1 \rightarrow a_{2k+1} = 2^k \frac{(1-\alpha)(3-\alpha)\cdots(2k-1-\alpha)}{(2k+1)!} a_1$$

Thus,

$$y = \sum_{j=0}^{\infty} a_j x^{j+s}$$

$$= \sum_{j=0}^{\infty} a_j x^{j} \qquad (s=0)$$

$$= \sum_{j\,\text{even}} a_j x^j + \sum_{j\,\text{odd}} a_j x^j$$

$$= \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}$$

$$= a_0 + a_2 x^2 + a_4 x^4 + - - -$$

$$+ a_1 + a_3 x^3 + a_5 x^5 + - - .$$

$$= a_0 \left[1 + \frac{2(-\alpha)}{2!} x^2 + \frac{2^2(-\alpha)(2-\alpha)}{4!} x^4 + \cdots \right]$$

$$+ a_1 \left[x + \frac{2(1-\alpha)}{3!} x^3 + \frac{2^2(1-\alpha)(3-\alpha)}{5!} x^5 + \cdots \right]$$

$= y_{even} + y_{odd}$, where

$$y_{even} = a_0\left[1 + \frac{2(-\alpha)}{2!}x^2 + \frac{2^2(-\alpha)(2-\alpha)}{4!}x^4 + \cdots\right]$$

$$= a_0 \sum_{k=0}^{\infty} \frac{2^k(-\alpha)(2-\alpha)\cdots(2k-2-\alpha)}{(2k)!} x^{2k}$$

and

$$y_{odd} = a_1\left[x + \frac{2(1-\alpha)}{3!}x^3 + \frac{2^2(1-\alpha)(3-\alpha)}{5!}x^5 + \cdots\right]$$

$$= a_1 \sum_{k=0}^{\infty} 2^k \frac{(1-\alpha)(3-\alpha)\cdots(2k-1-\alpha)}{(2k+1)!} x^{2k+1}$$

Since y contains two arbitrary constants a_0 and a_1, this is the general solution of the linear, 2nd order ODE (the Hermite Equation.)

Thus, the $s=1$ will not give us anything new. In the following, I will verify that $s=1$ gives us again y_{odd} (above!).

For $s=1$, $(*) \longrightarrow$ (Replacing a_j by b_j to avoid confusion with the $s=0$ case):

$$\cancel{0\, b_0\, x^{-1}} + 2\, b_1\, x^0$$

$$+ \sum_{j=0}^{\infty} \left[(j+3)(j+2)\, b_{j+2} - 2(j+1-\alpha)\, b_j \right] x^{j+1} = 0$$

$$\Rightarrow \begin{cases} b_0 \text{ is arbitrary,} \\ 2b_1 = 0, \text{ and} \\ b_{j+2} = 2\, \dfrac{j+1-\alpha}{(j+3)(j+2)}\, b_j \quad (***) \end{cases}$$

$\rightarrow$ for all $j \geq 0$

Since $b_1 = 0$, $(***) \Rightarrow b_{2k+1} = 0$ for all k;

that is: $b_1 = b_3 = b_5 = \cdots = 0$.

$$j = 0 \rightarrow b_2 = 2\, \frac{(1-\alpha)}{(3)(2)}\, b_0 = 2\, \frac{(1-\alpha)}{3!}\, b_0$$

$$j = 2 \rightarrow b_4 = 2\, \frac{(3-\alpha)}{(5)(4)}\, b_2 = 2^2\, \frac{(1-\alpha)(3-\alpha)}{5!}\, b_0$$

$$\vdots \qquad \vdots$$

$$b_{2k} = 2^k\, \frac{(1-\alpha)(3-\alpha)\cdots(2k-1-\alpha)}{(2k+1)!}\, b_0$$

Thus, $y = \sum_{j=0}^{\infty} b_j\, x^{j+s}$

$$= \sum_{j=0}^{\infty} b_j\, x^{j+1} \qquad (s=1)$$

$= b_0 x + b_2 x^3 + b_4 x^5 + \cdots$

$= b_0 \left[x + 2 \frac{(1-\alpha)}{3!} x^3 + 2^2 \frac{(1-\alpha)(3-\alpha)}{5!} x^5 + \cdots \right]$

$= y_{odd}$ from the $s=0$ case ✓

(with $b_0 \equiv a_1$ as the arbitrary constant.)

(b) In both series y_{even} and y_{odd}

$\frac{a_{j+2}}{a_j} = \frac{2(j-\alpha)}{(j+2)(j+1)}$. It follows that

$\lim_{j\to\infty} \frac{a_{j+2}}{a_j} = \lim_{j\to\infty} \frac{2}{j} = 0$.

By the ratio test, both series converge for all x.

In both series, the ratio of successive coeffs.

is $\frac{a_{n+2}}{a_n} = 2 \frac{n-\alpha}{(n+2)(n+1)}$, from the recursive relation (**)

$\sim \frac{2}{n}$ for large n.

Now: $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^j}{j!} + \frac{x^{j+1}}{(j+1)!} + \cdots$

$$\Rightarrow \quad e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \cdots + \frac{1}{j!} x^{2j} + \frac{1}{(j+1)!} x^{2j+2} + \cdots$$

The ratio of consecutive coeffs. in e^{x^2} is of the form $\frac{c_{2j+2}}{c_{2j}} = \frac{1}{(j+1)!} \Big/ \frac{1}{j!} = \frac{j!}{(j+1)!} = \frac{1}{j+1}$, or

Setting $n = 2j$, we get:

$\frac{c_{n+2}}{c_n} = \frac{1}{n/2 + 1} = \frac{2}{n+2} \sim \frac{2}{n}$. Thus, for large n, the two series in y_{even} and y_{odd} behave like e^{x^2} and this makes the Sturm-Liouville boundary condition $e^{-x^2}\left[v^* u' - (v^*)' u\right]\Big|_{-\infty}^{+\infty} = 0$ fail, unless the series terminate.

(c) Inspecting y_{even} and y_{odd}, we see that y_{even} terminates if $\alpha = 0, 2, 4, \ldots$, yielding an even polynomial of degree α; and y_{odd} terminates if $\alpha = 1, 3, 5, \ldots$, yielding an odd polynomial of degree α.

$\alpha = 0 \Rightarrow y_{even} = a_0$

$\alpha = 1 \rightarrow y_{odd} = a_1 x$

$\alpha = 2 \Rightarrow y_{even} = a_0 [1 - 2x^2]$

$\alpha = 3 \Rightarrow y_{odd} = a_1 [x - \frac{2}{3} x^3]$

$\alpha = 4 \rightarrow y_{even} = a_0 [1 - 4x^2 + \frac{4}{3} x^4]$

$\alpha = 5 \Rightarrow y_{odd} = a_1 [x - \frac{4}{3} x^3 + \frac{4}{15} x^5]$, etc...

If we choose a_0 and a_1 so that the highest order term in the polynomials obtained above is $(2x)^\alpha$, we get the Hermite polynomials. Thus,

$\alpha = 0 \Rightarrow H_0(x) = 1 \quad (a_0 = 1)$

$\alpha = 1 \rightarrow H_1(x) = 2x \quad (a_1 = 2)$

$\alpha = 2 \Rightarrow H_2(x) = \overbrace{4x^2}^{(2x)^2} - 2 \quad (a_0 = -2)$

$\alpha = 3 \rightarrow H_3(x) = 8x^3 - 12x \quad (a_1 = -12)$

$\alpha = 4 \rightarrow H_4(x) = 16x^4 - 48x^2 + 12 \quad (a_0 = 12)$

$\alpha = 5 \rightarrow H_5(x) = \underbrace{32x^5}_{(2x)^5} - 160x^3 + 120x \quad (a_1 = 120)$, ...

etc...