PHYS 3496

Assignment #7

Total: 46 marks

4 marks

11.6.3) $f_2(z) = f_2(z_0) + f_2'(z_0)(z-z_0) + \frac{f_2''(z_0)}{2!}(z-z_0)^2 + \cdots$

$= (z-z_0)\left[f_2'(z_0) + \frac{f_2''(z_0)}{2!}(z-z_0) + \cdots\right]$

$= (z-z_0)\, g(z)$, where

$g(z) = f_2'(z_0) + \frac{f_2''(z_0)}{2!}(z-z_0) + \cdots$

So $g(z_0) = f_2'(z_0) \neq 0$

$a_{-1} = \frac{1}{2\pi i}\oint_C f(w)\,dw$ where C is a closed contour contained within the region of convergence of the Laurent series (see Eqn 11.48 with $n=-1$; $z' \Leftrightarrow \omega$).

Thus, $$a_{-1} = \frac{1}{2\pi i} \oint_C \frac{f_1(\omega)}{f_2(\omega)} d\omega$$

$$= \frac{1}{2\pi i} \oint_C \frac{f_1(\omega)}{(\omega - z_0) g(\omega)} d\omega$$

$$= \frac{f_1(z_0)}{g(z_0)}$$ by Cauchy Integral formula (since $\frac{f_1}{g}$ is analytic on and within (C).)

But $g(z_0) = f_2'(z_0)$. Hence

$$a_{-1} = \frac{f_1(z_0)}{f_2'(z_0)}.$$

11.6.5 (5 marks) $$f(z) = z^{-1/3} + \frac{z^{-1/4}}{(z-3)^3} + (z-2)^{1/2}$$

- $z=0$ is a branch point of order 12 (the least common multiple of 3 and 4.)

- $z=3$ is a pole of order 3

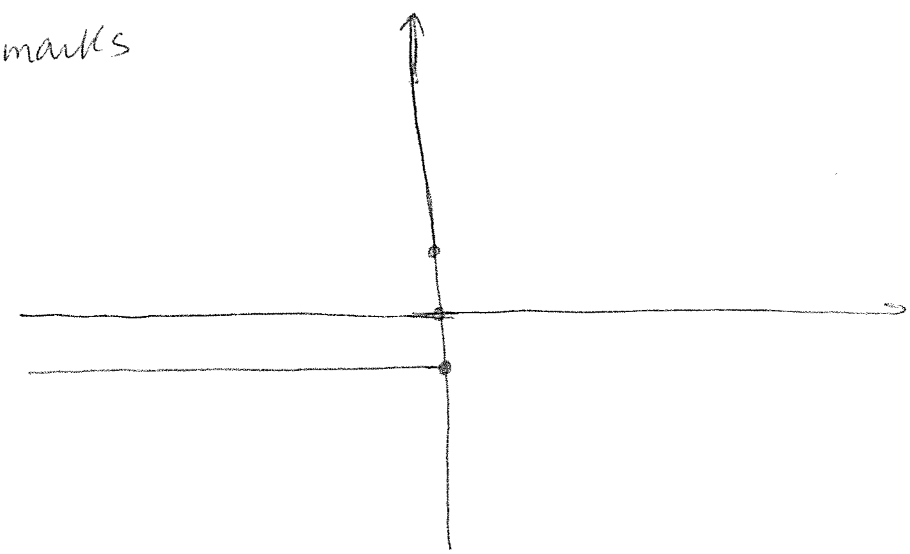
- $z=2$ is a branch point of order 2.
- To study the singularity of $f(z)$ at $z=\infty$, we look at $g(\omega) = f(1/\omega)$ at $\omega = 0$.

$$g(\omega) = \omega^{1/3} + \frac{\omega^{1/4}}{(\frac{1}{\omega} - 3)^3} + \left(\frac{1}{\omega} - 2\right)^{1/2}$$

$$g(\omega) = \omega^{1/3} + \omega^{1/4} \cdot \frac{\omega^3}{(1-3\omega)^3} + \frac{(1-2\omega)^{1/2}}{\omega^{1/2}}$$

$\omega = 0$ is a branch point of order 12 (the least common multiple of 2, 3 and 4.). Hence $z = \infty$ is a branch point of order 12 of $f(z)$.

11.6.6 5 marks



$$F(z) = \ln(z^2+1)$$

$$z^2+1 = (z^2 - i^2) = (z-i)(z+i) = r_1 e^{i\theta_1} r_2 e^{i\theta_2}$$

~~At z~~ $f(z) = \ln(z^2+1) = \ln\left[r_1 r_2 e^{i(\theta_1+\theta_2)}\right]$

$$= \mathrm{Ln}(r_1 r_2) + i(\theta_1 + \theta_2 + 2k\pi) \quad k \in \mathbb{Z}. \quad (*)$$

At $z = 0$, $z - i = -i = e^{-i\pi/2} \to \theta_1 = -\pi/2$
$z + i = +i = e^{i\pi/2} \to \theta_2 = \pi/2$ $\Big\} \to \theta_1 + \theta_2 = 0$

Since we want to use the branch of ln that gives $F(0) = -2\pi i$, $k = -1$ in $(*)$.

$f(\bar{z} - z) = ?$ (using the same branch with $k = -1$)

For $z = i-2$, $z - i = -2 = 2e^{-i\pi} \rightarrow \theta_1 = -\pi$

$z + i = 2i - 2 = 2(i-1) = 2\sqrt{2}\left[-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right]$

$= 2\sqrt{2}\, e^{i3\pi/4} \rightarrow \theta_2 = 3\pi/4$

$$F(i-2) = \ln(r_1 r_2) + i(\theta_1 + \theta_2 - 2\pi) \quad \leftarrow k = -1$$

$$= \ln(4\sqrt{2}) + i(-\pi + 3\pi/4 - 2\pi)$$

$$= \ln(4\sqrt{2}) - i\frac{9\pi}{4}.$$

5 marks

11.6.10 a) $f(z) = \frac{1}{z(z-1)} = \frac{1}{z-1}\,\frac{1}{z}$

$$= \frac{1}{z-1}\,\frac{1}{1+(z-1)}$$

$$= \frac{1}{z-1}\left[1 - (z-1) + (z-1)^2 - (z-1)^3 + - - - -\right]$$

$$= \frac{1}{z-1} - 1 + (z-1) - (z-1)^2 + - - - -$$

$$= \sum_{n=-1}^{\infty} (-1)^{n+1}(z-1)^n, \qquad 0 < |z-1| < 1$$

b) $f(z) = \frac{1}{z-1}\,\frac{1}{z} = \frac{1}{z-1}\,\frac{1}{1+(z-1)} = \frac{1}{(z-1)^2}\,\frac{1}{1+1/z-1}$

$$= \frac{1}{(z-1)^2}\left[1 - \frac{1}{z-1} + \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3} + - - - -\right] \quad \left(\frac{1}{|z-1|} < 1\right)$$

$$= \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3} + \frac{1}{(z-1)^4} - \frac{1}{(z-1)^5} + - - - -$$

$$= \sum_{n=2}^{\infty} (-1)^n (z-1)^{-n}, \qquad |z-1| > 1$$

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Homework #9

12 marks

11.7.1 (b) $f(z) = \frac{1}{(z^2+a^2)^2} = \frac{1}{(z-ia)^2(z+ia)^2}$

$z_1 = ia$ and $z_2 = -ia$ are both poles of order 2.

$$Res(f, z_1) = \lim_{z \to ia} \frac{1}{1!} \frac{d}{dz}\left[(z-ia)^2 f(z)\right]$$

$$= \lim_{z \to ia} \frac{d}{dz} \frac{1}{(z+ia)^2}$$

$$= \lim_{z \to ia} \frac{-2}{(z+ia)^3} = \frac{-2}{(2ia)^3} = \frac{-2}{-8ia^3}$$

$$= -\frac{i}{4a^3}.$$ similarly,

$$Res(f, z_2) = \lim_{z \to -ia} \frac{d}{dz}\left[\frac{1}{(z-ia)^2}\right] = \frac{-2}{(-2ia)^3} = \frac{i}{4a^3}$$

To check the behavior of f at $z = \infty$, we look at $g(w) = f(1/w)$ at $w = 0$.

$$g(w) = \frac{1}{\left(\frac{1}{w^2}+a^2\right)^2} = \frac{w^4}{(1+a^2w^2)^2}$$ is analytic at and near $w = 0$.

So,
$z = \infty$ is a regular pt for $f(z)$.

d) $f(z) = \frac{\sin(1/z)}{z^2+a^2} = \frac{\sin(1/z)}{(z+ia)(z-ia)}$

$z_1 = 0$ is an essential singularity

$z_2 = ia$ and $z_3 = -ia$ are both simple poles

Res(f, z_1):

$$f(z) = \frac{\sin(1/z)}{(z+ia)(z-ia)} = \frac{\sin(1/z)}{a^2[1+z^2/a^2]}$$

$$= a^{-2}\left[1 - z^2/a^2 + \frac{z^4}{a^4} - + - -\right]\left[\frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + - - -\right]$$

Res$(f, 0)$ = coefficient of $\frac{1}{z}$

$$= a^{-2}\left[1 + \frac{1}{3!a^2} + \frac{1}{5!a^4} + \cdots\right]$$

$$= a^{-1}\left[\frac{1}{a} + \frac{1}{3!}\frac{1}{a^3} + \frac{1}{5!}\frac{1}{a^5} + \cdots\right] = a^{-1}\sinh(1/a) = \frac{\sinh(1/a)}{a}$$

$$\text{Res}(f, z_2) = \lim_{z\to ia}\left[(z-ia)f(z)\right] = \lim_{z\to ia}\frac{\sin(1/z)}{z+ia}$$

$$= \frac{\sin(1/ia)}{2ia} = \frac{\sin(-i/a)}{2ia} = -\frac{i\sinh(1/a)}{2ia} = -\frac{\sinh(1/a)}{2a}$$

$$\text{Res}(f, z_3) = \lim_{z\to -ia}\left[(z+ia)f(z)\right] = \lim_{z\to -ia}\frac{\sin(1/z)}{z-ia}$$

$$= \frac{\sin(i/a)}{-2ia} = +\frac{i\sinh(1/a)}{-2ia} = -\frac{\sinh(1/a)}{2a}$$

At ∞: $g(w) = f(1/w) = \dfrac{\sin(w)}{\frac{1}{w^2} + a^2} = \dfrac{w^2\sin w}{1+a^2w^2}$

$w = 0$ is a regular pt of $g(w) \to z = \infty$ is a regular pt of $f(z)$.

g) $f(z) = \frac{z e^{iz}}{z^2 - a^2} = \frac{z e^{iz}}{(z-a)(z+a)}$

$z_1 = a$ and $z_2 = -a$ are both simple poles.

$$\text{Res}(f, z_1) = \lim_{z \to a} \frac{z e^{iz}}{z+a} = \frac{a e^{ia}}{2a} = \frac{e^{ia}}{2}$$

$$\text{Res}(f, z_2) = \lim_{z \to -a} \frac{z e^{iz}}{z-a} = \frac{-a e^{-ia}}{-2a} = \frac{e^{-ia}}{2}$$

At ∞:

$$g(\omega) = f(1/\omega) = \frac{1/\omega \; e^{i/\omega}}{\left(\frac{1}{\omega} - a\right)\left(\frac{1}{\omega} + a\right)} = \frac{\omega e^{i/\omega}}{(1-a\omega)(1+a\omega)}$$

$$= \frac{\omega e^{i/\omega}}{1 - a^2\omega^2}$$

$$= \omega\left[1 + a^2\omega^2 + a^4\omega^4 + \cdots\right]\left[1 + i\omega^{-1} - \frac{\omega^{-2}}{2!} - \frac{i\omega^{-3}}{3!} + \cdots\right] \quad (*)$$

$\omega = 0$ is an essential singularity of $g(\omega)$ $\Rightarrow$

$z = \infty$ is an essential singularity of $f(z)$.

$\text{Res}(f, \infty)$ = coefficient of $\frac{1}{\omega}$ in $(*)$:

$$-\frac{1}{2!} + \frac{a^2}{4!} - \frac{a^4}{6!} + \cdots = \frac{\cos a - 1}{a^2}$$

h) $f(z) = \frac{z^{-k}}{z+1}$

Branch points at $z = 0, \infty$.

$z = -1$ is a simple pole.

$$\text{Res}(f, -1) = \lim_{z \to -1 = e^{i\pi}} \left[(z-1) f(z)\right] = \lim_{z \to e^{i\pi}} z^{-k} = e^{-ik\pi}$$

5 marks

11.7.2 $f(z) = \dfrac{\pi \cot \pi z}{z(z+1)}$

Since $\cot \pi z = \dfrac{\cos \pi z}{\sin \pi z}$ has simple poles at $z =$ integer, it follows $z=0$ and $z=-1$ are both poles of order 2 for $f(z)$. To find the residues of f at $z=0$ and $z=-1$, we need to find a_{-1} in the Laurent expansions near 0 and -1.

Near $z=0$: $f(z) = \dfrac{\pi}{z} \dfrac{1}{1+z} (\cos \pi z) \dfrac{1}{\sin \pi z}$

$$= \frac{\pi}{z}(1 - z + z^2 - + \cdots) \frac{(1 - \pi^2 z^2/2 + \cdots)}{\pi z - \frac{\pi^3 z^3}{3!} + \cdots}$$

$$f(z) = \frac{\pi}{z}(1 - z + z^2 - + \cdots)(1 - \pi^2 z^2/2 + \cdots)\frac{1}{\pi z}\left(1 + \frac{\pi^2 z^2}{3!} - + \cdots\right)$$

$$= \frac{1}{z^2}(1 - z + z^2 - + \cdots)(1 - \pi^2 z^2/2 + \cdots)(1 + \pi^2 z^2/3! + \cdots)$$

$$= \frac{1}{z^2} - \frac{1}{z} + \cdots$$

$\rightarrow \operatorname{Res}(f, 0) = -1$

Near $z=-1$: $f(z) = \dfrac{\pi}{z+1} \dfrac{1}{(z+1)-1} \cdot \dfrac{\cos \pi (z+1)}{\sin \pi (z+1)}$

$$= -\frac{\pi}{(z+1)} \frac{1}{1-(z+1)} \frac{\cos \pi (z+1)}{\sin \pi (z+1)}$$

$$= \frac{-\pi}{(z+1)}\left[1 + (z+1) + (z+1)^2 + \cdots\right]\left[1 - \frac{\pi^2 (z+1)^2}{2} + \cdots\right]\frac{1}{\pi (z+1)}\left[1 + \frac{\pi^2 (z+1)^2}{3!} + \cdots\right]$$

$$= -\frac{1}{(z+1)^2} - \frac{1}{(z+1)} + \cdots \quad \rightarrow \operatorname{Res}(f, -1) = -1.$$

5 marks

11.7.3 $\quad E_i(x) = \int_{-\infty}^{x} \frac{e^t}{t}\,dt \qquad (x>0)$

$$= \lim_{\delta \to 0^+}\left(\int_{-\infty}^{-\delta} \frac{e^t}{t}\,dt + \int_{\delta}^{x} \frac{e^t}{t}\,dt\right)$$

$$= \lim_{\delta \to 0^+}\left(\int_{-\infty}^{-x} \frac{e^t}{t}\,dt + \int_{-x}^{-\delta} \frac{e^t}{t}\,dt + \int_{\delta}^{x} \frac{e^t}{t}\,dt\right)$$

$$= \int_{-\infty}^{-x} \frac{e^t}{t}\,dt + \lim_{\delta \to 0^+}\left[\int_{-x}^{-\delta} \frac{e^t}{t}\,dt + \int_{\delta}^{x} \frac{e^t}{t}\,dt\right] \quad (*)$$

But $\int_{-x}^{-\delta} \frac{e^t}{t}\,dt = \int_{x}^{\delta} \frac{e^{-u}}{(-u)}\,d(-u)$

$$= \int_{x}^{\delta} \frac{e^{-u}}{u}\,du = \int_{x}^{\delta} \frac{e^{-t}}{t}\,dt = -\int_{\delta}^{x} \frac{e^{-t}}{t}\,dt$$

Hence $\int_{-x}^{-\delta} \frac{e^t}{t}\,dt + \int_{\delta}^{x} \frac{e^t}{t}\,dt = \int_{\delta}^{x} \frac{e^t - e^{-t}}{t}\,dt$

Since $\frac{e^t - e^{-t}}{t} = \frac{(1+t+t^2/2!+\cdots) - (1-t+t^2/2!-+\cdots)}{t}$

$$= 2 + \tfrac{1}{3}t^2 + \cdots,$$

$t=0$ is a regular pt (not a singular pt) of $\frac{e^t - e^{-t}}{t}$

and hence the limit in $(*)$ exists. Thus,

$\int_{-\infty}^{x} \frac{e^t}{t}\,dt$ converges for positive x.

[$\int_{-\infty}^{-x} \frac{e^t}{t}\,dt$ converges for $x>0$ since 0 is not in the interval of integration.]

5 marks

(11.7.9) $f(z) = \dfrac{z^2 - 3z + 2}{z} = \dfrac{(z-1)(z-2)}{z} = \dfrac{z-1}{z} + \dfrac{z-2}{z}$

$$f'(z) = \frac{z-2}{z} + \frac{z-1}{z} - \frac{(z-1)(z-2)}{z^2}$$

$$\frac{f'(z)}{f(z)} = \frac{1}{z-1} + \frac{1}{z-2} - \frac{1}{z}$$

Note that $z=0$ is a pole, and $z=1$ and $z=2$ are zeros of $f(z)$. However, all three points are simple poles of $\dfrac{f'(z)}{f(z)}$ with residues -1, 1 and 1 ~~respe~~ at $z = 0, 1, 2$, respectively. Therefore, the integral of f'/f on any contour that encloses $z=0$ will have a contribution $-2\pi i$ from the $\left(-\frac{1}{z}\right)$ term, while the integral will have contributions $2\pi i$ for each of $z=1$ and $z=2$ that are enclosed by (C). This is consistent with the formula

$$\oint_C \frac{f'(z)}{f(z)}\, dz = 2\pi i \,(N_f - P_f).$$