PHYS 3496, Assignment 6
Total: 31 marks

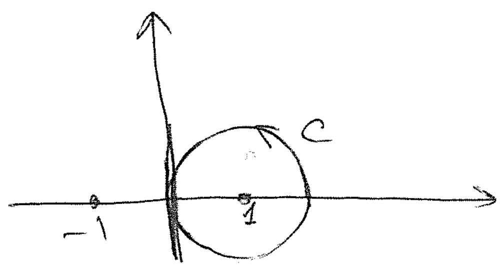
11.4.1 2 marks

$$\frac{1}{2\pi i}\oint_C z^{m-n-1}\,dz = \begin{cases} 0 & \text{if } m-n-1 \neq -1 \\ 1 & \text{if } m-n-1 = -1 \end{cases}, \text{ using Eqn (11.29)}$$

$$= \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

$$= \delta_{mn}.$$

11.4.2 3 marks



$$\oint_C \frac{dz}{z^2-1} = \oint_C \frac{f(z)\,dz}{z-1} \quad \text{where } f(z) = \frac{1}{z+1} \text{ is}$$

analytic on and within (C). Thus, by Cauchy Integral Formula,

$$\oint_C \frac{dz}{z^2-1} = 2\pi i\, f(1) = 2\pi i \frac{1}{1+1} = \pi i.$$

11.4.3 3 marks

Since f is analytic on and within (C), so is f'. Hence, applying the Cauchy Integral formula to f', we get

$$\oint_C \frac{f'(z)}{z-z_0}\,dz = 2\pi i\, f'(z_0). \quad (*)$$

From Equation (11.32), we have

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz \quad (**)$$

Substituting (**) into (*), we get: $\oint_C \frac{f'(z)}{z-z_0}\,dz = \oint_C \frac{f(z)}{(z-z_0)^2}\,dz.$

11.4.6 3 marks Recall that $f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$

for (C) any closed contour encircling z_0 once,

and f any function that is analytic on and within (C).

Take $n=2$, $z_0 = 0$, (C) the square in the problem,

and $f(z) = e^{iz}$:

$$f''(0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{z^3}\, dz \rightarrow \oint_C \frac{f(z)}{z^3}\, dz = \pi i\, f''(0)$$

$$\rightarrow \oint_C \frac{e^{iz}}{z^3}\, dz = \pi i \left.\frac{d^2}{dz^2}(e^{iz})\right|_{z=0} = -\pi i$$

11.4.9 5 marks Using partial fractions, we find

$$\frac{1}{z(2z+1)^2} = \frac{1}{z} - \frac{2}{2z+1} - \frac{2}{(2z+1)^2}$$

$$\rightarrow \oint_C \frac{f(z)}{z(2z+1)^2}\, dz = \oint_C \frac{f(z)}{z}\, dz - \oint_C \frac{f(z)\, dz}{z+1/2} - \frac{1}{2} \oint_C \frac{f(z)\, dz}{(z+\frac{1}{2})^2}$$

$$= 2\pi i\, f(0) - 2\pi i\, f(-1/2) - \pi i\, f'(-1/2)$$

[both 0 and $-1/2$ are inside (C); and we assumed $f(z)$ is analytic on and inside (C) so we could apply Cauchy Integral formula].

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~~[illegible]~~

(11.5.1) 3 marks $f(z) = \ln(1+z)$ is analytic at $z=0$ (and in the disk $|z|<1$ around 0).

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

$$f(0) = \ln 1 = 0$$

$$f'(z) = \frac{1}{1+z} \rightarrow f'(0) = 1$$

$$f''(z) = -\frac{1}{(1+z)^2} \rightarrow f''(0) = -1$$

$$f'''(z) = +\frac{2!}{(1+z)^3} \rightarrow f'''(0) = 2!$$

$$\vdots$$

using induction, we get:

$$f^{(n)}(z) = (-1)^{n-1} \frac{(n-1)!}{(1+z)^n} \rightarrow f^{(n)}(0) = (-1)^{n-1}(n-1)!$$

Thus, $f(z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(n-1)!}{n!} z^n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$, $|z|<1$.

(11.5.2) 4 marks $f(z) = (1+z)^m$

Note that for m not a positive integer (or zero), $z=-1$ is a singular pt of $f(z)$. So the Taylor series we will find will have a radius of convergence $=1$ (distance from

0 to -1.)

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

$$f(0) = 1^m = 1$$

$$f'(z) = m(1+z)^{m-1} \longrightarrow f'(0) = m$$

$$f''(z) = m(m-1)(1+z)^{m-2} \longrightarrow f''(0) = m(m-1)$$

$\vdots$

using induction, we can show that

$$f^{(n)}(z) = m(m-1) \cdots (m-n+1)(1+z)^{m-n} \longrightarrow$$

$$f^{(n)}(0) = m(m-1) \cdots (m-n+1)$$

$$= \frac{\Gamma(m+1)}{\Gamma(m-n+1)}$$

(Remember Γ function from PHYS $\left\{ \begin{matrix} \cancel{2490}? \\ 2496 \end{matrix} \right\}$)

$$\rightarrow \frac{f^{(n)}(0)}{n!} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)\, n!} = \binom{m}{n}$$

[see problem 3 Section 5, Chapter 11 in Boas, one of the homework problems for PHYS 2490 and PHYS 2496.]

Thus, $f(z) = 1 + \sum_{n=1}^{\infty} \binom{m}{n} z^n$.

This converges for $|z| < 1$, as explained above, since the distance from 0 to the nearest (in this case the only) singular pt $(z=-1)$ is equal to 1.

11.5.7 4 marks

$$f(z) = \frac{z e^z}{z-1} = \frac{1}{z-1}\left[(z-1)+1\right] e \, e^{z-1}$$

$$= \frac{e}{z-1}\left[(z-1)e^{z-1} + e^{z-1}\right]$$

$$= \frac{e}{z-1}\left[\sum_{n=0}^{\infty}\frac{(z-1)^{n+1}}{n!} + \sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}\right]$$

$$= \frac{e}{z-1}\left[\sum_{n=0}^{\infty}\frac{(z-1)^{n+1}}{n!} + 1 + \sum_{n=1}^{\infty}\frac{(z-1)^n}{n!}\right]$$

$$= \frac{e}{z-1} + \frac{e}{z-1}\left[\sum_{n=0}^{\infty}\frac{(z-1)^{n+1}}{n!} + \sum_{n=1}^{\infty}\frac{(z-1)^n}{n!}\right]$$

$$= \frac{e}{z-1} + \frac{e}{z-1}\left[\sum_{n=0}^{\infty}\frac{(z-1)^{n+1}}{n!} + \sum_{n=0}^{\infty}\frac{(z-1)^{n+1}}{(n+1)!}\right]$$

$$= \frac{e}{z-1} + e\sum_{n=0}^{\infty}\left[\frac{1}{n!} + \frac{1}{(n+1)!}\right](z-1)^n$$

$$= \frac{e}{z-1} + e\sum_{n=0}^{\infty}\frac{(n+1)! + n!}{n!\,(n+1)!}(z-1)^n$$

$$= \frac{e}{z-1} + e\sum_{n=0}^{\infty}\frac{(n+1)+1}{(n+1)!}(z-1)^n$$

$$= \frac{e}{z-1} + e\sum_{n=0}^{\infty}\frac{n+2}{n+1}\frac{(z-1)^n}{n!}$$

$$= \frac{e}{z-1} + e\left[2 + \frac{3}{2}(z-1) + \frac{2}{3}(z-1)^2 + \cdots\right]$$

11.5.8 4 marks

$$f(z) = (z-1)e^{1/z} = (z-1)\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{z^n}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{z^{n-1}} - \sum_{n=0}\frac{1}{n!}\frac{1}{z^n}$$

$$= z + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{1}{z^{n-1}} - \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n}$$

$$= z + \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \frac{1}{z^n} - \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n}$$

$$= z + \sum_{n=0}^{\infty} \left[\frac{1}{(n+1)!} - \frac{1}{n!}\right] \frac{1}{z^n}$$

$$= z - \sum_{n=0}^{\infty} \left[\frac{1}{n!} - \frac{1}{(n+1)!}\right] z^{-n}$$

$$= z - \sum_{n=1}^{\infty} \left[\frac{1}{n!} - \frac{1}{(n+1)!}\right] z^{-n}$$

[The $n=0$ term vanishes $\frac{1}{0!} - \frac{1}{1!} = 0.$]

$$= z - \sum_{n=1}^{\infty} \left[\frac{(n+1)! - n!}{n!\,(n+1)!}\right] z^{-n}$$

$$= z - \sum_{n=1}^{\infty} \frac{n+1-1}{(n+1)!} z^{-n}$$

$$= z - \sum_{n=1}^{\infty} \frac{n}{n+1} \frac{z^{-n}}{n!}$$

$$= z - \left[\frac{1}{2}\frac{1}{z} + \frac{1}{3}\frac{1}{z^2} + \frac{1}{8}\frac{1}{z^3} + \cdots\right]$$