PHYS 3496
Homework # 5
52 ~~30~~ marks

11.2.1 2 marks

$f(z) = \mathrm{Re}(z) = x \rightarrow u(x,y) = x, \quad v(x,y) = 0$
$= u(x,y) + i\,v(x,y)$

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = 0$$

Since $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ for all (x,y), it follows that f is not differentiable for ~~all z~~ any z and hence f is nowhere analytic.

11.2.3 4 marks a) $u(x,y) = x^3 - 3xy^2$

We want $w(z) = u(x,y) + i\,v(x,y)$ to be analytic. So the Cauchy-Riemann conditions have to hold:

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} & (1) \\[2ex] \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} & (2) \end{cases}$$

We integrate (1) and (2) to find $v(x,y)$.

$$(1) \rightarrow \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2$$

$$\rightarrow v(x,y) = 3x^2y - y^3 + h(x) \quad (*)$$

$$(2) \rightarrow \frac{\partial v}{\partial x} = 6xy$$

$$(*) \rightarrow \frac{\partial v}{\partial x} = 6xy + h'(x)$$

$\Big\} \rightarrow h'(x) = 0$

$\rightarrow h(x) = K$ (constant).

Thus, $w(z) = u(x,y) + i\, v(x,y)$

$$= (x^3 - 3xy^2) + i(3x^2y - y^3 + K)$$

$$= (x^3 - 3xy^2) + i(3x^2y - y^3) + iK$$

$$= (x+iy)^3 + iK$$

$$w(z) = z^3 + iK \quad (K \text{ real constant})$$

4 marks

b) $v(x,y) = e^{-y}\sin x$

$w(z) = u(x,y) + i\,v(x,y)$ with (C-R conditions):

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = -e^{-y}\sin x \quad (1) \\ \text{and} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} = -e^{-y}\cos x \quad (2) \end{cases}$$

$$(1) \rightarrow u(x,y) = e^{-y}\cos x + h(y) \quad (*)$$

$$(2) \rightarrow \frac{\partial u}{\partial y} = -e^{-y}\cos x$$

$$= -e^{-y}\cos x + h'(y) \quad (\text{by } (*))$$

Hence $h'(y) = 0$ and $h(y) = K$ (constant).

Thus, $u(x,y) = e^{-y}\cos x + K$ and hence

$w(z) = e^{-y}\cos x + K + i\,e^{-y}\sin x$

$w(z) = e^{-y}[\cos x + i\sin x] + K$

$= e^{-y} e^{+ix} + K$

$= e^{ix-y} + K$

$= e^{i(x+iy)} + K$

$w(z) = e^{iz} + K$ (K real constant).

4 marks

11.2.4 Suppose that $w_1(z) = u(x,y) + iv(x,y)$ and $w_2(z) = w^*(z) = u(x,y) - iv(x,y)$ are both analytic in some region R of the complex plane.

Then applying the Cauchy-Riemann conditions to $w_1(z)$, we obtain

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} & (1) \\ \text{and} & \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} & (2) \end{cases}$$

The Cauchy-Riemann conditions for $w_2(z)$ are obtained from those for $w_1(z)$ by replacing v by $-v$. Thus,

$$\begin{cases} \dfrac{\partial u}{\partial x} = -\dfrac{\partial v}{\partial y} & (3) \\ \text{and} & \\ \dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x} & (4) \end{cases}$$

(1) and (3) $\longrightarrow \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = 0$

(2) and (4) $\rightarrow \dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x} = 0$.

From $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y} = 0$, we obtain that $u(x,y) = c_1$ (const.)

Similarly, from $\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} = 0$, we infer that $v(x,y) = c_2$ (constant)

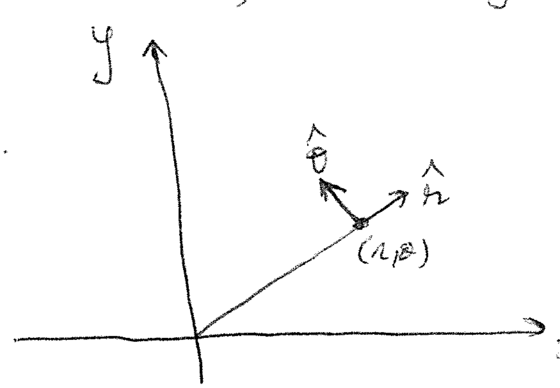
11.2.7 8 marks

$$f(re^{i\theta}) = R(r,\theta)\, e^{i\Theta(r,\theta)}$$

$$= R(r,\theta)\left[\cos\Theta(r,\theta) + i\sin\Theta(r,\theta)\right]$$

$$= \underbrace{R(r,\theta)\cos\Theta(r,\theta)}_{Re(f)} + i\,\underbrace{R(r,\theta)\sin\Theta(r,\theta)}_{Im(f)}$$

The real and imaginary parts of an analytic function (in this case f) must satisfy the Cauchy-Riemann equations for an arbitrary orientation of the (orthogonal) coordinate system. We can take these coordinate directions to be $\hat{r}$ and $\hat{\theta}$.



A change of r by dr at (r,θ) entails a ~~change~~ displacement of dr in the $\hat{r}$ direction, while a ~~disp~~ change of θ by $d\theta$ entails a displacement of $r\,d\theta$ in the $\hat{\theta}$

direction. Thus, the partial derivatives in the $\hat{r}$ and $\hat{\theta}$ directions take the form $\frac{\partial}{\partial r}$ and $\frac{1}{r}\frac{\partial}{\partial \theta}$, respectively

Hence the Cauchy-Riemann equations take the form:

$$\begin{cases} \frac{\partial}{\partial r}\left[R\cos\Theta\right] = \frac{1}{r}\frac{\partial}{\partial \theta}\left[R\sin\Theta\right] & (1) \\ \text{and} & \\ \frac{1}{r}\frac{\partial}{\partial \theta}\left[R\cos\Theta\right] = -\frac{\partial}{\partial r}\left[R\sin\Theta\right] & (2) \end{cases}$$

$$(1) \Rightarrow \cos\Theta\frac{\partial R}{\partial r} - R\sin\Theta\frac{\partial\Theta}{\partial r} = \frac{1}{r}\left[\sin\Theta\frac{\partial R}{\partial \theta} + R\cos\Theta\frac{\partial\Theta}{\partial\theta}\right]$$

Rearranging terms and dividing by $\cos\Theta$, we obtain

$$\left[\frac{\partial R}{\partial r} - \frac{R}{r}\frac{\partial\Theta}{\partial\theta}\right] = \tan\Theta\left[R\frac{\partial\Theta}{\partial r} + \frac{1}{r}\frac{\partial R}{\partial\theta}\right] \quad (*)$$

$$(2) \rightarrow \sin\Theta\frac{\partial R}{\partial r} + R\cos\Theta\frac{\partial\Theta}{\partial r} = -\frac{1}{r}\left[\cos\Theta\frac{\partial R}{\partial\theta} - R\sin\Theta\frac{\partial\Theta}{\partial\theta}\right]$$

Rearranging terms and dividing by $\sin\Theta$, we get:

$$\left[\frac{\partial R}{\partial r} - \frac{R}{r}\frac{\partial\Theta}{\partial\theta}\right] = -\cot\Theta\left[R\frac{\partial\Theta}{\partial r} + \frac{1}{r}\frac{\partial R}{\partial\theta}\right] \quad (**)$$

Multiplying ~~together~~ the left-hand sides of $(*)$ and $(**)$ and setting the result equal to the product of the right-hand sides, we get:

$$\left[\frac{\partial R}{\partial r} - \frac{R}{r}\frac{\partial \Theta}{\partial \theta}\right]^2 = -\left[R\frac{\partial \Theta}{\partial r} + \frac{1}{r}\frac{\partial R}{\partial \theta}\right]^2 \quad (***)$$

The quantities in square brackets are real; so the only way for the left-hand side of (***), which ≥ 0, to be equal to the right-hand side, which is ≤ 0, is for both to equal 0. Thus,

$$\frac{\partial R}{\partial r} - \frac{R}{r}\frac{\partial \Theta}{\partial \theta} = 0 \quad \text{and} \quad R\frac{\partial \Theta}{\partial r} + \frac{1}{r}\frac{\partial R}{\partial \theta} = 0$$

or

$$\begin{cases} \dfrac{\partial R}{\partial r} = \dfrac{R}{r}\dfrac{\partial \Theta}{\partial \theta}; \quad \text{and} \\[2ex] \dfrac{1}{r}\dfrac{\partial R}{\partial \theta} = -R\dfrac{\partial \Theta}{\partial r} \end{cases}$$

11.2.8 6 marks Recall the two Cauchy - Riemann Equations in polar coordinates from Exercise 11.2.7:

$$\begin{cases} \dfrac{\partial R}{\partial r} = \dfrac{R}{r} \dfrac{\partial \Theta}{\partial \theta} & (1) \\ \dfrac{1}{r} \dfrac{\partial R}{\partial \theta} = -R \dfrac{\partial \Theta}{\partial r} & (2) \end{cases}$$

Differentiating (1) w.r.t. θ, we get

$$\frac{\partial^2 R}{\partial \theta \partial r} = \frac{1}{r} \frac{\partial R}{\partial \theta} \frac{\partial \Theta}{\partial \theta} + \frac{R}{r} \frac{\partial^2 \Theta}{\partial \theta^2}$$

$$\rightarrow \frac{1}{r^2} \frac{\partial^2 \Theta}{\partial \theta^2} = \frac{1}{rR} \frac{\partial^2 R}{\partial \theta \partial r} - \frac{1}{r^2 R} \frac{\partial R}{\partial \theta} \frac{\partial \Theta}{\partial \theta}$$

$$\rightarrow \frac{1}{r^2} \frac{\partial^2 \Theta}{\partial \theta^2} = \frac{1}{rR} \frac{\partial^2 R}{\partial \theta \partial r} + \frac{1}{R} \frac{\partial R}{\partial r} \frac{\partial \Theta}{\partial r} \quad (*) \quad \left(\text{using (1) and (2)}\right)$$

Also from (2),

$$\frac{1}{r} \frac{\partial \Theta}{\partial r} = -\frac{1}{r^2 R} \frac{\partial R}{\partial \theta} \qquad (**)$$

Now, differentiating (2) w.r.t. r, we get:

$$-\frac{1}{r^2} \frac{\partial R}{\partial \theta} + \frac{1}{r} \frac{\partial^2 R}{\partial r \partial \theta} = -\frac{\partial R}{\partial r} \frac{\partial \Theta}{\partial r} - R \frac{\partial^2 \Theta}{\partial r^2}$$

from which we get

$$\frac{\partial^2 \Theta}{\partial r^2} = -\frac{1}{R}\frac{\partial R}{\partial r}\frac{\partial \Theta}{\partial r} + \frac{1}{r^2 R}\frac{\partial R}{\partial \theta} - \frac{1}{rR}\frac{\partial^2 R}{\partial r \partial \theta} \quad (***)$$

Adding (*), (**) and (***), we finally get:

$$\frac{\partial^2 \Theta}{\partial r^2} + \frac{1}{r}\frac{\partial \Theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Theta}{\partial \theta^2} = 0$$

11.2.9) 8 marks (2 for each part)

c) $f(z) = \frac{1}{z(z+1)} = \frac{1}{z} - \frac{1}{z+1}$

The functions z and $z+1$ are analytic for all finite z

Thus, $\frac{1}{z}$ is analytic for $z \neq 0$ and $\frac{1}{z+1}$ is analytic for $z \neq -1$. Hence $f(z)$ is analytic for all finite $z \neq 0, -1$.

To check the behavior of $f(z)$ at infinity, we look at $g(w) = f(1/w)$ when $w \to 0$.

$$g(w) = w - \frac{1}{1/w + 1} = w - \frac{w}{1+w} \xrightarrow[w \to 0]{} 0$$

Thus, f is analytic for all $z \neq 0, -1$ (including at ∞).

$$f'(z) = -\frac{1}{z^2} + \frac{1}{(z+1)^2}$$

d) $f(z) = e^{-1/z}$

By the Chain Rule, $f(z)$ is analytic everywhere (in the finite complex plane) except at $z=0$, and

$$f'(z) = e^{-1/z} \cdot \frac{d}{dz}\left(-1/z\right) = \frac{1}{z^2}\, e^{-1/z}$$

At infinity: $g(w) = f(1/w) = e^{-w} \underset{w \to 0}{\longrightarrow} 1$; so f is analytic at infinity.

e) $f(z) = z^2 - 3z + 2$ is analytic for all finite z

with $f'(z) = 2z - 3$

At infinity: $g(w) = f(1/w) = \frac{1}{w^2} - \frac{3}{w} + 2$ is singular at $w=0$; hence $f(z)$ is singular at infinity.

g) $f(z) = \tanh(z) = \dfrac{\sinh z}{\cosh z} \quad \left(= \dfrac{e^z - e^{-z}}{e^z + e^{-z}}\right)$

analytic for all finite z such that $\cosh z \neq 0$

(since $\sinh z$ and $\cosh z$ are analytic everywhere except at ∞).

But $\cosh z = 0 \iff \cos(iz) = 0 \iff iz = (2k+1)\frac{\pi}{2} \quad (k \in \mathbb{Z})$

$$\iff z = -(2k+1)\frac{\pi}{2} i$$
$$\iff z = (2\ell+1)\frac{\pi}{2} i \quad (\ell \in \mathbb{Z})$$

At infinity: $g(w) = f(1/w) = \dfrac{e^{1/w} - e^{-1/w}}{e^{1/w} + e^{-1/w}}$ is singular at $w=0 \to f(z)$ is singular at $z=\infty$.

Thus, $f(z)$ is analytic everywhere except at

$z = \infty$ and $z = (l + \frac{1}{2})\pi i$, $l = 0, \pm 1, \pm 2, \cdots$

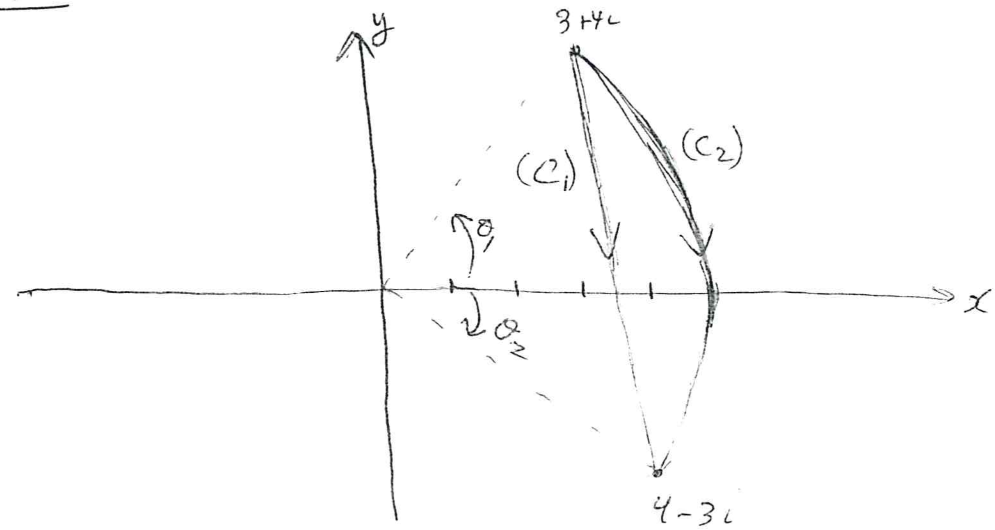
$$f'(z) = \frac{\cosh z \cdot \cosh z - \sinh z \, \sinh z}{\cosh^2 z} \quad \text{[Quotient Rule]}$$

$$= \frac{\cosh^2 z - \sinh^2 z}{\cosh^2 z}$$

$$= \frac{1}{\cosh^2 z} .$$

[redacted]

11.3.3) **8 marks**



a) Evaluate the integral along (C_1).

On (C_1): $y = -7x + 25$ and $dy = -7\,dx$

Thus, $$\int_{C_1} (4z^2 - 3iz)\,dz = \int_{C_1} \left[4(x^2 - y^2) + i8xy + 3y - 3ix\right]\left[dx + i\,dy\right]$$

$$= \int_{x=3}^{4} \left[4x^2 - 4(-7x+25)^2 + i\,8x(-7x+25) + 3(-7x+25) - 3ix\right]\left[1 - 7i\right]dx$$

$$= (1-7i)\int_{x=3}^{4} \left[4x^2 - 4(49x^2 - 350x + 625) + i(-56x^2 + 200x) - 21x + 75 - 3ix\right]dx$$

$$= (1-7i)\int_3^4 \left[-192x^2 + 1379x - 2425 + i(-56x^2+197x)\right]dx$$

$$= \cdots = \frac{76-707i}{3}$$

b) Along (C_2): $z = 5e^{i\theta}$

$$\rightarrow \quad z^2 = 25e^{i2\theta} \quad \text{and} \quad dz = 5ie^{i\theta}d\theta$$

$$\int_{C_2}(4z^2 - 3iz)\,dz = \int_{\theta_1}^{\theta_2}\left(100e^{i2\theta} - 15ie^{i\theta}\right)5ie^{i\theta}d\theta$$

$$= \int_{\theta_1}^{\theta_2}\left[500ie^{i3\theta} + 75e^{i2\theta}\right]d\theta$$

$$= \frac{500}{3}\left(e^{i3\theta_2} - e^{i3\theta_1}\right) - \frac{75}{2}i\left(e^{i2\theta_2} - e^{i2\theta_1}\right) \quad (*)$$

Note that $e^{i\theta_1} = \cos\theta_1 + i\sin\theta_1 = \frac{3}{5} + \frac{4}{5}i$

and $e^{i\theta_2} = \cos\theta_2 + i\sin\theta_2 = \frac{4}{5} - \frac{3}{5}i$

Thus,

$$\left.\begin{aligned} e^{i2\theta_1} &= \left(e^{i\theta_1}\right)^2 = \left(\frac{3}{5}+\frac{4}{5}i\right)^2 = \frac{-7+24i}{25} \\ e^{i2\theta_2} &= \left(e^{i\theta_2}\right)^2 = \left(\frac{4}{5}-\frac{3i}{5}\right)^2 = \frac{7-24i}{25} \\ e^{i3\theta_1} &= \left(e^{i\theta_1}\right)^3 = \left(\frac{3}{5}+\frac{4}{5}i\right)^3 = \frac{-117+44i}{125} \\ \text{and } e^{i3\theta_2} &= \left(e^{i\theta_2}\right)^3 = \left(\frac{4}{5}-\frac{3i}{5}\right)^3 = \frac{-44-117i}{125} \end{aligned}\right\} (**)$$

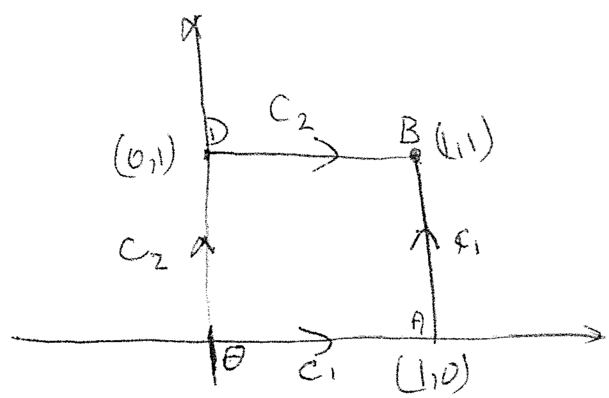
Substituting (**) into (*), we get

$$\int_{C_2} (4z^2 - 3iz)\, dz = \frac{76 - 707i}{3}$$

Note: It is not surprising that we got the same answer in parts a) and b) as $f(z) = 4z^2 - 3iz$ is an ~~analy~~ entire function, so its integral between any two points is independent of the path. Of course the easiest way to compute the integral is to use an antiderivative

$$\int_{3+4i}^{4-3i} (4z^2 - 3iz)\, dz = \left(\frac{4}{3} z^3 - i\, \frac{3z^2}{2}\right)\Bigg|_{3+4i}^{4-3i} = \frac{76 - 707i}{3}$$

11.3.6 <u>5 marks</u>



$$\int_{C_1} z^* dz = \int_{C_1} (x - iy)(dx + i\,dy) = \int_{OA} + \int_{AB}$$

on OA: $y = 0$ and $dy = 0 \rightarrow \int_{OA} = \int_{x=0}^{1} x\, dx = \frac{1}{2}$

on AB: $x = 1$ and $dx = 0$

$$\rightarrow \int_{AB} = \int_{y=0}^{1} (1 - iy)\, i\, dy = \int_{y=0}^{1} (y + i)\, dy = \frac{1}{2} + i$$

Hence $\displaystyle\int_{C_1} z^* dz = \frac{1}{2} + \left(\frac{1}{2} + i\right) = 1 + i$

$$\int_{C_2} z^* dz = \int_{OD} (x - iy)(dx + i dy) + \int_{DB} (x - iy)(dx + i dy)$$

on OD: $x = 0$ and $dx = 0$

$$\rightarrow \int_{OD} = \int_{y=0}^{1} y\, dy = \frac{1}{2}$$

on DB: $y = 1$ and $dy = 0$

$$\rightarrow \int_{DB} = \int_{x=0}^{1} (x - i)\, dx = \frac{1}{2} - i$$

Hence $\displaystyle\int_{C_2} z^* dz = \frac{1}{2} + \left(\frac{1}{2} - i\right) = 1 - i \neq \int_{C_1} z^* dz = 1 + i$

11.3.7 2 marks

$$\oint_C \frac{dz}{z^2 + z} = \oint_C \left(\frac{1}{z} - \frac{1}{z+1}\right) dz$$

$$= \oint_C \frac{dz}{z} - \oint_C \frac{dz}{z+1}$$

$$= 2\pi i - 2\pi i = 0$$

[0 and -1 are both inside (C) for $R > 1$]