PHYS 3496
Homework # 4
28 marks

8.2.2 $\quad y'' - 2xy' + 2\alpha y = 0 \quad (*)$

4 marks

$$w(x) = \frac{1}{P_0} e^{\int \frac{P_1}{P_0} dx} \quad ; \text{ here } P_0(x) = 1, \quad P_1(x) = -2x$$

$$\Rightarrow w(x) = \frac{1}{1} e^{\int(-2x)dx} = e^{-x^2}$$

Multiplying by e^{-x^2}, we get

$$e^{-x^2} y'' - 2x e^{-x^2} y' + 2\alpha e^{-x^2} y = 0 \quad (**)$$, which

is self-adjoint since

$\frac{d}{dx}\left[e^{-x^2}\right] = -2x e^{-x^2}$ and hence $(**)$ can

be rewritten in the form

$$\frac{d}{dx}\left[e^{-x^2} y'\right] + 2\alpha e^{-x^2} y = 0.$$

$$\langle v . u \rangle = \int_{-\infty}^{+\infty} v^*(x) w(x) u(x) dx = \int_{-\infty}^{+\infty} v^*(x) u(x) e^{-x^2} dx$$

Alternatively, let $\psi = e^{-x^2/2} y$. Then

$y = e^{x^2/2} \psi$. Thus,

$$y' = e^{x^2/2} \psi' + x e^{x^2/2} \psi \quad \text{and}$$

$$y'' = e^{x^2/2} \psi'' + 2x e^{x^2/2} \psi' + [x^2+1] e^{x^2/2} \psi$$

Substituting into (*), we get

$$e^{x^2/2}\left[\psi'' + \cancel{2x\psi'} + (x^2+1)\psi\right]$$

$$-2x\,e^{x^2/2}\left[\cancel{\psi'} + x\psi\right] + 2\alpha e^{x^2/2}\psi = 0$$

$\Rightarrow \psi'' + (2\alpha + 1 - x^2)\psi = 0$, which is self-adjoint

Note that $\displaystyle\int_{-\infty}^{+\infty} \psi_m(x)\,\psi_n(x)\,dx = \int_{-\infty}^{+\infty} y_m(x)\,y_n(x)\,e^{-x^2}dx$

weight function $=1$ weight function: e^{-x^2}

8.2.3 $\quad (1-x^2)y'' - xy' + n^2 y = 0 \quad (*) \quad -1 \le x \le +1$

Arfken's

$$w(x) = \frac{1}{1-x^2}\, e^{\int \frac{-x}{1-x^2}dx}$$

$$= \frac{1}{1-x^2}\, e^{\frac{1}{2}\ln(1-x^2)} = \frac{1}{1-x^2}\, e^{\ln\sqrt{1-x^2}}$$

$$= \frac{\sqrt{1-x^2}}{1-x^2} = \frac{1}{\sqrt{1-x^2}}$$

Multiplying (*) by $w(x) = (1-x^2)^{-1/2} = \frac{1}{\sqrt{1-x^2}}$, we get

$$(1-x^2)^{1/2}\,y'' - \frac{x}{\sqrt{1-x^2}}\,y' + \frac{n^2}{\sqrt{1-x^2}}\,y = 0 \quad \text{or}$$

$$\frac{d}{dx}\left[\sqrt{1-x^2}\,y'\right] + \frac{n^2}{\sqrt{1-x^2}}\,y = 0 \quad \text{which is}$$

self-adjoint.

The corresponding inner product (for the space of solutions) should include the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$: $\langle v, u \rangle = \int_{-1}^{+1} v^*(x)\, u(x) \frac{1}{\sqrt{1-x^2}}\, dx$

[based on what we did in class.]

8.2.4 We want $\left[v^* \bar{P}_0(x)\, u' - (v^*)' \bar{P}_0(x)\, u \right]_a^b = 0$

8 marks

where u and v are solutions of the ODE and $\bar{P}_0(x) = P_0(x)\, w(x)$. Here u, v are polynomials.

a) Legendre Equation: $P_0(x) = (1-x^2)$ and $w(x) = 1$

$$\left[v^* \bar{P}_0(x)\, u' - (v^*)' \bar{P}_0(x)\, u \right]_{-1}^{+1}$$

$$= \left[v^* (1-x^2)\, u' - (v^*)' (1-x^2)\, u \right]_{-1}^{+1} = 0$$

since $(1-x^2) = 0$ for $x = \pm 1$ and $u, v^*, u', (v^*)'$ are finite (they being polynomials) at ± 1.

(b) Chebyshev equation: $P_0(x) = (1-x^2)$, $w(x) = \frac{1}{\sqrt{1-x^2}}$

$\rightarrow \bar{P}_0(x) = \sqrt{1-x^2} \rightarrow \bar{P}_0(\pm 1) = 0$.

$\rightarrow \left[v^* \bar{P}_0(x)\, u' - (v^*)' \bar{P}_0(x)\, u \right]_{-1}^{+1} = 0$

just as in part (a) above.

(c) Hermite Equation : $P_0(x) = 1$, $w(x) = e^{-x^2}$

$\rightarrow \bar{P}_0(x) = e^{-x^2}$. Thus,

$$\bar{P}_0(x) \cdot [\text{any polynomial}] \xrightarrow[x \to \pm\infty]{} 0; \text{ and}$$

hence $\left[v^* \bar{P}_0(x) u' - (v^*)' \bar{P}_0(x) u \right]_{-\infty}^{+\infty} = 0$

(d) Laguerre Equation : $P_0(x) = x$, $w(x) = e^{-x}$

$\rightarrow \bar{P}_0(x) = x e^{-x}$. Thus,

$$\left(\bar{P}_0(x) \cdot [\text{any polynomial}] \right)(x=0) = 0$$

and $\lim_{x \to +\infty} \left[\bar{P}_0(x) \cdot (\text{any polynomial}) \right] = 0$

Hence $\left[v^* \bar{P}_0(x) u' - (v^*)' \bar{P}_0(x) u \right]_0^{\infty} = 0$.

8.2.5 H hermitian operator

3 marks

$H u_1 = \lambda_1 u_1$ and $H u_2 = \lambda_2 u_2$ with $\lambda_1 \neq \lambda_2$

We want to show that u_1 and u_2 are linearly independent. Suppose not. Then $u_2 = K u_1$ for some $K \neq 0$. Thus,

$H u_2 = H(K u_1) = K\, H u_1$; and hence

$$\lambda_2 u_2 = k\lambda_1 u_1 = \lambda_1 (k u_1) = \lambda_1 u_2 .$$

It follows that $(\lambda_2 - \lambda_1) u_2 = 0$ and hence

$\lambda_2 = \lambda_1$, a contradiction. Therefore,

u_1 and u_2 must be linearly independent.

8.2.6 a) $\displaystyle\int_{-1}^{+1} P_1(x)\, Q_0(x)\, dx = \int_{-1}^{+1} \frac{x}{2} \ln\left(\frac{1+x}{1-x}\right) dx$

5 marks

$$= \left(\frac{x^2-1}{4}\right) \ln\left(\frac{1+x}{1-x}\right) \Bigg|_{-1}^{+1} - \int_{-1}^{+1} \frac{x^2-1}{4} \frac{d}{dx} \ln\left(\frac{1+x}{1-x}\right) dx$$

(integration by parts)

Since $\lim_{x\to 1} (x-1)\ln(1-x) = \lim_{x\to -1} (x+1)\ln(1+x) = 0$,

we have $\displaystyle\frac{x^2-1}{4} \ln\left(\frac{1+x}{1-x}\right)\Bigg|_{-1}^{+1} = 0$. Thus,

$$\int_{-1}^{+1} P_1(x)\, Q_0(x)\, dx = -\int_{-1}^{+1} \frac{x^2-1}{4} \left[\frac{1}{1+x} + \frac{1}{1-x}\right] dx$$

$$= \int_{-1}^{+1} \frac{1-x^2}{4} \, \frac{1-\cancel{x} + 1+\cancel{x}}{1-x^2}\, dx = \frac{1}{2}\int_{-1}^{+1} dx = 1 \neq 0$$

(b) $Q_0(x)$ is singular at $x = \pm 1$; so the boundary condition $\left[P_1^*(x)\,(1-x^2)\, Q_0'(x) - P_1^{*\prime}(x)\,(1-x^2)\, Q_0(x) \right]_{-1}^{+1} = 0$

does <u>not</u> hold. In fact:

$$\left[P_1^*(x)(1-x^2)\,Q_0'(x) - (P_1^*)'(x)\,(1-x^2)\,Q_0(x)\right]_{-1}^{+1}$$

$$=\left\{x\,(1-x^2)\,\frac{1}{2}\left(\frac{1}{1+x}+\frac{1}{1-x}\right) - \cancelto{0}{1\,(1-x^2)\,\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)}\right]_{-1}^{+1}$$

$$=\left[x\,(1-x^2)\,\frac{1}{2}\,\frac{1-x+1+x}{1-x^2}\right]_{-1}^{+1} = \left[x\right]_{-1}^{+1} = 2 \neq 0.$$

8.2.9 Assume $\psi_1, \dots, \psi_n$ are linearly dependent

4 marks

then ~~If~~ there exists j, $1 < j \leq n$ such that

$\{\psi_1, \dots, \psi_{j-1}\}$ are linearly independent and

$\psi_j = \sum_{i=1}^{j-1} \alpha_i \psi_i$ ~~Then~~, with $\alpha_i \neq 0$ for some $i \in \{1, \dots, j-1\}$

Then, $A\psi_j = \lambda_j \psi_j = A\left(\sum_{i=1}^{j-1} \alpha_i \psi_i\right)$; and hence

$\lambda_j \psi_j = \sum_{i=1}^{j-1} \alpha_i A\psi_i$. Thus,

$\lambda_j \sum_{i=1}^{j-1} \alpha_i \psi_i = \sum_{i=1}^{j-1} \alpha_i \lambda_i \psi_i$ and hence

$$\sum_{i=1}^{j-1} (\lambda_j - \lambda_i)\alpha_i \psi_i = 0$$

Since $\{\psi_1, \dots, \psi_{j-1}\}$ are linearly independent, it follows that $(\lambda_j - \lambda_i)\alpha_i = 0$ for all $i = 1, \dots, j-1$.

~~$\rightarrow \alpha_i = 0$ for all $i = 1, 2, \dots, j-1$ (since $\lambda_j - \lambda_i \neq 0$)~~

$\rightarrow \psi$

$\rightarrow \lambda_j = \lambda_i$ for those i in $\{1, \dots, j-1\}$ for which $\alpha_i \neq 0$. That is a ~~contraction~~ contradiction to the fact that $\lambda_j \neq \lambda_i$ for all $i \in \{1, \dots, j-1\}$.

Therefore, $\psi_1, \dots, \psi_n$ must be linearly independent.

Remark: Exercise 8.2.5 is a special case of Exercise 8.2.9 with $n = 2$. The assumption ~~that op~~ the operator is Hermitian was not needed in Exercise 8.2.5!