Not graded

~~4 marks~~

PHYS 3496
Homework #3
~~5~~3 marks

3.10.4 a) using Equation 3.141, with $V_1 = 1$ and $V_2 = V_3 = 0$,

we get $\vec{\nabla} \cdot \hat{e}_1 = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial q_1}(h_2 h_3)$

b) Using 3.143, with $B_1 = 1$ and $B_2 = B_3 = 0$, we get

$$\vec{\nabla} \times \hat{e}_1 = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 & 0 & 0 \end{vmatrix}$$

$$= \frac{1}{h_1 h_2 h_3} \left[h_1 \hat{e}_1 (0 - 0) - h_2 \hat{e}_2 \left(0 - \frac{\partial h_1}{\partial q_3}\right) + h_3 \hat{e}_3 \left(0 - \frac{\partial h_1}{\partial q_2}\right) \right]$$

$$= \frac{1}{h_1 h_2 h_3} \left[h_2 \frac{\partial h_1}{\partial q_3} \hat{e}_2 - h_3 \frac{\partial h_1}{\partial q_2} \hat{e}_3 \right]$$

$$= \frac{1}{h_1} \left[\frac{1}{h_3} \frac{\partial h_1}{\partial q_3} \hat{e}_2 - \frac{1}{h_2} \frac{\partial h_1}{\partial q_2} \hat{e}_3 \right]$$

2 marks

3.10.8 We have, by Exercise 3.10.6, that

$\hat{e}_\rho = \hat{e}_x \cos\varphi + \hat{e}_y \sin\varphi$ and
$\hat{e}_\varphi = -\hat{e}_x \sin\varphi + \hat{e}_y \cos\varphi$. Then

$\frac{\partial \hat{e}_\rho}{\partial \varphi} = -\hat{e}_x \sin\varphi + \hat{e}_y \cos\varphi$ ($\hat{e}_x$ and $\hat{e}_y$ are constant vectors).

$= \hat{e}_\varphi$

$$\frac{\partial \hat{e}_\varphi}{\partial \varphi} = -\hat{e}_x \cos\varphi - \hat{e}_y \sin\varphi = -\hat{e}_\rho .$$

Not graded ~~Answer~~

3.10.9 $$\vec{\nabla}\cdot\vec{V} = \left(\hat{e}_\rho \frac{\partial}{\partial \rho} + \hat{e}_\varphi \frac{1}{\rho}\frac{\partial}{\partial \varphi} + \hat{e}_z \frac{\partial}{\partial z}\right)\cdot\left(V_\rho \hat{e}_\rho + V_\varphi \hat{e}_\varphi + V_z \hat{e}_z\right)$$

$$= \hat{e}_\rho\cdot\hat{e}_\rho \frac{\partial V_\rho}{\partial \rho} + V_\rho\, \hat{e}_\rho\cdot \cancel{\frac{\partial \hat{e}_\rho}{\partial \rho}}^{\,0} + V_\varphi\, \hat{e}_\rho \cdot \cancel{\frac{\partial \hat{e}_\varphi}{\partial \rho}}^{\,0} + V_z\, \hat{e}_\rho\cdot \cancel{\frac{\partial \hat{e}_z}{\partial \rho}}^{\,0}$$

$$+ \frac{V_\rho}{\rho}\hat{e}_\varphi\cdot \frac{\partial \hat{e}_\rho}{\partial \varphi} + \frac{1}{\rho}\frac{\partial V_\varphi}{\partial \rho}\hat{e}_\varphi\cdot\hat{e}_\varphi + \frac{V_\varphi}{\rho}\hat{e}_\varphi\cdot\frac{\partial \hat{e}_\varphi}{\partial \varphi} + \frac{V_z}{\rho}\hat{e}_\varphi\cdot \cancel{\frac{\partial \hat{e}_z}{\partial \varphi}}^{\,0}$$

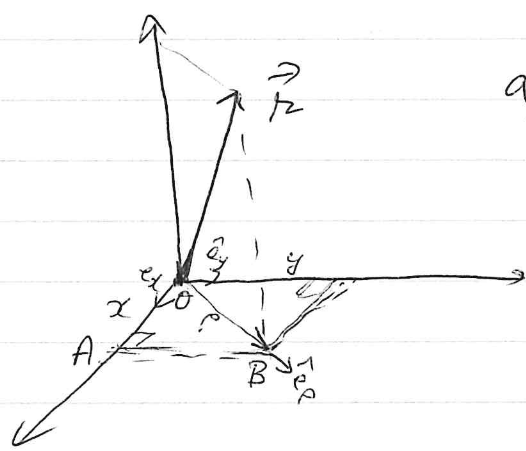
$$+ V_\rho\, \hat{e}_z\cdot \cancel{\frac{\partial \hat{e}_\rho}{\partial z}}^{\,0} + V_\varphi\, \hat{e}_z\cdot \cancel{\frac{\partial \hat{e}_\varphi}{\partial z}}^{\,0} + \frac{\partial V_z}{\partial z}\hat{e}_z\cdot\hat{e}_z + V_z\, \hat{e}_z\cdot \cancel{\frac{\partial \hat{e}_z}{\partial z}}^{\,0}$$

$$= \frac{\partial V_\rho}{\partial \rho} + \frac{V_\rho}{\rho}\hat{e}_\varphi\cdot\hat{e}_\varphi + \frac{1}{\rho}\frac{\partial V_\varphi}{\partial \rho} + \frac{V_\varphi}{\rho}\cancel{\hat{e}_\varphi\cdot(-\hat{e}_\rho)}^{\,0} + \frac{\partial V_z}{\partial z}$$

$$= \frac{\partial V_\rho}{\partial \rho} + \frac{1}{\rho}V_\rho + \frac{1}{\rho}\frac{\partial V_\varphi}{\partial \rho} + \frac{\partial V_z}{\partial z}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho V_\rho) + \frac{1}{\rho}\frac{\partial V_\varphi}{\partial \rho} + \frac{\partial V_z}{\partial z}$$, which is (Eq. 3.148)

3.10.10 ~~Answer~~ Not graded



a) $\vec{r} = (x\,\hat{e}_x + y\,\hat{e}_y) + z\,\hat{e}_z$
$= (\vec{OA} + \vec{AB}) + z\,\hat{e}_z$
$= \vec{OB} + z\,\hat{e}_z$
$= \rho\,\hat{e}_\rho + z\,\hat{e}_z \quad (*)$

b) Using $(*)$ and Equation (3.148), with $V_\rho = \rho$, $V_\varphi = 0$ and $V_z = z$, we get

$$\vec{\nabla}\cdot\vec{r} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^2) + \frac{\partial(z)}{\partial z} = \frac{2\rho}{\rho} + 1 = 3$$

Also, using Equation (3.150), with $V_\rho = \rho$, $V_\varphi = 0$ and $V_z = z$,

we get $$\vec{\nabla}\times\vec{r} = \frac{1}{\rho}\begin{vmatrix} \hat{e}_\rho & \rho\hat{e}_\varphi & \hat{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ \rho & 0 & z \end{vmatrix} = \vec{0}$$

Not graded

3.10.11 a) under a parity operation

$$x \longrightarrow -x$$
$$y \longrightarrow -y$$
$$z \longrightarrow -z$$

~~Since~~ $\rho = \sqrt{x^2+y^2} \longrightarrow \sqrt{(-x)^2+(-y)^2} = \sqrt{x^2+y^2} = \rho$

Since $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$,

for both x and y to change sign (ρ remains unchanged)

$$\left.\begin{matrix}\cos\varphi \longrightarrow -\cos\varphi \text{ and} \\ \sin\varphi \longrightarrow -\sin\varphi\end{matrix}\right\} \Rightarrow \varphi \text{ changes by } \pi \text{; so}$$

$$\varphi \longrightarrow \varphi \pm \pi .$$

Finally $z \longrightarrow -z$ (as in cartesian coordinate system.)

b) $\hat{e}_x, \hat{e}_y, \hat{e}_z$ remain constant. In particular

$\hat{e}_z \longrightarrow \hat{e}_z$ (even parity)

since $\cos\varphi \to -\cos\varphi$ and
$\sin\varphi \to -\sin\varphi$ under a parity operation,

it follows that

$$\hat{e}_\rho = \hat{e}_x \cos\varphi + \hat{e}_y \sin\varphi \to -\hat{e}_x \cos\varphi - \hat{e}_y \sin\varphi = -\hat{e}_\rho$$

and

$$\hat{e}_\varphi = -\hat{e}_x \sin\varphi + \hat{e}_y \cos\varphi \to \hat{e}_x \sin\varphi - \hat{e}_y \cos\varphi = -\hat{e}_\varphi$$

Therefore, $\hat{e}_\rho$ and $\hat{e}_\varphi$ have odd parity while $\hat{e}_z$ has even parity.

3 marks

3.10.12 $\quad \vec{\omega} = \omega \hat{e}_z \qquad \vec{r} = \rho \hat{e}_\rho + z \hat{e}_z$

a) $\vec{v} = \vec{\omega} \times \vec{r} = \omega \hat{e}_z \times (\rho \hat{e}_\rho + z \hat{e}_z)$
$= \omega\rho\, \hat{e}_z \times \hat{e}_\rho = \omega\rho\, \hat{e}_\varphi$.

b) $$\vec{\nabla} \times \vec{v} = \frac{1}{\rho} \begin{vmatrix} \hat{e}_\rho & \rho \hat{e}_\varphi & \hat{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ 0 & \omega\rho^2 & 0 \end{vmatrix}$$

$\leftarrow$ Equation 3.150 with $V_\rho = V_z = 0$, $V_\varphi = \omega\rho$

$$= \frac{1}{\rho}\left[\hat{e}_\rho (0-0) - \rho \hat{e}_\varphi (0-0) + \hat{e}_z (2\omega\rho - 0)\right]$$

$$= 2\omega \hat{e}_z = 2\vec{\omega}.$$

4 marks

3.10.13 Using the results of Exercise 3.10.8, we have that $\frac{d\hat{e}_\rho}{dt} = \frac{\partial \hat{e}_\rho}{\partial \varphi}\dot{\varphi} = \hat{e}_\varphi \dot{\varphi}$ and

$$\frac{d\hat{e}_\varphi}{dt} = \frac{\partial \hat{e}_\varphi}{\partial \varphi}\dot{\varphi} = -\hat{e}_\rho \dot{\varphi}.$$ Thus,

$$\vec{v} = \frac{d}{dt}\vec{r} = \frac{d}{dt}(\rho\hat{e}_\rho + z\hat{e}_z)$$

$$= \dot{\rho}\hat{e}_\rho + \rho\frac{d\hat{e}_\rho}{dt} + \dot{z}\hat{e}_z$$

$$= \dot{\rho}\hat{e}_\rho + \rho\dot{\varphi}\hat{e}_\varphi + \dot{z}\hat{e}_z$$

$\rightarrow$ $v_\rho = \dot{\rho}$, $v_\varphi = \rho\dot{\varphi}$ and $v_z = \dot{z}$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left[\dot{\rho}\hat{e}_\rho + \rho\dot{\varphi}\hat{e}_\varphi + \dot{z}\hat{e}_z\right]$$

$$= \ddot{\rho}\hat{e}_\rho + \dot{\rho}\frac{d\hat{e}_\rho}{dt} + \dot{\rho}\dot{\varphi}\hat{e}_\varphi + \rho\ddot{\varphi}\hat{e}_\varphi + \rho\dot{\varphi}\frac{d\hat{e}_\varphi}{dt} + \ddot{z}\hat{e}_z$$

$$= \ddot{\rho}\hat{e}_\rho + \dot{\rho}\dot{\varphi}\hat{e}_\varphi + \dot{\rho}\dot{\varphi}\hat{e}_\varphi + \rho\ddot{\varphi}\hat{e}_\varphi - \rho\dot{\varphi}^2\hat{e}_\rho + \ddot{z}\hat{e}_z$$

$$= (\ddot{\rho} - \rho\dot{\varphi}^2)\hat{e}_\rho + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\hat{e}_\varphi + \ddot{z}\hat{e}_z$$

$\rightarrow$ $a_\rho = \ddot{\rho} - \rho\dot{\varphi}^2$, $a_\varphi = \rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi}$ and $a_z = \ddot{z}$.

2 marks

3.10.14 $\vec{V} = V_\rho(\rho,\varphi)\,\hat{e}_\rho + V_\varphi(\rho,\varphi)\,\hat{e}_\varphi$

$$\vec{\nabla}\times\vec{V} = \frac{1}{\rho}\begin{vmatrix} \hat{e}_\rho & \rho\hat{e}_\varphi & \hat{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ V_\rho(\rho,\varphi) & \rho V_\varphi(\rho,\varphi) & 0 \end{vmatrix} = \frac{1}{\rho}\Big[\hat{e}_\rho(0-0) - \rho\hat{e}_\varphi(0-0) + \hat{e}_z\Big(\frac{\partial}{\partial\rho}(\rho V_\varphi) - \frac{\partial V_\rho}{\partial\varphi}\Big)\Big]$$

$$= \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho V_\varphi) - \frac{\partial V_\rho}{\partial\varphi}\right]\hat{e}_z.$$

5 marks

3.10.16 a) $\vec{F} = -\hat{e}_x \dfrac{y}{x^2+y^2} + \hat{e}_y \dfrac{x}{x^2+y^2}$

$$= \frac{-\hat{e}_x \rho \sin\varphi + \hat{e}_y \rho \cos\varphi}{\rho^2}$$

$$= \frac{-\hat{e}_x \sin\varphi + \hat{e}_y \cos\varphi}{\rho} = \frac{\hat{e}_\varphi}{\rho} \quad \text{(see Exercise 3.10.6)}$$

b) $$\vec{\nabla} \times \vec{F} = \frac{1}{\rho}\begin{vmatrix} \hat{e}_\rho & \rho\hat{e}_\varphi & \hat{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ 0 & 1 & 0 \end{vmatrix} = \vec{0} \quad \text{for } \rho \neq 0 \ (x^2+y^2 \neq 0)$$

c) $$\oint_C \vec{F}\cdot d\vec{r} = \oint_C F_\rho \cancelto{0}{d\rho} + F_\varphi \rho\, d\varphi + \cancelto{0}{F_z\, dz} = \oint_C 1\, d\varphi = \int_0^{2\pi} d\varphi = 2\pi$$

d) That $\oint_C \vec{F}\cdot d\vec{r} \neq 0$ does not contradict Stokes' Theorem since $\vec{F}$ and its partial derivatives are not defined and continuous everywhere inside (C). So Stokes' Theorem does not hold in this case and we can <u>not</u> write

$$\oint_C \vec{F}\cdot d\vec{r} = \int_S \vec{\nabla}\times\vec{F}\cdot\hat{n}\, d\sigma$$ where S is the surface enclosed by (C).

2 marks

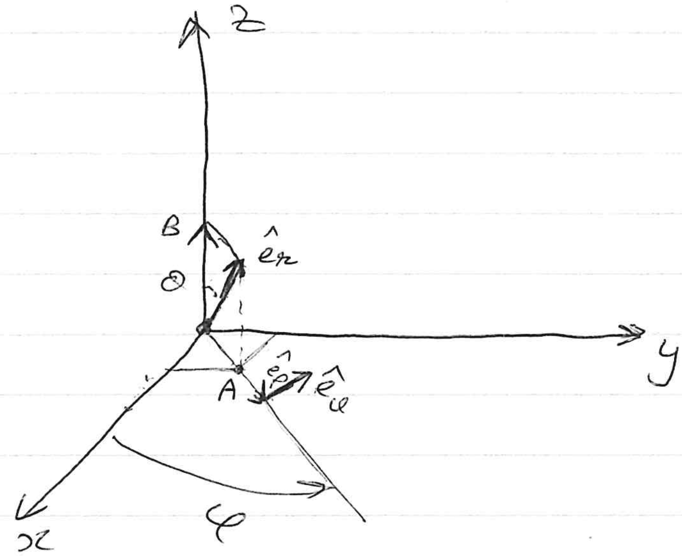
3.10.17 $\vec{B} = B_\varphi(\rho)\,\hat{e}_\varphi$ $\hat{e}_\varphi$

$$(\vec{B}\cdot\vec{\nabla})\vec{B} = \left[B_\varphi(\rho)\hat{e}_\varphi \cdot \left(\hat{e}_\rho\frac{\partial}{\partial\rho} + \hat{e}_\varphi \frac{1}{\rho}\frac{\partial}{\partial\varphi} + \hat{e}_z\frac{\partial}{\partial z}\right)\right]\left(B_\varphi(\rho)\,\hat{e}_\varphi\right)$$

$$= \frac{B_\varphi(\rho)}{\rho}\frac{\partial}{\partial\varphi}\left(B_\varphi(\rho)\,\hat{e}_\varphi\right) = \frac{B_\varphi^2(\rho)}{\rho}\frac{\partial}{\partial\varphi}(\hat{e}_\varphi)$$

$$= -\frac{B_\varphi^2(\rho)}{\rho}\hat{e}_\rho \quad \text{(using Exercise 3.10.8)}$$

3.10.18 ~~4 marks~~ Not graded (I did in class)



$\hat{e}_r = \vec{OA} + \vec{OB}$

$= |\hat{e}_r| \sin\theta \, \hat{e}_\rho + |\hat{e}_r| \cos\theta \, \hat{e}_z$

$= \sin\theta \, \hat{e}_\rho + \cos\theta \, \hat{e}_z$

$= \sin\theta \, [\cos\varphi \, \hat{e}_x + \sin\varphi \, \hat{e}_y] + \cos\theta \, \hat{e}_z$ (Exercise 3.10.6)

$\hat{e}_r = \sin\theta \cos\varphi \, \hat{e}_x + \sin\theta \sin\varphi \, \hat{e}_y + \cos\theta \, \hat{e}_z$ (1)

$\hat{e}_\theta$ is obtained from $\hat{e}_r$ by replacing θ by $\theta + \pi/2$:

$\hat{e}_\theta = \sin(\theta + \pi/2) \cos\varphi \, \hat{e}_x + \sin(\theta + \pi/2) \sin\varphi \, \hat{e}_y + \cos(\theta + \pi/2) \hat{e}_z$

$\rightarrow \hat{e}_\theta = \cos\theta \cos\varphi \, \hat{e}_x + \cos\theta \sin\varphi \, \hat{e}_y - \sin\theta \, \hat{e}_z$ (2)

$\hat{e}_\varphi = (\hat{e}_\varphi \cdot \hat{e}_x) \, \hat{e}_x + (\hat{e}_\varphi \cdot \hat{e}_y) \, \hat{e}_y + (\hat{e}_\varphi \cdot \hat{e}_z) \, \hat{e}_z$

$= \cos(\varphi + \pi/2) \, \hat{e}_x + \cos\varphi \, \hat{e}_y + 0 \, \hat{e}_z$

$\hat{e}_\varphi = -\sin\varphi \, \hat{e}_x + \cos\varphi \, \hat{e}_y$ (3)

5 marks

3.10.19 From Exercise 3.10.18, we have that

$$\hat{e}_r = \sin\theta\cos\varphi\,\hat{e}_x + \sin\theta\sin\varphi\,\hat{e}_y + \cos\theta\,\hat{e}_z \quad (1)$$

$$\hat{e}_\theta = \cos\theta\cos\varphi\,\hat{e}_x + \cos\theta\sin\varphi\,\hat{e}_y - \sin\theta\,\hat{e}_z \quad (2)$$

$$\hat{e}_\varphi = -\sin\varphi\,\hat{e}_x + \cos\varphi\,\hat{e}_y \quad (3)$$

Using (1), (2), (3), we get:

$$\hat{e}_r\sin\theta\cos\varphi + \hat{e}_\theta\cos\theta\cos\varphi - \hat{e}_\varphi\sin\varphi$$

$$= \sin^2\theta\cos^2\varphi\,\hat{e}_x + \sin^2\theta\sin\varphi\cos\varphi\,\hat{e}_y + \cancel{\sin\theta\cos\theta\cos\varphi\,\hat{e}_z}$$

$$+ \cos^2\theta\cos^2\varphi\,\hat{e}_x + \cos^2\theta\cos\varphi\sin\varphi\,\hat{e}_y - \cancel{\sin\theta\cos\theta\cos\varphi\,\hat{e}_z}$$

$$+ \sin^2\varphi\,\hat{e}_x - \sin\varphi\cos\varphi\,\hat{e}_y$$

$$= \underbrace{\left[\cos^2\varphi\underbrace{(\sin^2\theta + \cos^2\theta)}_{1} + \sin^2\varphi\right]}_{1}\hat{e}_x + \left[\cos\varphi\sin\varphi\overbrace{(\sin^2\theta + \cos^2\theta)}^{1} - \sin\varphi\cos\varphi\right]\hat{e}_y$$

$$= \hat{e}_x.$$ Thus,

$$\boxed{\hat{e}_x = \sin\theta\cos\varphi\,\hat{e}_r + \cos\theta\cos\varphi\,\hat{e}_\theta - \sin\varphi\,\hat{e}_\varphi}$$

Again from (1), (2) and (3), we get:

$$\sin\theta\sin\varphi\,\hat{e}_r + \cos\theta\sin\varphi\,\hat{e}_\theta + \cos\varphi\,\hat{e}_\varphi$$

$$= \sin^2\theta\sin\varphi\cos\varphi\,\hat{e}_x + \sin^2\theta\sin^2\varphi\,\hat{e}_y + \cancel{\sin\theta\cos\theta\sin\varphi\,\hat{e}_z}$$
$$+ \cos^2\theta\sin\varphi\cos\varphi\,\hat{e}_x + \cos^2\theta\sin^2\varphi\,\hat{e}_y - \cancel{\sin\theta\cos\theta\sin\varphi\,\hat{e}_z}$$
$$- \sin\varphi\cos\varphi\,\hat{e}_x + \cos^2\varphi\,\hat{e}_y$$

$$= \left[(\sin^2\theta + \cos^2\theta) - 1\right] \sin\varphi \cos\varphi \, \hat{e}_x$$

$$+ \left[(\sin^2\theta + \cos^2\theta)\sin^2\varphi + \cos^2\varphi\right] \hat{e}_y = \hat{e}_y.$$ Thus,

$$\boxed{\hat{e}_y = \sin\theta \sin\varphi \, \hat{e}_r + \cos\theta \sin\varphi \, \hat{e}_\theta + \cos\varphi \, \hat{e}_\varphi}$$

Finally, from (1) and (2), we get:

$$\cos\theta \, \hat{e}_r - \sin\theta \, \hat{e}_\theta =$$

$$\cancel{\cos\theta \sin\theta \cos\varphi \, \hat{e}_x} + \cancel{\cos\theta \sin\theta \sin\varphi \, \hat{e}_y} + \cos^2\theta \, \hat{e}_z$$
$$- \cancel{\sin\theta \cos\theta \cos\varphi \, \hat{e}_x} - \cancel{\sin\theta \cos\theta \sin\varphi \, \hat{e}_y} + \sin^2\theta \, \hat{e}_z$$

$$= (\cos^2\theta + \sin^2\theta)\hat{e}_z = \hat{e}_z.$$ Thus,

$$\boxed{\hat{e}_z = \cos\theta \, \hat{e}_r - \sin\theta \, \hat{e}_\theta}$$

4 marks

3.10.23) $\vec{\omega} = \omega \hat{e}_z = \omega\cos\theta \, \hat{e}_r - \omega \sin\theta \, \hat{e}_\theta$ (Exercise 3.10.19)

a) $\vec{v} = \vec{\omega} \times \vec{r} = (\omega\cos\theta \, \hat{e}_r - \omega\sin\theta \, \hat{e}_\theta) \times (r \, \hat{e}_r)$

$$= -\omega r \sin\theta \, \hat{e}_\theta \times \hat{e}_r = -\omega r \sin\theta \, (-\hat{e}_\varphi)$$
$$= \omega r \sin\theta \, \hat{e}_\varphi$$ (in agreement with 3.10.12 a) since $\rho = r\sin\theta$)

b) $$\vec{\nabla} \times \vec{v} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & r\,\hat{e}_\theta & r\sin\theta \, \hat{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ 0 & 0 & \omega r^2 \sin^2\theta \end{vmatrix}$$ (Eqn 3.159)

$$\vec{\nabla}\times\vec{v} = \frac{1}{r^2\sin\theta}\left[\hat{e}_r \frac{\partial}{\partial\theta}\left(\omega r^2 \sin^2\theta\right) - r\,\hat{e}_\theta \frac{\partial}{\partial r}\left(\omega r^2 \sin^2\theta\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[2\omega r^2 \sin\theta\cos\theta\,\hat{e}_r - r\left(2\omega r\sin^2\theta\right)\hat{e}_\theta\right]$$

$$= \frac{1}{r^2\sin\theta}\left[2\omega r^2 \sin\theta\cos\theta\,\hat{e}_r - 2\omega r^2\sin^2\theta\,\hat{e}_\theta\right]$$

$$= 2\omega\cos\theta\,\hat{e}_r - 2\omega\sin\theta\,\hat{e}_\theta$$

$$= 2\omega\left(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta\right)$$

$$= 2\omega\,\hat{e}_z$$

$\vec{\nabla}\times\vec{v} = 2\vec{\omega}$ (in agreement with 3.10.12 b), of course!)

Not graded

PHYS 3496
Homework # 3

3.10.26 a) $\vec{A}\cdot\vec{\nabla}\vec{r} = \left(A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z}\right)(x\hat{e}_x + y\hat{e}_y + z\hat{e}_z)$

$$= A_x \hat{e}_x + A_y \hat{e}_y + A_z \hat{e}_z = \vec{A}$$

b) $\vec{A}\cdot\vec{\nabla}\vec{r} = \left(A_r \frac{\partial}{\partial r} + \frac{A_\theta}{r}\frac{\partial}{\partial \theta} + \frac{A_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)(r\hat{e}_r)$

$$= A_r \hat{e}_r + A_r r \frac{\partial \hat{e}_r}{\partial r} + A_\theta \frac{\partial \hat{e}_r}{\partial \theta} + \frac{A_\varphi}{\sin\theta}\frac{\partial \hat{e}_r}{\partial \varphi}$$

from $\hat{e}_r = \hat{e}_x \sin\theta\cos\varphi + \hat{e}_y \sin\theta\sin\varphi + \hat{e}_z\cos\theta$

$\hat{e}_\theta = \hat{e}_x \cos\theta\cos\varphi + \hat{e}_y \cos\theta\sin\varphi - \hat{e}_z \sin\theta$

$\hat{e}_\varphi = -\hat{e}_x \sin\varphi + \hat{e}_y \cos\varphi$, we get

$\frac{\partial \hat{e}_r}{\partial r} = \vec{0}$

$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta$ and $\frac{\partial \hat{e}_r}{\partial \varphi} = \hat{e}_\varphi \sin\theta$. Thus,

$\vec{A}\cdot\vec{\nabla}\vec{r} = A_r\hat{e}_r + A_r r\vec{0} + A_\theta\hat{e}_\theta + \frac{A_\varphi}{\sin\theta}(\hat{e}_\varphi\sin\theta)$

$$= A_r\hat{e}_r + A_\theta\hat{e}_\theta + A_\varphi\hat{e}_\varphi = \vec{A}.$$

3.10.27 10 marks $\vec{r}(t) = r(t)\hat{e}_r(t)$

$\vec{v}(t) = \frac{d\vec{r}}{dt} = \dot{r}(t)\hat{e}_r(t) + r(t)\frac{d\hat{e}_r}{dt}$

$\frac{d\hat{e}_r}{dt} = \hat{e}_x \cos\theta\cos\varphi\,\dot{\theta} - \hat{e}_x \sin\theta\sin\varphi\,\dot{\varphi}$

$$+\hat{e}_y \cos\theta \sin\varphi\, \dot{\theta} + \hat{e}_y \sin\theta \cos\varphi\, \dot{\varphi} - \hat{e}_z \sin\theta\, \dot{\theta}$$

$$= [\hat{e}_x \cos\theta \cos\varphi + \hat{e}_y \cos\theta \sin\varphi - \hat{e}_z \sin\theta]\, \dot{\theta}$$

$$+ [-\hat{e}_x \sin\varphi + \hat{e}_y \cos\varphi] \sin\theta\, \dot{\varphi}$$

$$= \hat{e}_\theta \dot{\theta} + \hat{e}_\varphi \sin\theta\, \dot{\varphi}.$$ Thus,

$$\vec{v}(t) = \dot{r}\, \hat{e}_r + r[\hat{e}_\theta \dot{\theta} + \hat{e}_\varphi \sin\theta\, \dot{\varphi}]$$

$$= \dot{r}\, \hat{e}_r + r\dot{\theta}\, \hat{e}_\theta + r\sin\theta\, \dot{\varphi}\, \hat{e}_\varphi.$$ So

$$v_r = \dot{r}\, ; \quad v_\theta = r\dot{\theta} \quad \text{and} \quad v_\varphi = r\sin\theta\, \dot{\varphi}$$

Alternatively, $\vec{v} = \frac{d\vec{r}}{dt}$

$$d\vec{r} = dr\, \hat{e}_r + r\, d\theta\, \hat{e}_\theta + r\sin\theta\, d\varphi\, \hat{e}_\varphi \qquad (\text{Eq. } 3.154)$$

$$\Rightarrow \vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt} \hat{e}_r + r\frac{d\theta}{dt} \hat{e}_\theta + r\sin\theta \frac{d\varphi}{dt} \hat{e}_\varphi$$

$$= \dot{r}\, \hat{e}_r + r\dot{\theta}\, \hat{e}_\theta + r\sin\theta\, \dot{\varphi}\, \hat{e}_\varphi. \checkmark$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \ddot{r}\, \hat{e}_r + \dot{r}\frac{d\hat{e}_r}{dt} + \dot{r}\dot{\theta}\, \hat{e}_\theta + r\ddot{\theta}\, \hat{e}_\theta + r\dot{\theta}\frac{d\hat{e}_\theta}{dt}$$

$$+ \dot{r}\sin\theta\, \dot{\varphi}\, \hat{e}_\varphi + r\cos\theta\, \dot{\theta}\dot{\varphi}\, \hat{e}_\varphi + r\sin\theta\, \ddot{\varphi}\, \hat{e}_\varphi + r\sin\theta\, \dot{\varphi}\frac{d\hat{e}_\varphi}{dt}$$

$$\frac{d\hat{e}_r}{dt} = \hat{e}_\theta \dot{\theta} + \hat{e}_\varphi \sin\theta\, \dot{\varphi}$$

$$\frac{d\hat{e}_\theta}{dt} = -\hat{e}_x \sin\theta \cos\varphi\, \dot{\theta} - \hat{e}_x \cos\theta \sin\varphi\, \dot{\varphi} \; \cancel{\hat{e}_z \cos\theta\, \dot{\theta}}$$

$$- \hat{e}_y \sin\theta \sin\varphi\, \dot{\theta} + \hat{e}_y \cos\theta \cos\varphi\, \dot{\varphi}$$

$$- \hat{e}_z \cos\theta\, \dot{\theta}$$

$$\frac{d\hat{e}_\theta}{dt} = -\left[\hat{e}_x \sin\theta\cos\varphi + \hat{e}_y \sin\theta\sin\varphi + \hat{e}_z \cos\theta\right]\dot{\theta} + \left[-\hat{e}_x \sin\varphi + \hat{e}_y \cos\varphi\right]\cos\theta\,\dot{\varphi}$$

$$\frac{d\hat{e}_\theta}{dt} = -\hat{e}_r\,\dot{\theta} + \hat{e}_\varphi \cos\theta\,\dot{\varphi}$$

$$\frac{d\hat{e}_\varphi}{dt} = -\hat{e}_x \cos\varphi\,\dot{\varphi} - \hat{e}_y \sin\varphi\,\dot{\varphi} = -(\hat{e}_x\cos\varphi + \hat{e}_y \sin\varphi)\dot{\varphi}$$

$$= -\left[+\hat{e}_r \sin\theta + \hat{e}_\theta \cos\theta\right]\dot{\varphi}$$

$$= -\hat{e}_r \sin\theta\,\dot{\varphi} - \hat{e}_\theta \cos\theta\,\dot{\varphi}\,.$$ Thus,

$$\vec{a}(t) = \ddot{r}\,\hat{e}_r + \dot{r}\,(\hat{e}_\theta\,\dot{\theta} + \hat{e}_\varphi \sin\theta\,\dot{\varphi})$$

$$+ \dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta + r\dot{\theta}\left[-\hat{e}_r\,\dot{\theta} + \hat{e}_\varphi \cos\theta\,\dot{\varphi}\right]$$

$$+ (\dot{r}\sin\theta\,\dot{\varphi} + r\cos\theta\,\dot{\theta}\dot{\varphi} + r\sin\theta\,\ddot{\varphi})\,\hat{e}_\varphi$$

$$+ r\sin\theta\,\dot{\varphi}\left[-\hat{e}_r \sin\theta\,\dot{\varphi} - \hat{e}_\theta \cos\theta\,\dot{\varphi}\right]$$

$$= \left[\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\varphi}^2\right]\hat{e}_r$$

$$+ \left[r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\,\dot{\varphi}^2\right]\hat{e}_\theta$$

$$+ \left[r\sin\theta\,\ddot{\varphi} + 2\dot{r}\sin\theta\,\dot{\varphi} + 2r\cos\theta\,\dot{\theta}\dot{\varphi}\right]\hat{e}_\varphi$$

Hence $a_r = \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\varphi}^2$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\,\dot{\varphi}^2$$

$$a_\varphi = r\sin\theta\,\ddot{\varphi} + 2\dot{r}\sin\theta\,\dot{\varphi} + 2r\cos\theta\,\dot{\theta}\dot{\varphi}$$

2 marks

3.10.29 Using the results of Exercise 3.10.28, we get

$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} =$$

$$= r\sin\theta\cos\varphi\left[\sin\theta\sin\varphi\frac{\partial}{\partial r} + \cos\theta\sin\varphi\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right]$$

$$-r\sin\theta\sin\varphi\left[\sin\theta\cos\varphi\frac{\partial}{\partial r} + \cos\theta\cos\varphi\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right]$$

$$= (\cos^2\varphi + \sin^2\varphi)\frac{\partial}{\partial\varphi} = \frac{\partial}{\partial\varphi}$$

$$\rightarrow \underbrace{-i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)}_{L_z} = -i\frac{\partial}{\partial\varphi}.$$

10 marks

3.10.32 a) $\vec{L} = -i\vec{r}\times\vec{\nabla}$

$$= -i\, r\,\hat{e}_r \times \left[\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right]$$

$$= -i\left[\hat{e}_\varphi\frac{\partial}{\partial\theta} - \hat{e}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right]$$

$$= i\left[\hat{e}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} - \hat{e}_\varphi\frac{\partial}{\partial\theta}\right]$$

b) $$\vec{L} = i\left[\frac{1}{\sin\theta}(\hat{e}_x\cos\theta\cos\varphi + \hat{e}_y\cos\theta\sin\varphi - \hat{e}_z\sin\theta)\frac{\partial}{\partial\varphi} - (-\sin\varphi\,\hat{e}_x + \cos\varphi\,\hat{e}_y)\frac{\partial}{\partial\theta}\right]$$

$$= \hat{e}_x\left(i\sin\varphi\frac{\partial}{\partial\theta} + i\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$$

$$+ \hat{e}_y\left(-i\cos\varphi\frac{\partial}{\partial\theta} + i\cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)$$

$$+ \hat{e}_z\left(-i\frac{\partial}{\partial\varphi}\right)$$

Thus, $L_x = i\sin\varphi \frac{\partial}{\partial\theta} + i\cot\theta\cos\varphi \frac{\partial}{\partial\varphi}$

$L_y = -i\cos\varphi \frac{\partial}{\partial\theta} + i\cot\theta\sin\varphi \frac{\partial}{\partial\varphi}$

$L_z = -i \frac{\partial}{\partial\varphi}$ (which we already know from 3.10.29)

c) $L^2 = L_x^2 + L_y^2 + L_z^2$

$$= -\left(\sin\varphi \frac{\partial}{\partial\theta} + \cot\theta\cos\varphi \frac{\partial}{\partial\varphi}\right)\left(\sin\varphi \frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$$

$$-\left(\cos\varphi \frac{\partial}{\partial\theta} - \cot\theta\sin\varphi \frac{\partial}{\partial\varphi}\right)\left(\cos\varphi \frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)$$

$$-\frac{\partial^2}{\partial\varphi^2}$$ [I used $i^2 = -1$]

$$= -\Big[\sin^2\varphi \frac{\partial^2}{\partial\theta^2} + \cos\varphi\sin\varphi(-\csc^2\theta)\frac{\partial}{\partial\varphi} + \sin\varphi\cos\varphi\cot\theta\frac{\partial^2}{\partial\theta\partial\varphi}$$

$$+ \cancel{\cot\theta\cos\varphi\sin\varphi\frac{\partial^2}{\partial\varphi\partial\theta}} + \cot\theta\cos^2\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cos\varphi\cot^2\theta\frac{\partial}{\partial\varphi}$$

$$+ \cot^2\theta\cos^2\varphi\frac{\partial^2}{\partial\varphi^2}\Big]$$

$$-\Big[\cos^2\varphi\frac{\partial^2}{\partial\theta^2} + \sin\varphi\cos\varphi\csc^2\theta\frac{\partial}{\partial\varphi} - \sin\varphi\cos\varphi\cot\theta\frac{\partial^2}{\partial\varphi\partial\theta}$$

$$+ \cot\theta\sin^2\varphi\frac{\partial}{\partial\theta} + \cot^2\theta\sin^2\varphi\frac{\partial^2}{\partial\varphi^2} + \sin\varphi\cos\varphi\cot^2\theta\frac{\partial}{\partial\varphi}\Big]$$

$$-\frac{\partial^2}{\partial\varphi^2}$$

$$= -\frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} - \cot^2\theta\frac{\partial^2}{\partial\varphi^2} - \frac{\partial^2}{\partial\varphi^2}$$

$$= -\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\right) - \left(\frac{\cos^2\theta}{\sin^2\theta} + 1\right)\frac{\partial^2}{\partial\varphi^2}$$

$$= -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$= -r^2\left[\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

$$= -r^2\left\{\frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\varphi^2}\right] - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right]$$

$$= -r^2\left\{\frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\varphi^2}\right]\right\} + \frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$$

$$= -r^2\nabla^2 + \frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$$

where we have used Eq. 3.158 for ∇^2 in spherical coordinates.

Not graded

3.10.30 Using the results of Exercise 3.10.32, we have that

$$L_x = i\sin\varphi\frac{\partial}{\partial\theta} + i\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}$$

and $L_y = -i\cos\varphi\frac{\partial}{\partial\theta} + i\cot\theta\sin\varphi\frac{\partial}{\partial\varphi}$. Hence

a) $$L_x + iL_y = i\sin\varphi\frac{\partial}{\partial\theta} + i\cot\theta\cos\varphi\frac{\partial}{\partial\varphi} + \cos\varphi\frac{\partial}{\partial\theta} + i\cot\theta(i\sin\varphi)\frac{\partial}{\partial\varphi}$$

$$= (\cos\varphi + i\sin\varphi)\frac{\partial}{\partial\theta} + i\cot\theta\left[\cos\varphi + i\sin\varphi\right]\frac{\partial}{\partial\varphi}$$

$$L_x + i L_y = e^{i\varphi} \frac{\partial}{\partial \theta} + i \cot\theta \, e^{i\varphi} \frac{\partial}{\partial \varphi}$$
$$= e^{i\varphi} \left[\frac{\partial}{\partial \theta} + i \cot\theta \frac{\partial}{\partial \varphi} \right].$$

b) $$L_x - i L_y = i \sin\varphi \frac{\partial}{\partial \theta} + i \cot\theta \cos\varphi \frac{\partial}{\partial \varphi}$$
$$- \cos\varphi \frac{\partial}{\partial \theta} + i \cot\theta (-i \sin\varphi) \frac{\partial}{\partial \varphi}$$
$$= -(\cos\varphi - i\sin\varphi) \frac{\partial}{\partial \theta} + i \cot\theta (\cos\varphi - i\sin\varphi) \frac{\partial}{\partial \varphi}$$
$$= -e^{-i\varphi} \frac{\partial}{\partial \theta} + i \cot\theta \, e^{-i\varphi} \frac{\partial}{\partial \varphi}$$
$$= -e^{-i\varphi} \left[\frac{\partial}{\partial \theta} - i \cot\theta \frac{\partial}{\partial \varphi} \right].$$

4 marks

3.10.34 a) $$\nabla^2 \psi(r) = \frac{1}{r^2 \sin\theta} \left[\sin\theta \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin\theta} \frac{\partial^2 \psi}{\partial \varphi^2} \right]$$ Eq. 3.158

Since $\psi = \psi(r)$, $\frac{\partial \psi}{\partial \theta} = \frac{\partial \psi}{\partial \varphi} = 0$ and $\frac{\partial \psi}{\partial r} = \frac{d\psi}{dr}$.

$$\rightarrow \nabla^2 \psi(r) = \frac{1}{r^2 \sin\theta} \left[\sin\theta \frac{d}{dr} \left(r^2 \frac{d\psi}{dr} \right) + 0 + 0 \right]$$
$$= \frac{1}{r^2} \frac{d}{dr} \left[r^2 \frac{d\psi}{dr} \right]$$

b) $$\nabla^2 \psi(r) = \frac{1}{r^2} \frac{d}{dr} \left[r \left(r \frac{d\psi}{dr} \right) \right]$$
$$= \frac{1}{r} \frac{d}{dr} \left(r \frac{d\psi}{dr} \right) + \frac{1}{r} \frac{d\psi}{dr} = \frac{1}{r} \frac{d}{dr} \left[r \frac{d\psi}{dr} + \psi \right]$$
$$= \frac{1}{r} \frac{d}{dr} \left[\frac{d}{dr} (r\psi) \right] = \frac{1}{r} \frac{d^2}{dr^2} (r\psi)$$

c) $\nabla^2 \psi(r) = \frac{1}{r^2} \frac{d}{dr} \left[r^2 \frac{d\psi}{dr} \right]$ (part a)

$= \frac{1}{r^2} \left[r^2 \frac{d^2\psi}{dr^2} + 2r \frac{d\psi}{dr} \right]$ (product Rule)

$= \frac{d^2\psi}{dr^2} + \frac{2}{r} \frac{d\psi}{dr}$.