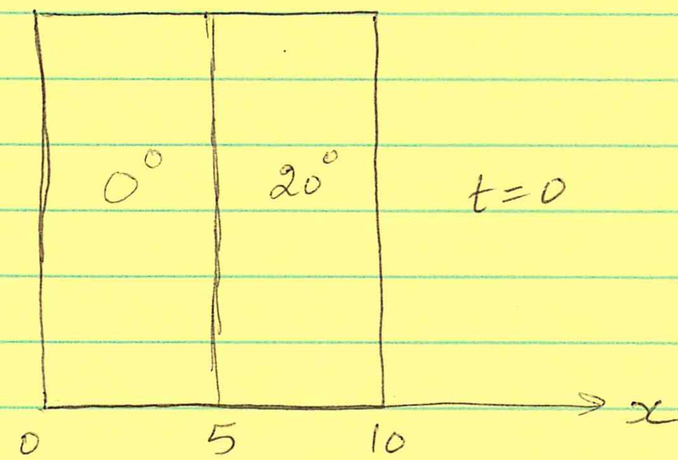
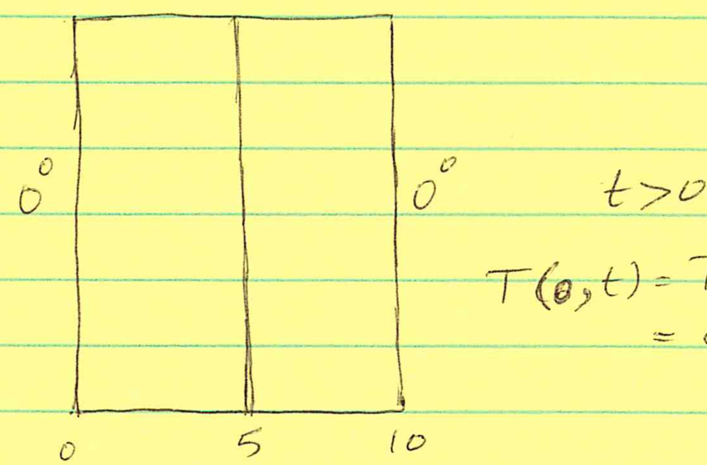
PHYS 3496 Homework # 2 Total: 36 marks

3.4

10 marks



Since T is independent of y, z, we take $T = T(x,t)$, with $T(x,0) = \begin{cases} 0^\circ & \text{if } 0 < x < 5 \\ 20^\circ & \text{if } 5 < x < 10 \end{cases}$



$T(0,t) = T(10,t)$
$= 0^\circ$

We must solve $\nabla^2 T = \frac{1}{\alpha^2}\frac{\partial T}{\partial t}$ (here $\nabla^2 T = \frac{\partial^2 T}{\partial x^2}$)

Assume $T(x,t) = X(x)\,G(t)$. Then

$$G(t)\frac{d^2X}{dx^2} = \frac{1}{\alpha^2}X(x)\frac{dG}{dt}$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{\alpha^2 G}\frac{dG}{dt} = -k^2$$

$$\frac{dG}{dt} = -\alpha^2 k^2 G \qquad (1)$$

$$\frac{d^2X}{dx^2} = -k^2 X \qquad (2)$$

$$(1) \Rightarrow G(t) = A e^{-\alpha^2 k^2 t}$$

$$(2) \Rightarrow X(x) = B \sin kx + C \cos kx$$

$T(0,t) = 0 \Rightarrow X(0) = 0 \Rightarrow C = 0$; so

$$X(x) = B \sin kx$$

$T(10,t) = 0 \Rightarrow X(10) = 0 \Rightarrow k(10) = n\pi$

$\Rightarrow k = \frac{n\pi}{10}$. Therefore,

$$T(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{10} x \, e^{-\alpha^2 \frac{n^2\pi^2}{100} t}$$

But $T(x,0) = \begin{cases} 0 & \text{if } 0 < x < 5 \\ 20 & \text{if } 5 < x < 10 \end{cases} = f(x)$

Thus, $f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{10} x$. This is a Fourier sine series for $f(x)$ of period $(2l) = 20$.

$$A_n = \frac{2}{10}\int_0^{10} f(x) \sin \frac{n\pi}{10} x \, dx$$

$$A_n = \frac{1}{5}\int_5^{10} 20 \sin \frac{n\pi}{10} x \, dx$$

$$= 4\left[-\frac{10}{n\pi} \cos \frac{n\pi}{10}x \Big|_5^{10} \right]$$

$$= \frac{-40}{n\pi}\left[\cos n\pi - \cos \frac{n\pi}{2} \right]$$

$$= \begin{cases} \frac{40}{n\pi} & \text{if } n \text{ is odd} \\ \frac{-80}{n\pi} & \text{if } n = 2, 6, 10, \ldots \\ 0 & \text{if } n = 4, 8, 12, \ldots \end{cases}$$

$$T(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{10} x \, e^{-\alpha^2 \frac{n^2\pi^2}{100} t}$$

$$= \frac{40}{\pi}\left[\sin \frac{\pi}{10} x \, e^{-\alpha^2 \frac{\pi^2}{100} t} - \sin \frac{2\pi x}{10} e^{-\frac{\alpha^2 4\pi^2}{100} t} \right.$$

$$\left. + \frac{1}{3} \sin \frac{3\pi}{10} x \, e^{-\frac{\alpha^2 9\pi^2}{100} t} - - - \right]$$

7.3) $u(1, \theta, \varphi) = \cos\theta - 3\sin^2\theta$

10 marks

$\nabla^2 u = 0$ in spherical coordinates has solution

$$u(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=0}^{l} c_{l,m} r^l P_l^m(\cos\theta) \begin{cases} \sin m\varphi \\ \cos m\varphi \end{cases} \quad (*$$

(we ignore r^{-l-1} since that diverges at $r=0$).

Since $u(1, \theta, \varphi)$ is independent of φ, only the $m=0$

terms should appear in the solution in (*).

Thus, $u(r,\theta,\varphi) = u(r,\theta)$

$$= \sum_{l=0}^{\infty} c_{l0}\, r^{l}\, P_l^{0}(\cos\theta)$$

$$= \sum_{l=0}^{\infty} c_l\, r^{l}\, P_l(\cos\theta)$$

$$u(1,\theta) = \sum_{l=0}^{\infty} c_l\, P_l(\cos\theta)$$

$$= \cos\theta - 3\sin^2\theta$$

$$= \cos\theta - 3(1-\cos^2\theta)$$

$$= 3\cos^2\theta + \cos\theta - 3$$

$$= (3\cos^2\theta - 1) + (\cos\theta) - 2$$

$$= 2P_2(\cos\theta) + P_1(\cos\theta) - 2P_0(\cos\theta)$$

Therefore, $c_0 = -2$, $c_1 = 1$, $c_2 = 2$ and

$c_l = 0$ for $l \geq 3$. Therefore,

$$u(r,\theta) = -2 + r\,P_1(\cos\theta) + 2r^2 P_2(\cos\theta)$$

$$= -2 + r\cos\theta + r^2(3\cos^2\theta - 1)$$

10 marks

7.14 Since u is independent of φ on the boundary ($r=1$ and $r=2$), u is independent of φ throughout the spherical shell. But now the origin is not included in the region of interest; so the r part must include both r^{ℓ} and $r^{-\ell-1}$. Thus,

$$u(r,\theta) = \sum_{\ell=0}^{\infty} \left[A_\ell r^{\ell} + B_\ell r^{-\ell-1} \right] P_\ell(\cos\theta)$$

$$u(1,\theta) = \sum_{\ell=0}^{\infty} \left[A_\ell + B_\ell \right] P_\ell(\cos\theta)$$

$= 0 \Rightarrow B_\ell = -A_\ell$ for all ℓ. Therefore

$$u(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell \left[r^{\ell} - r^{-\ell-1} \right] P_\ell(\cos\theta)$$

$$u(2,\theta) = \sum_{\ell=0}^{\infty} A_\ell \left[2^{\ell} - 2^{-\ell-1} \right] P_\ell(\cos\theta)$$

$$= \sum_{\ell=0}^{\infty} c_\ell P_\ell(\cos\theta) = f(\cos\theta)$$

with $A_\ell = \dfrac{c_\ell}{2^{\ell} - 2^{-\ell-1}}$; and

$$f(\cos\theta) = \begin{cases} 100 & 0 < \cos\theta < 1 \quad (0 < \theta < \frac{\pi}{2}) \\ 0 & -1 < \cos\theta < 0 \quad (\frac{\pi}{2} < \theta < \pi) \end{cases}$$

$$c_\ell = \frac{2\ell+1}{2} \int_{-1}^{+1} f(x) P_\ell(x)\, dx = \frac{2\ell+1}{2} \int_0^1 100\, P_\ell(x)\, dx$$

$$c_0 = 50 \int_0^1 dx = 50$$

For $\ell \geq 1$, use Equation 5.8(e) of section 5, chapter 12 to get

$$c_\ell = 50 \int_0^1 (2\ell+1) P_\ell(x)\, dx$$

$$= 50 \left[\left(P_{\ell+1}(x) - P_{\ell-1}(x) \right) \Big|_0^1 \right]$$

$$= 50 \left[P_{\ell+1}(1) - P_{\ell+1}(0) - P_{\ell-1}(1) + P_{\ell-1}(0) \right]$$

But $P_{\ell+1}(1) = P_{\ell-1}(1) = 1$; so

$$c_\ell = 50 \left[P_{\ell-1}(0) - P_{\ell+1}(0) \right]$$

For ℓ even, $\ell-1$ and $\ell+1$ are both odd and hence $P_{\ell-1}(0) = P_{\ell+1}(0) = 0$. So $c_\ell = 0$ for ℓ even; i.e. for $\ell = 2, 4, 6, \ldots$

$$c_1 = 50 \left[P_0(0) - P_2(0) \right] = 50 \left[1 - \left(-\tfrac{1}{2}\right) \right] = 75$$

$$c_3 = 50 \left[P_2(0) - P_4(0) \right] = 50 \left[-\tfrac{1}{2} - \tfrac{3}{8} \right] = \frac{-350}{8} = \frac{-175}{4}$$

$\vdots$

$$A_0 = \frac{c_0}{1 - \frac{1}{2}} = 2c_0 = 100$$

$$A_1 = \frac{c_1}{2 - \frac{1}{4}} = \frac{4\, c_1}{7} = \frac{300}{7}$$

$$A_3 = \frac{c_3}{8 - \frac{1}{16}} = \frac{16}{127}(c_3) = \frac{16}{127}\left(\frac{-175}{4}\right) = \frac{-700}{127}$$

Thus, $u(r,\theta,\varphi) = u(r,\theta)$

$$= 100\left(1 - \frac{1}{r}\right) + \frac{300}{7}\left(r - \frac{1}{r^2}\right) P_1(\cos\theta)$$

$$- \frac{700}{127}\left(r^3 - \frac{1}{r^4}\right) P_3(\cos\theta) + \cdots$$

3.11 (8 marks)

We start with Equation 3-26 with $\ell = \pi$:

$$\psi(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \, e^{-iE_n t/\hbar} \quad : \quad E_n = \frac{\hbar^2}{2m} n^2$$

$$\psi(x,0) = \sum_{n=1}^{\infty} b_n \sin nx = 1 \qquad 0 < x < \pi$$

This is a Fourier sine series for 1 on $(0, \pi)$; so

$$b_n = \frac{2}{\pi} \int_0^{\pi} 1 \cdot \sin nx \, dx$$

$$= -\frac{2}{\pi} \frac{\cos nx}{n} \Big|_0^{\pi} = \frac{2}{n\pi} \left[1 - \cos n\pi \right] = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{n\pi} & \text{if } n \text{ is odd.} \end{cases}$$

So $$\psi(x,t) = \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n} \sin nx \, e^{-iE_n t/\hbar},$$

where $E_n = \frac{\hbar^2}{2m} n^2$