PHYS 3496 Homework # 1 Total: 34 marks

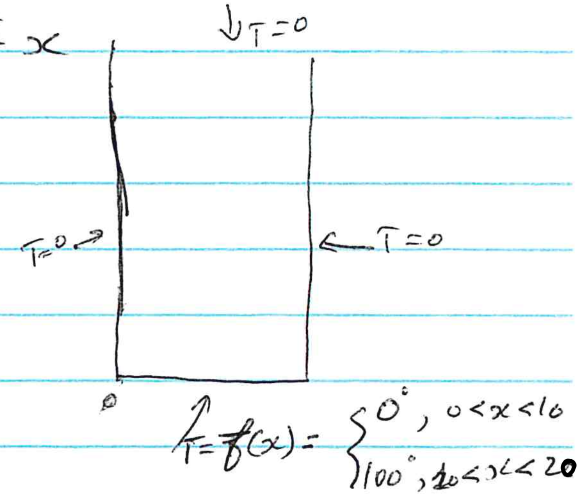
Chapter 13

(2.2) The solution of $\nabla^2 T = 0$ which satisfies

10 marks

$T = 0$ at $x = 0$, $x = 20$, and $y \to \infty$ is obtained from Equation (2.9), replacing 10 by 20; so

$$T(x,y) = \sum_{n=1}^{\infty} b_n e^{-\frac{n\pi}{20}y} \sin \frac{n\pi}{20} x$$



It remains to match the boundary condition at the bottom edge of the plate $(y=0)$.

$$T(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{20} x$$

$$= f(x)$$

So we have a fourier-sine series of $f(x)$ of period $2l = 40$. The coefficients b_n are given by:

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi}{l} x \, dx$$

$$= \frac{2}{20} \int_0^{20} f(x) \sin \frac{n\pi}{20} x \, dx$$

$$= \frac{1}{10} \int_{10}^{20} 100 \sin \frac{n\pi}{20} x \, dx$$

$$= (10) \left[-\frac{20}{n\pi} \cos \frac{n\pi}{20} x \Big|_{10}^{20} \right]$$

$$= -\frac{200}{n\pi}\left[\cos n\pi - \cos\frac{n\pi}{2}\right]$$

$$= \begin{cases} \dfrac{200}{n\pi} & \text{if } n = 1, 3, 5, \ldots \quad (n \text{ is odd}) \\[2ex] -\dfrac{400}{n\pi} & \text{if } n = 2, 6, 10, \ldots \quad (n = 4k-2;\ k = 1,2,3\ldots) \\[2ex] 0 & \text{if } n = 4, 8, 12, \ldots \quad (n = 4k;\ k = 1,2,3,\ldots) \end{cases}$$

Hence $T(x,y) = \frac{200}{\pi}\Big[e^{-\pi y/20}\sin\frac{\pi x}{20} - e^{-\pi y/10}\sin\frac{\pi x}{10}$

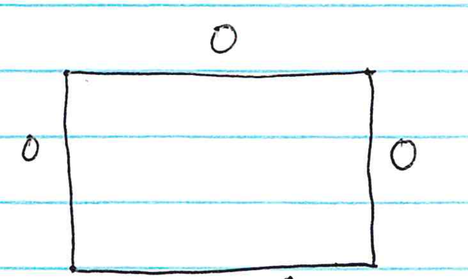
$+ \frac{1}{3}e^{-3\pi y/20}\sin\frac{3\pi x}{20} + \frac{1}{5}e^{-5\pi y/20}\sin\frac{5\pi x}{20}$

$- \frac{1}{3}e^{-3\pi y/10}\sin\frac{3\pi x}{10} + \cdots\Big]$

$$= \frac{200}{\pi}\sum_{n,\,odd}\frac{1}{n}e^{-n\pi y/20}\sin\frac{n\pi x}{20} - \frac{400}{\pi}\sum_{n=4k-2}\frac{1}{n}e^{-n\pi y/20}\sin\frac{n\pi x}{20}$$

2.9

6 marks



$T(x,10) = f(x)$ same as in problem 2.2

See Equation (2.15) in the book; here the width is 20 instead of 10 and the height is 10 instead of 30.

$$T(x,y) = \sum_{n=1}^{\infty} B_n \sinh\frac{n\pi}{20}(10-y)\sin\frac{n\pi x}{20}$$

$$T(x,0) = \sum_{n=1}^{\infty} B_n \sinh\frac{n\pi}{2} \sin\frac{n\pi x}{20}$$

$$= f(x) = \begin{cases} 0, & 0 < x < 10 \\ 100, & 10 < x < 20. \end{cases}$$

Thus, $B_n \sinh\frac{n\pi}{2} = b_n$ of problem 2.2 above!

Hence $B_n = \dfrac{b_n}{\sinh\frac{n\pi}{2}}$ Therefore,

$$T(x,y) = \frac{200}{\pi} \sum_{n\ \text{odd}} \frac{1}{n \sinh\left(\frac{n\pi}{2}\right)} \sinh\frac{n\pi}{20}(10-y) \sin\frac{n\pi x}{20}$$

$$- \frac{400}{\pi} \sum_{n=4k-2} \frac{1}{n \sinh\left(\frac{n\pi}{2}\right)} \sinh\frac{n\pi}{20}(10-y) \sin\frac{n\pi x}{20}$$

5.2 b) Using Equation (5.17) in the book, we

10 marks

have that

$$u(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(k_{nm} r/a)\left[A_{nm}\cos n\theta + B_{nm}\sin n\theta\right] e^{-k_{nm} z/a}$$

At $z=0$

$$u(r,\theta,0) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(k_{nm} r/a)\left[A_{nm}\cos n\theta + B_{nm}\sin n\theta\right]$$

$$= r\sin\theta.$$

Using Equations (5.19) and (5.20) for the coefficients A_{nm} and B_{nm} and noting that

$$\int_0^{2\pi} \sin\theta \sin n\theta \, d\theta = \pi \, \delta_{n1} \quad \text{and}$$

$$\int_0^{2\pi} \sin\theta \cos n\theta \, d\theta = 0 \quad \text{for all } n,$$ it follows that

$A_{nm} = 0$ for all n and for all m; and

$B_{nm} = 0$ if $n \neq 1$. Therefore,

$$u(r,\theta,z) = \sum_{m=1}^{\infty} B_{1m} \, J_1(K_{1m} r/a) \sin\theta \, e^{-K_{1m} z/a}$$

~~Denoting~~ Renaming the zeros of J_1 K_m instead of K_{1m} and using c_m instead of B_{1m}; we thus obtain

$$u(r,\theta,z) = \sum_{m=1}^{\infty} c_m \, J_1(K_m r/a) \, e^{-K_m z/a} \sin\theta$$

$$u(r,\theta,0) = \sum_{m=1}^{\infty} c_m \, J_1(K_m r/a) \sin\theta$$

$$= r \sin\theta,$$ from which we obtain

$$\sum_{m=1}^{\infty} c_m \, J_1(K_m r/a) = r \qquad (*)$$

Multiply both sides of $(*)$ by $r\, J_1(K_\mu r/a)$; and integrate from 0 to a:

$$\sum_{m=1}^{\infty} c_m \int_0^a r \, J_1(K_m r/a) \, J_1(K_\mu r/a) \, dr = \int_0^a r^2 J_1(K_\mu r/a)$$

$$\sum_{m=1}^{\infty} c_m \int_0^a r J_1(K_m r/a) J_1(K_\mu r/a)\, dr$$

$$= \sum_{m=1}^{\infty} c_m \frac{a^2}{2} J_2^2(K_m)\, \delta_{m\mu} = \frac{a^2}{2} c_\mu J_2^2(K_\mu)$$

Next we evaluate $\int_0^a r^2 J_1(K_\mu r/a)\, dr$

By Equation (15.1) of chapter 12, we have that

$\frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x)$. Therefore,

$\frac{d}{dx}[x^2 J_2(x)] = x^2 J_1(x)$. Replacing x by

$K_\mu r/a$, we get:

$$\frac{a}{K_\mu} \frac{d}{dr}\left[\frac{K_\mu^2}{a^2} r^2 J_2(K_\mu r/a)\right] = \frac{K_\mu^2}{a^2} r^2 J_1(K_\mu r/a)$$

and hence

$r^2 J_1(K_\mu r/a) = \frac{a}{K_\mu} \frac{d}{dr}[r^2 J_2(K_\mu r/a)]$. Thus,

$$\int_0^a r^2 J_1(K_\mu r/a)\, dr = \frac{a}{K_\mu} r^2 J_2(K_\mu r/a) \Big|_{r=0}^{r=a}$$

$= \frac{a^3}{K_\mu} J_2(K_\mu)$. Therefore,

$\frac{a^2}{2} c_\mu J_2^2(K_\mu) = \frac{a^3}{K_\mu} J_2(K_\mu)$; and hence

$$c_\mu = \frac{2a}{K_\mu J_2(K_\mu)}$$

Therefore,

$$u(r,\theta,z) = \sum_{m=1}^{\infty} \frac{2a}{K_m J_2(K_m)} J_1(K_m r/a)\, e^{-K_m z/a} \sin\theta \,, \quad \text{where}$$

K_m is the m^{th} root (zero) of $\underline{\underline{J_1}}$.

Not to be graded!

$$a = 2 \Rightarrow u\left(r=1, \theta=\frac{\pi}{2}, z=1\right) =$$

$$= \sum_{m=1}^{\infty} \frac{4}{K_m J_2(K_m)} J_1(K_m/2)\, e^{-K_m/2}$$

$$= 4\left[\frac{1}{K_1 J_2(K_1)} J_1(K_1/2)\, e^{-K_1/2} + \frac{1}{K_2 J_2(K_2)} J_1(K_2/2)\, e^{-K_2/2}\right.$$

$$\left. + \frac{1}{K_3 J_2(K_3)} J_1(K_3/2)\, e^{-K_3/2} + \cdots\right] \simeq 0.211$$

5.3 b)

6 marks

Since the height of the cylinder is finite, the z-dependence now has to be a linear combination of e^{Kz} and e^{-Kz} ($K = K_m/a$) rather than just e^{-Kz} as in the case of a semi-infinite cylinder. To make such a linear combination vanish at $z = H$, we use

$$\sinh K(H-z) = \frac{1}{2}\left[e^{K(H-z)} - e^{-K(H-z)}\right]$$

$$= \frac{1}{2} e^{KH} e^{-Kz} - \frac{1}{2} e^{-KH} e^{Kz}.$$

So Equation (5.11) now becomes:

$$u = \sum_{m=1}^{\infty} c_m' J_0(\kappa_m r/a) \sinh \kappa_m (H-z)/a$$

$$u\Big|_{z=0} = \sum_{m=1}^{\infty} c_m' J_0(\kappa_m r/a) \sinh(\kappa_m H/a) = 100$$

Comparing with Equation (5.12), we see that

$$c_m' \sinh(\kappa_m H/a) = c_m = \frac{200}{\kappa_m J_1(\kappa_m)} \quad \text{(Equation (5.16))}$$

Thus,

$$c_m' = \frac{200}{\kappa_m J_1(\kappa_m) \sinh(\kappa_m H/a)}.$$ Therefore,

$$u = 200 \sum_{m=1}^{\infty} \frac{1}{\kappa_m J_1(\kappa_m) \sinh(\kappa_m H/a)} J_0(\kappa_m r/a) \sinh[\kappa_m (H-z)/a]$$

where κ_m is the m^{th} zero of J_0.