Supplemental Material - Acoustic double negativity induced by position correlations within a disordered set of monopolar resonators

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I. SCATTERING OF A PAIR OF MONOPOLES

We consider a plane wave impinging on a pair of scatterers placed at $\boldsymbol{r_A}$ and $\boldsymbol{r_B}$ (see Fig. S1). The pressure scattered

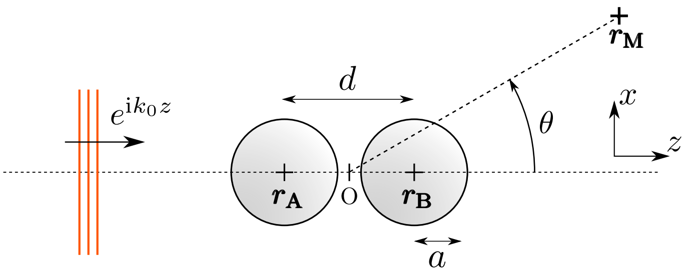

FIG. S1: A pair of scatterers of radius a, separated by a distance d.

at point $\boldsymbol{r_M}$ can be written:

$$p(\boldsymbol{r_M}) = p_0 + p_A f \frac{e^{-\mathrm{i}k_0||\boldsymbol{r_M}-\boldsymbol{r_A}||}}{||\boldsymbol{r_M}-\boldsymbol{r_A}||} + p_B f \frac{e^{-\mathrm{i}k_0||\boldsymbol{r_M}-\boldsymbol{r_B}||}}{||\boldsymbol{r_M}-\boldsymbol{r_B}||} \tag{S1}$$

with

$$||\ \boldsymbol{r}_\mathbf{M} - \boldsymbol{r}_\mathbf{A}\ || = \sqrt{\frac{d^2}{4} + r_\mathrm{M}^2 + r_\mathrm{M} d\cos\theta} \tag{S2}$$

and

$$||\ \boldsymbol{r}_\mathbf{M} - \boldsymbol{r}_\mathbf{B}\ || = \sqrt{\frac{d^2}{4} + r_\mathrm{M}^2 - r_\mathrm{M} d\cos\theta}. \tag{S3}$$

where r is the distance to the center of the pair and $\theta = 0$ in the forward direction. The pressures exerted on A and B can be obtained by inverting the multiple scattering matrix of the system:

$$\begin{bmatrix} p_A \\ p_B \end{bmatrix} = \begin{bmatrix} 1 & -f(\omega)\frac{e^{\mathrm{i}k_0 d}}{d} \\ -f(\omega)\frac{e^{\mathrm{i}k_0 d}}{d} & 1 \end{bmatrix}^{-1} \times \begin{bmatrix} p_{0A} \\ p_{0B} \end{bmatrix} \tag{S4}$$

with, here,

$$p_{0A} = e^{-\mathrm{i}k_0 d/2} \qquad \text{et} \qquad p_{0B} = e^{\mathrm{i}k_0 d/2}. \tag{S5}$$

Finally, we extract the scattering function of a pair by dividing the scattered pressure by $e^{\mathrm{i}k_0 r_M}/r_\mathrm{M}$. A zero order expansion in r_M/d yields the following expression:

$$f_d(\theta) = \frac{d^2 f}{d^2 - f^2 e^{2\mathrm{i}k_0 d}} \Big[e^{-\mathrm{i}k_0 \frac{d}{2}(1-\cos\theta)} \big(1 + \frac{f}{d} e^{2\mathrm{i}k_0 d}\big) + e^{\mathrm{i}k_0 \frac{d}{2}(1-\cos\theta)} \big(1 + \frac{f}{d}\big) \Big]. \tag{S6}$$

We can then easily determine analytic expressions for the forward and backward scattering:

$$f_d(0) = \frac{d^2 f}{d^2 - f^2 e^{2\mathrm{i}k_0 d}} \Big[2 + \frac{f}{d}\big(1 + e^{2\mathrm{i}k_0 d}\big) \Big] \tag{S7}$$

$$f_d(\pi) = \frac{d^2 f}{d^2 - f^2 e^{2\mathrm{i}k_0 d}}\left[2\cos(k_0 d) + 2\frac{f}{d}e^{\mathrm{i}k_0 d}\right] \tag{S8}$$

As in Eq. (2) of the main document, the expression for the scattering function is:

$$f = \frac{-a}{1 - \omega_0^2/\omega^2 + \mathrm{i}(k_0 a + \delta)}. \tag{S9}$$

After substituting this expression for f in Eqs. (S7) and (S8), one can obtain the symmetric and antisymmetric parts of the scattering function:

$$f_s = \frac{f_d(0) + f_d(\pi)}{2} = \frac{2a}{\left(\frac{\omega_0}{\omega}\right)^2 - (1 + \frac{a}{d}) - \mathrm{i}(2k_0 a + \delta)} \tag{S10}$$

$$f_a = \frac{f_d(0) - f_d(\pi)}{2} = \frac{k_0^2 d^2 a/2}{\left(\frac{\omega_0}{\omega}\right)^2 - (1 - \frac{a}{d}) - \mathrm{i}(k_0^3 a d^2/6 + \delta)}. \tag{S11}$$

II. NEGATIVE REFRACTION

Knowing the forward and backward scattering functions for the pairs of bubbles, we can now apply Waterman and Truell model to the assembly of pair-correlated bubbles. The full multiple scattering process occurring within a pair is included (thanks to f_a and f_s). However, we neglect the recurrent sequences (loops) and the position correlations between distinct pairs. One then obtains

$$\frac{\chi_{\text{eff}}}{\chi_0} = 1 + \frac{4\pi(n/2)}{k_0^2} f_s, \tag{S12a}$$

$$\frac{\rho_{\text{eff}}}{\rho_0} = 1 + \frac{4\pi(n/2)}{k_0^2} f_a \tag{S12b}$$

where the $n/2$ term comes from the fact that pairs are half as concentrated as single scatterers.
Let us introduce the following parameters:

$$\begin{aligned}
\omega_1 &= \omega_0/\sqrt{1 + a/d} \\
\Omega_s &= (\omega_0/\omega)^2 - (\omega_0/\omega_1)^2 \\
B_s &= 4\pi n a/k_0^2 \\
\Delta_s &= 2k_0 a + \delta
\end{aligned}$$

and

$$\begin{aligned}
\omega_2 &= \omega_0/\sqrt{1 - a/d} \\
\Omega_a &= (\omega_0/\omega)^2 - (\omega_0/\omega_2)^2 \\
B_a &= \pi n d^2 a \\
\Delta_a &= k_0^3 a d^2/6 + \delta
\end{aligned}$$

Equations (S12) then become

$$\frac{\chi_{\text{eff}}}{\chi_0} = 1 + \frac{B_s}{\Omega_s - \mathrm{i}\Delta_s}, \tag{S13a}$$

$$\frac{\rho_{\text{eff}}}{\rho_0} = 1 + \frac{B_a}{\Omega_a - \mathrm{i}\Delta_a}. \tag{S13b}$$

Their arguments can be written as

$$\begin{aligned}
\arg[\chi_{\text{eff}}] &= \arg\Big[\frac{\Omega_s + B_s - \mathrm{i}\Delta_s}{\Omega_s - \mathrm{i}\Delta_s}\Big] \\
&= \arg\big[\big(\Omega_s + B_s - \mathrm{i}\Delta_s\big)\big(\Omega_s + \mathrm{i}\Delta_s\big)\big],
\end{aligned} \tag{S14}$$

$$\arg[\rho_{\mathrm{eff}}] = \arg\Big[\frac{\Omega_a + B_a - \mathrm{i}\Delta_a}{\Omega_a - \mathrm{i}\Delta_a}\Big] \\ = \arg\big[\big(\Omega_a + B_a - \mathrm{i}\Delta_a\big)\big(\Omega_a + \mathrm{i}\Delta_a\big)\big]. \tag{S15}$$

Both expressions have the same form, but they contain a significant difference: while B_s can be large (because it is proportional to $1/k_0^2$), B_a is small. We will see that the condition for negative density will thus be more difficult to fulfill.

The condition for the real part of χ_{eff} or ρ_{eff} to be negative takes the following form in both cases:

$$\Omega^2 + B\Omega + \Delta^2 < 0, \tag{S16}$$

where the two cases can be distinguished by adding subscript s for χ, and a for ρ. The roots of this equation are

$$\Omega = -\frac{B}{2} \pm \frac{\sqrt{B^2 - 4\Delta^2}}{2}, \tag{S17}$$

leading to a simple criterion for obtaining negativity:

$$B > 2\Delta. \tag{S18}$$

i) Negative compressibility
As B_s can be large, criterion (S18) is easy to satisfy. For instance, for the case considered in Fig. 1 (main document), at resonance ($\omega = \omega_0$), $B_s \simeq 4$ and $\Delta_s \simeq 0.2 + \delta$. Except in the case of very large dissipation, we therefore have $B_s \gg 2\Delta_s$, and the condition $\mathrm{Re}(\chi_{\mathrm{eff}}) < 0$ can be satisfied over a large frequency range.

ii) Negative density
The same condition for density is not as easy to satisfy, because B_a is smaller. In the example of Fig. 1 (main document), $B_a \simeq 0.05$. The radiative part of Δ_a is also much smaller than in the symmetrical case ($k_0^3 a d^2/6 \simeq 6 \times 10^{-4}$), which means that negative density is possible when dissipation is neglected, as shown in Figs. 1, 2 and 3. For the realistic case with losses, however, dissipation makes criterion (S18) unsatisfied.

ii) Negative index
Negative refraction does not require double-negativity. It is enough to satisfy condition $\arg[\rho_{\mathrm{eff}}\chi_{\mathrm{eff}}] > \pi$. This condition is easier to satisfy close to ω_2, the frequency of the antisymmetrical mode, where we have

$$\arg[\rho_{\mathrm{eff}}\chi_{\mathrm{eff}}] \simeq \pi + \frac{\Delta_s}{\Omega_s} + \frac{B_a}{\Delta_a}, \tag{S19}$$

from which we can establish the following criterion for negative refraction:

$$\pi n d^2 a > \frac{(\delta + 2k_0 a)(\delta + k_0^3 a d^2/6)}{1 + a/d - \omega_0^2/\omega^2} \tag{S20}$$