Supplemental Material - Acoustic double negativity induced by position correlations within a disordered set of monopolar resonators

Maxime Lanoy,1,2,3 John H. Page,1 Geoffroy Lerosey,2 Fabrice Lemoult,2 Arnaud Tourin,2 and Valentin Leroy3

1Department of Physics and Astronomy, University of Manitoba, Winnipeg, Manitoba R3T 2N2, Canada
2Institut Langevin, ESPCI ParisTech, CNRS (UMR 7587), PSL Research University, Paris, France
3Laboratoire Matière et Systèmes Complexes, Université Paris-Diderot, CNRS (UMR 7057), Paris, France

(Dated: November 27, 2017)

I. SCATTERING OF A PAIR OF MONOPOLES

We consider a plane wave impinging on a pair of scatterers placed at \( \mathbf{r}_A \) and \( \mathbf{r}_B \) (see Fig. S1). The pressure scattered at point \( \mathbf{r}_M \) can be written:

\[
p(\mathbf{r}_M) = p_0 + p_A e^{-ik_0 ||\mathbf{r}_M - \mathbf{r}_A||} + p_B e^{-ik_0 ||\mathbf{r}_M - \mathbf{r}_B||}
\]

with

\[
|| \mathbf{r}_M - \mathbf{r}_A || = \sqrt{d^2/4 + r_M^2 + r_M d \cos \theta}
\]

and

\[
|| \mathbf{r}_M - \mathbf{r}_B || = \sqrt{d^2/4 + r_M^2 - r_M d \cos \theta}
\]

where \( r \) is the distance to the center of the pair and \( \theta = 0 \) in the forward direction. The pressures exerted on \( A \) and \( B \) can be obtained by inverting the multiple scattering matrix of the system:

\[
\begin{bmatrix}
\begin{bmatrix}
p_A \\
p_B 
\end{bmatrix}
\end{bmatrix}
= \begin{bmatrix}
1 \\
-f(\omega) e^{ik_0 d/2}
\end{bmatrix}^{-1}
\begin{bmatrix}
p_0A \\
p_0B
\end{bmatrix}
\]

with, here,

\[
p_{0A} = e^{-ik_0 d/2} \quad \text{et} \quad p_{0B} = e^{ik_0 d/2}
\]

Finally, we extract the scattering function of a pair by dividing the scattered pressure by \( e^{ik_0 r_M / r_M} \). A zero order expansion in \( r_M / d \) yields the following expression:

\[
f_d(\theta) = \frac{d^2 f}{d^2 - f^2 e^{2ik_0 d}} e^{-ik_0 \frac{d}{2} (1 - \cos \theta)} \left( 1 + \frac{f}{d} e^{2ik_0 d} \right)
\]

We can then easily determine analytic expressions for the forward and backward scattering:

\[
f_d(0) = \frac{d^2 f}{d^2 - f^2 e^{2ik_0 d}} \left( 2 + \frac{f}{d} \right)
\]

FIG. S1: A pair of scatterers of radius \( a \), separated by a distance \( d \).
\begin{equation}
    f_d(\pi) = \frac{d^2 f}{d^2 - f^2 e^{2ikd}} \left[ 2 \cos(k_0 d) + 2 \frac{f}{d} e^{i k d} \right] \tag{S8}
\end{equation}

As in Eq. (2) of the main document, the expression for the scattering function is:

\begin{equation}
    f = \frac{-a}{1 - \omega_0^2/\omega^2 + i(k_0 a + \delta)} \tag{S9}
\end{equation}

After substituting this expression for \( f \) in Eqs. (S7) and (S8), one can obtain the symmetric and antisymmetric parts of the scattering function:

\begin{equation}
    f_s = \frac{f_d(0) + f_d(\pi)}{2} = \frac{2a}{(\omega_0/\omega)^2 - (1 + \frac{a}{2}) - i(2k_0 a + \delta)} \tag{S10}
\end{equation}

\begin{equation}
    f_a = \frac{f_d(0) - f_d(\pi)}{2} = \frac{k_0^2 d^2 a/2}{(\omega_0/\omega)^2 - (1 - \frac{a}{2}) - i(k_0^3 a d^2/6 + \delta)} \tag{S11}
\end{equation}

\section{II. NEGATIVE REFRACTION}

Knowing the forward and backward scattering functions for the pairs of bubbles, we can now apply Waterman and Truell model to the assembly of pair-correlated bubbles. The full multiple scattering process occurring within a pair is included (thanks to \( f_a \) and \( f_s \)). However, we neglect the recurrent sequences (loops) and the position correlations between distinct pairs. One then obtains

\begin{equation}
    \frac{\chi_{\text{eff}}}{\chi_0} = 1 + \frac{4\pi(n/2)}{k_0^2} f_s, \tag{S12a}
\end{equation}

\begin{equation}
    \frac{\rho_{\text{eff}}}{\rho_0} = 1 + \frac{4\pi(n/2)}{k_0^2} f_a, \tag{S12b}
\end{equation}

where the \( n/2 \) term comes from the fact that pairs are half as concentrated as single scatterers.

Let us introduce the following parameters:

\begin{align*}
    \omega_1 &= \omega_0/\sqrt{1 + a/d} \\
    \Omega_s &= (\omega_0/\omega)^2 - (\omega_0/\omega_1)^2 \\
    B_s &= 4\pi na/k_0^2 \\
    \Delta_s &= 2k_0 a + \delta \\
\end{align*}

and

\begin{align*}
    \omega_2 &= \omega_0/\sqrt{1 - a/d} \\
    \Omega_a &= (\omega_0/\omega)^2 - (\omega_0/\omega_2)^2 \\
    B_a &= \pi na d^2 a \\
    \Delta_a &= k_0^3 a d^2 / 6 + \delta \\
\end{align*}

Equations (S12) then become

\begin{equation}
    \frac{\chi_{\text{eff}}}{\chi_0} = 1 + \frac{B_s}{\Omega_s - i\Delta_s}, \tag{S13a}
\end{equation}

\begin{equation}
    \frac{\rho_{\text{eff}}}{\rho_0} = 1 + \frac{B_a}{\Omega_a - i\Delta_a}. \tag{S13b}
\end{equation}

Their arguments can be written as

\begin{equation}
    \arg[\chi_{\text{eff}}] = \arg\left\{ \frac{\Omega_s + B_s - i\Delta_s}{\Omega_s - i\Delta_s} \right\} = \arg\left[ (\Omega_s + B_s - i\Delta_s)(\Omega_s + i\Delta_s) \right], \tag{S14}
\end{equation}
\[
\arg[\rho_{\text{eff}}] = \arg \left[ \frac{\Omega_a + B_a - i\Delta_a}{\Omega_a - i\Delta_a} \right] = \arg \left[ (\Omega_a + B_a - i\Delta_a)(\Omega_a + i\Delta_a) \right].
\]

(S15)

Both expressions have the same form, but they contain a significant difference: while \( B_s \) can be large (because it is proportional to \( 1/k_{0}^2 \)), \( B_a \) is small. We will see that the condition for negative density will thus be more difficult to fulfill.

The condition for the real part of \( \chi_{\text{eff}} \) or \( \rho_{\text{eff}} \) to be negative takes the following form in both cases:

\[
\Omega^2 + B\Omega + \Delta^2 < 0,
\]

(S16)

where the two cases can be distinguished by adding subscript \( s \) for \( \chi \), and \( a \) for \( \rho \). The roots of this equation are

\[
\Omega = -\frac{B}{2} \pm \frac{\sqrt{B^2 - 4\Delta^2}}{2},
\]

(S17)

leading to a simple criterion for obtaining negativity:

\[
B > 2\Delta.
\]

(S18)

i) Negative compressibility

As \( B_s \) can be large, criterion (S18) is easy to satisfy. For instance, for the case considered in Fig. 1 (main document), at resonance (\( \omega = \omega_0 \)), \( B_s \simeq 4 \) and \( \Delta_s \simeq 0.2 + \delta \). Except in the case of very large dissipation, we therefore have \( B_s \gg 2\Delta_s \), and the condition \( \text{Re}(\chi_{\text{eff}}) < 0 \) can be satisfied over a large frequency range.

ii) Negative density

The same condition for density is not as easy to satisfy, because \( B_a \) is smaller. In the example of Fig. 1 (main document), \( B_a \simeq 0.05 \). The radiative part of \( \Delta_a \) is also much smaller than in the symmetrical case (\( k_0^2\alpha d^2/6 \simeq 6 \times 10^{-4} \)), which means that negative density is possible when dissipation is neglected, as shown in Figs. 1, 2 and 3. For the realistic case with losses, however, dissipation makes criterion (S18) unsatisfied.

ii) Negative index

Negative refraction does not require double-negativity. It is enough to satisfy condition \( \arg[\rho_{\text{eff}}\chi_{\text{eff}}] > \pi \). This condition is easier to satisfy close to \( \omega_2 \), the frequency of the antisymmetrical mode, where we have

\[
\arg[\rho_{\text{eff}}\chi_{\text{eff}}] \simeq \pi + \frac{\Delta_s}{\Omega_s} + \frac{B_a}{\Delta_a},
\]

(S19)

from which we can establish the following criterion for negative refraction:

\[
\pi nd^2 a > \frac{(\delta + 2k_0\alpha)(\delta + k_0^2\alpha d^2/6)}{1 + a/d - \omega_0^2/\omega^2}
\]

(S20)