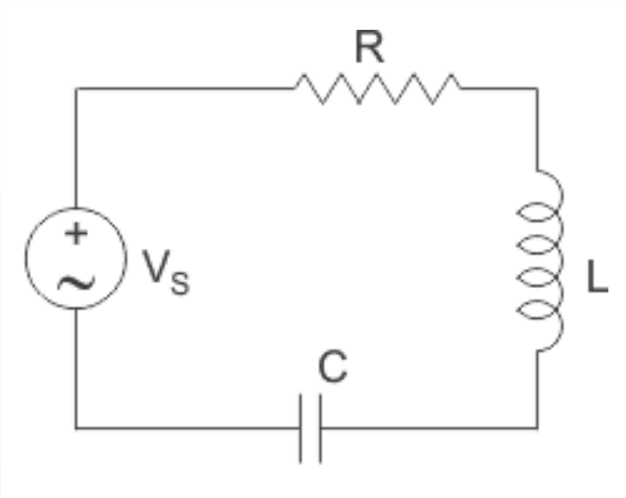
7) Resonance LRC circuit



$$z = R + j\left(\omega L - \frac{1}{\omega C}\right) = |Z|e^{j\theta}$$

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \qquad \tan\theta = \frac{\omega L - \frac{1}{\omega C}}{R}$$

Using $v = iz$, if

$$\boxed{v = Ve^{j\omega t}}$$ then $$\boxed{i = Ie^{j(\omega t - \theta)}}$$ with $$I = \frac{V}{|Z|}$$

$$i = Ie^{j(\omega t-\theta)}$$

$$I = \frac{V}{\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}}$$

$$\tan\theta = \frac{\omega L-\frac{1}{\omega C}}{R}$$

(a)Resonance

For $\omega\to\infty, \quad |Z|\to\infty, \quad i\to 0, \quad \theta\to\pi/2$

$\omega\to 0, \quad |Z|\to\infty, \quad i\to 0, \quad \theta\to -\pi/2$

current approaches zero at high and low frequencies

Maximum occurs when $\omega L = \dfrac{1}{\omega C}$

frequency dependent terms cancel

Then, $\theta\to 0, \quad |Z|\to R, \quad z\to R$, so $i = \dfrac{v}{R}$

i.e. the circuit is purely resistive — the effect of *L* and *C* cancel

Resonance frequency: $\omega_0 L = \dfrac{1}{\omega_0 C} \quad \to \omega_0{}^2 = \dfrac{1}{LC}$

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$$

$$I = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$$\tan\theta = \frac{\omega L - \frac{1}{\omega C}}{R}$$

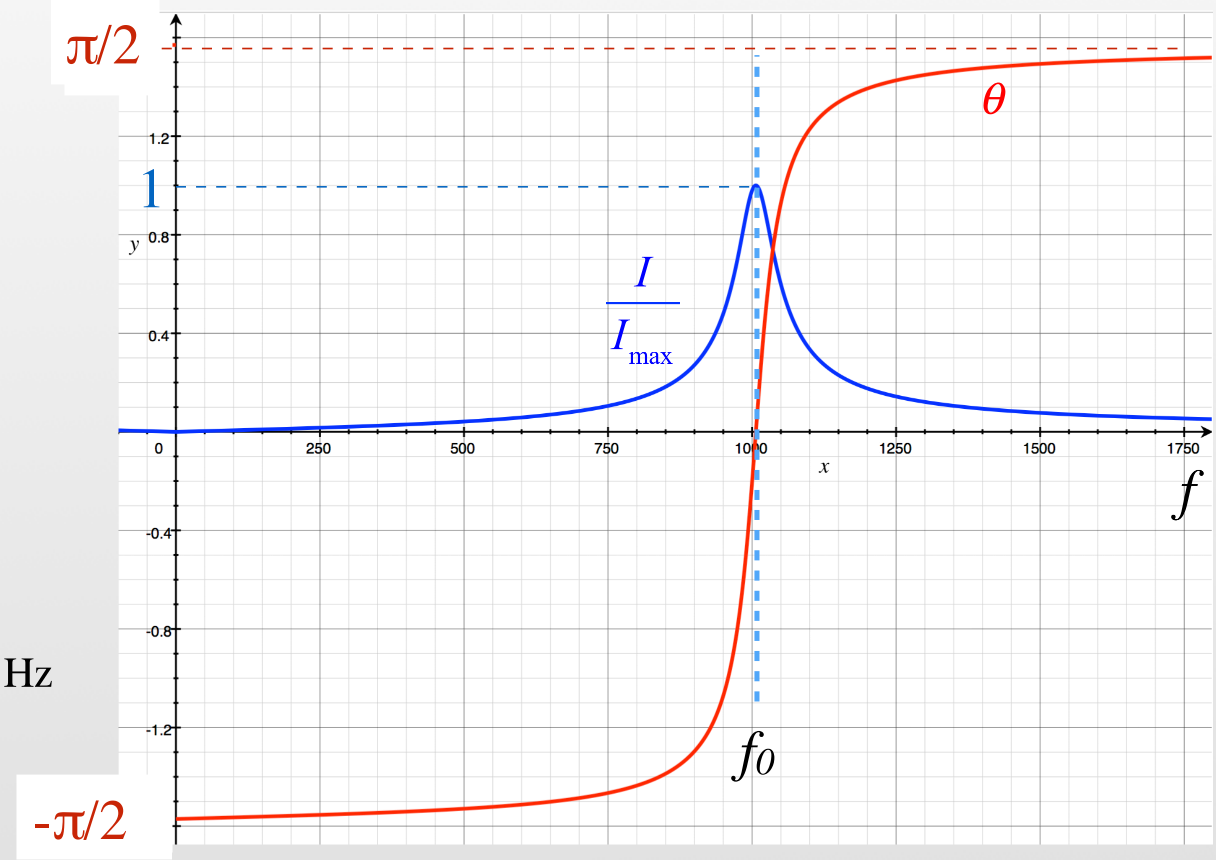
$$\boxed{\omega = \omega_0 = \frac{1}{\sqrt{LC}}}$$

$$\rightarrow \quad I(\omega_0) = I_{\max} = \frac{V}{R}$$

$$\rightarrow \quad \frac{I}{I_{\max}} = \frac{1}{\sqrt{1 + \left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)^2}}$$

L = 250 mH
C = 0.1 μF
R = 100 Ω

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}} = 1000\ \text{Hz}$$



(b) Resistor voltage

$$v_R = V_R e^{j(\omega t-\theta)} = \frac{VR}{|Z|} e^{j(\omega t-\theta)}$$

$$v_R = iR \rightarrow V_R = IR \rightarrow V_{R\max} = I_{\max} R$$

$$\rightarrow \frac{V_R}{V_{R\max}} = \frac{I}{I_{\max}}$$

e.g.

$R = 100\Omega$

$C = 0.1\mu\text{F}$

$L = 250\text{ mH}$

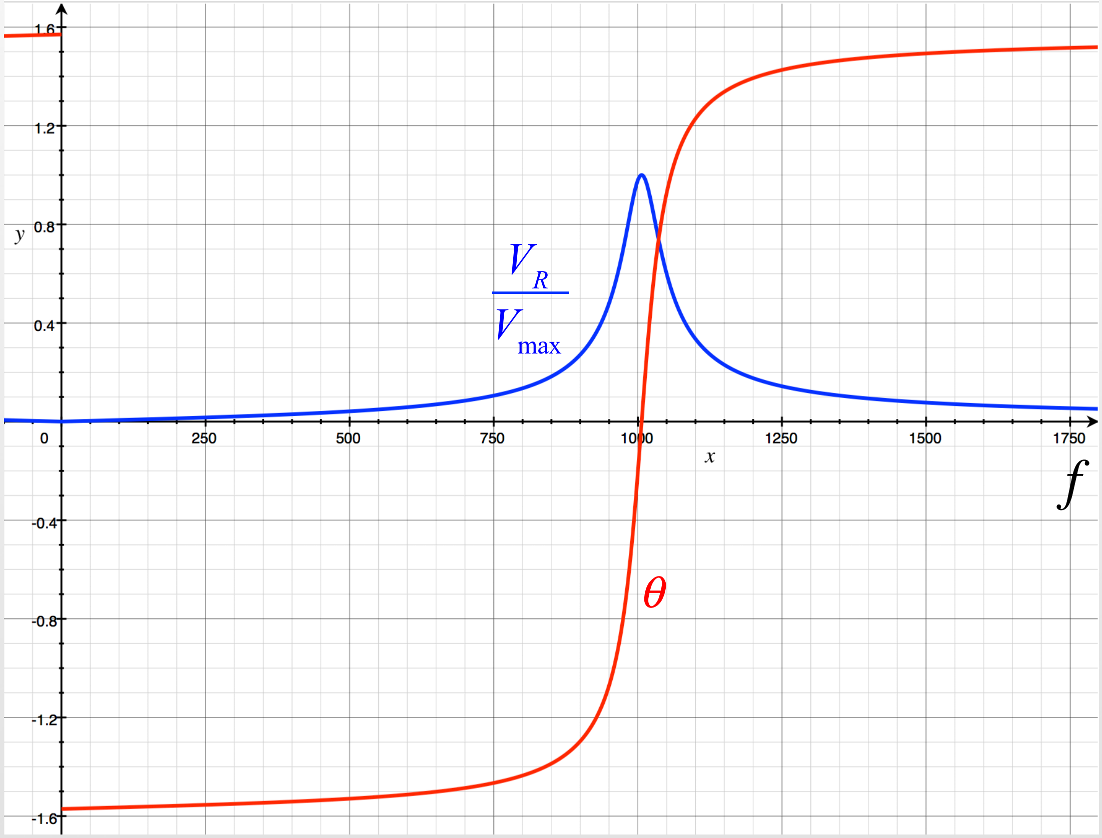
$V = 10\text{ V}$

Resonant freq:

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}} = 1006.6\text{ Hz}$$

$$I_{\max} = \frac{V}{R} = 0.10\text{ A}$$

$$V_{R\max} = I_{\max} R = \ 10\text{ V}$$



(c) Q-factor

- shape depends on the component values, *R* in particular

- *R* cannot be removed (because of *L*)

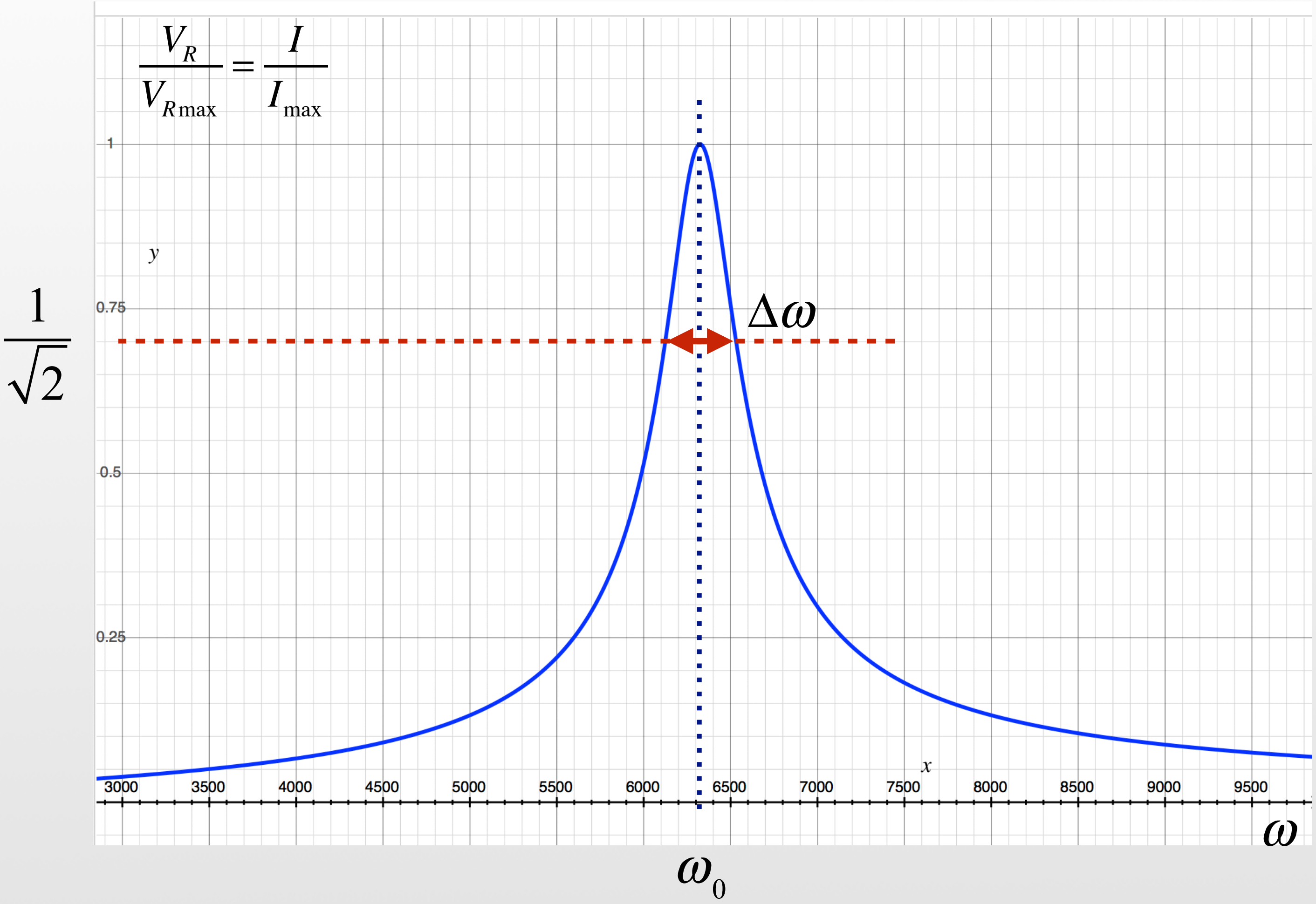
- sharper is better for resonance circuit ==> higher *Q*

$$Q \equiv \frac{\omega_0}{\Delta\omega}$$

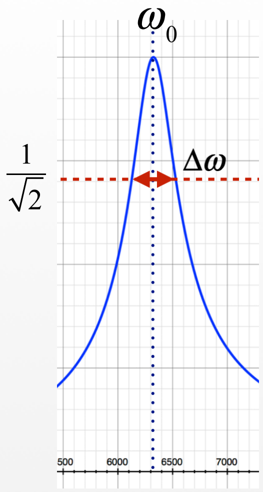
("resolution" in spectroscopy)

$\Delta\omega$ is the width at 1/2 power, or 1/√2 amplitude

$Q = \dfrac{\omega_0}{\Delta\omega}$ $\Delta\omega$ is the width at 1/2 power, or $1/\sqrt{2}$ amplitude



To find $Q(R,L,C)$, find frequencies where $\dfrac{V_R}{V_{R\max}} = \dfrac{I}{I_{\max}} = \dfrac{1}{\sqrt{2}}$



Recall $$\frac{I}{I_{\max}}=\frac{1}{\sqrt{1+\left(\frac{\omega L}{R}-\frac{1}{\omega RC}\right)^2}}$$

so, $$\frac{I}{I_{\max}}=\frac{1}{\sqrt{2}}$$

when $$\left(\frac{\omega L}{R}-\frac{1}{\omega RC}\right)=1$$

$$\frac{\omega L}{R}\left(1-\frac{1}{\omega^2 LC}\right)=1 \qquad \times\omega^2$$

$$\left(1-\frac{1}{\omega^2/\omega_0{}^2}\right)=\frac{R}{\omega L}$$

$$\left(\omega^2-\omega_0{}^2\right)=\frac{R\omega}{L}$$

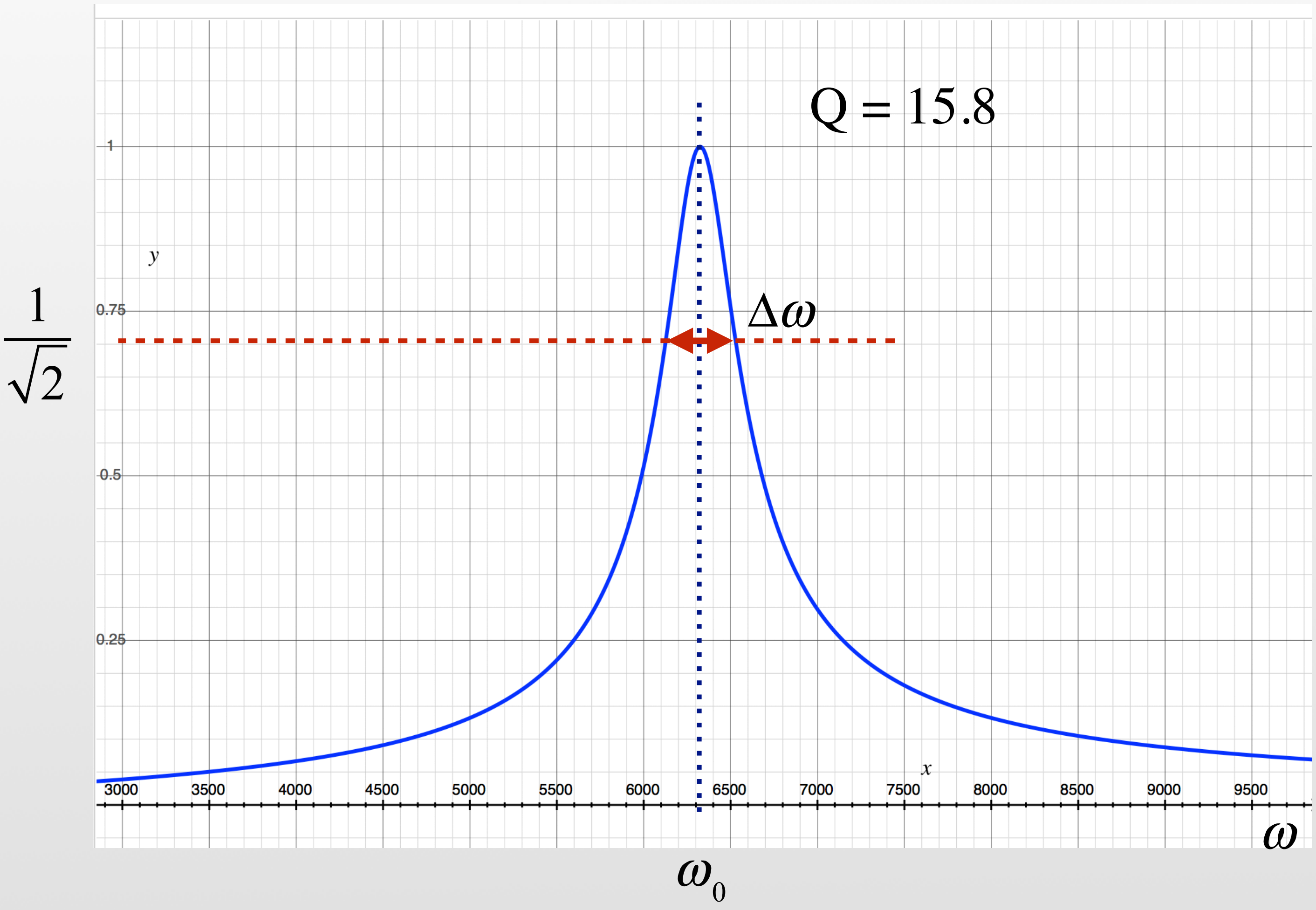
$$\underbrace{\left(\omega-\omega_0\right)}_{\frac{\Delta\omega}{2}}\underbrace{\left(\omega+\omega_0\right)}_{2\omega}=\frac{R\omega}{L}$$

For $\omega \gg \Delta\omega, \quad \omega \cong \omega_0, \quad \omega+\omega_0 \cong 2\omega_0$

$$\rightarrow \Delta\omega \cong \frac{R}{L} \qquad \rightarrow Q=\frac{\omega_0}{\Delta\omega}=\boxed{\frac{\omega_0 L}{R}}$$

$$\rightarrow \qquad \boxed{Q=\frac{\sqrt{L/C}}{R}}$$

$$Q = \frac{\omega_0}{\Delta\omega} \rightarrow \boxed{Q = \frac{1}{R}\sqrt{\frac{L}{C}}}$$



L = 250 mH

C = 0.1 μF

R = 100 Ω

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

smaller R, higher Q, sharper peak

L = 250 mH

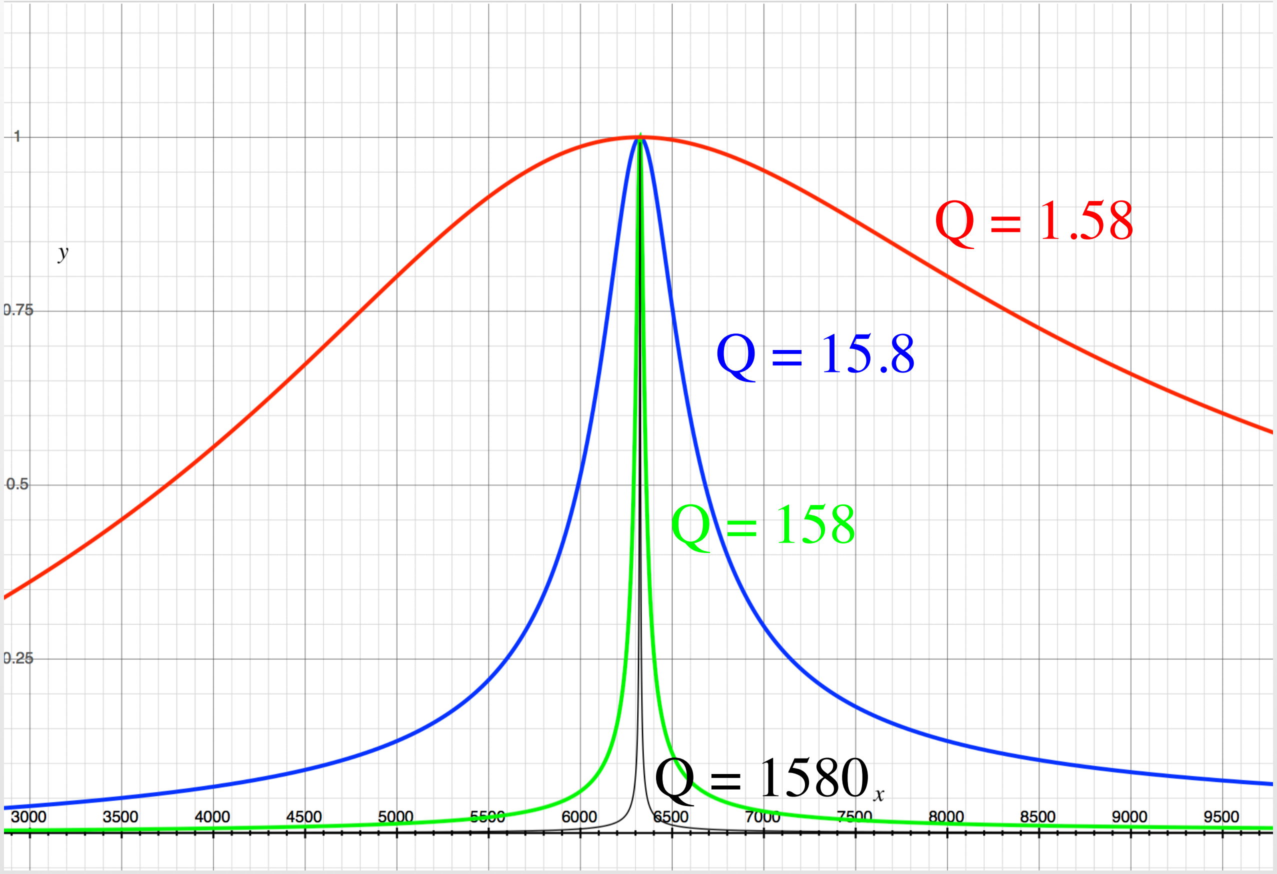
C = 0.1 μF

R = 1000 Ω

R = 100 Ω

R = 10 Ω

R = 1 Ω



(d) Capacitor voltage

Peak Value (numerically):

$$V_{c\max} = I_{\max} X_C = \frac{I_{\max}}{\omega_0 C} = \frac{0.10\text{A}}{2\pi(1000\text{Hz})(0.1\mu\text{F})} = 158\ \text{V}$$

function of time:

$$v_C = i z_C = \frac{v}{z} z_C = \frac{v}{j\omega C\ z}$$

but $z = |Z| e^{j\theta}, \quad j\omega C = \omega C e^{j\pi/2}$

so $$v_C = \frac{V e^{j\omega t}}{\omega C |Z| e^{j(\theta+\pi/2)}}$$

$$\boxed{v_C = \frac{V}{\omega C |Z|} e^{j(\omega t \underbrace{-\theta-\pi/2}_{\beta})}}$$

$$\boxed{v_C = \frac{V}{\omega C|Z|} e^{\underbrace{j(\omega t - \theta - \pi/2)}_{\beta}}}$$

Amplitude:

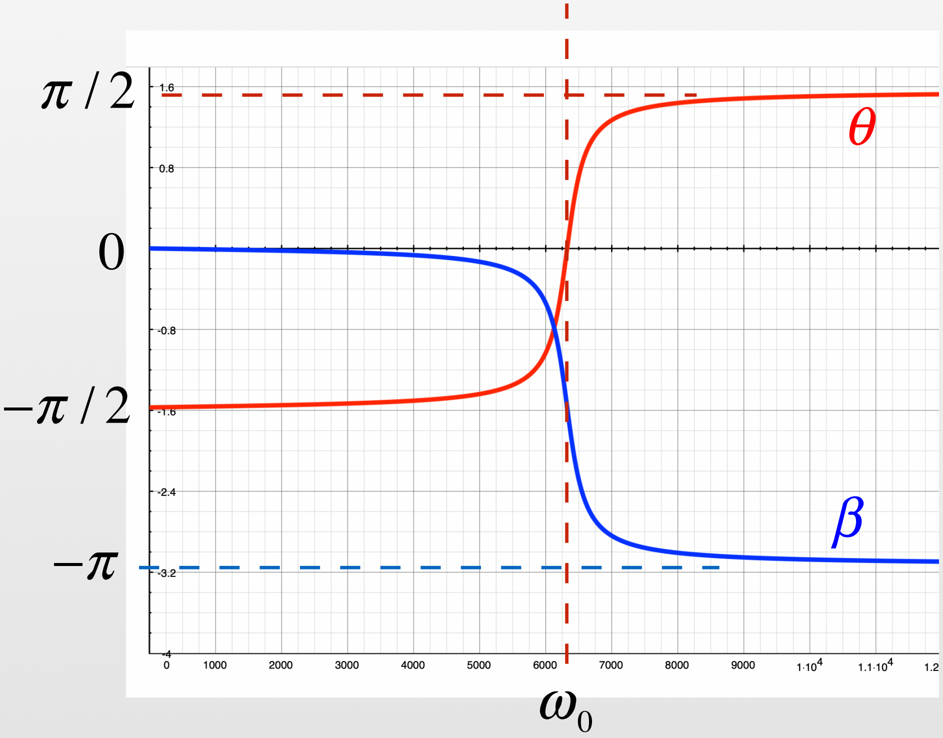
$$V_C = \frac{V}{\omega C|Z|} = \frac{V}{\sqrt{(\omega RC)^2 + (1-\omega^2 LC)^2}}$$

At resonance, $\dfrac{V_C}{V} = \dfrac{1}{\omega_0 RC} = \dfrac{1}{R}\sqrt{\dfrac{L}{C}} = Q$

$$v_C = \frac{V}{\omega C|Z|} e^{j(\omega t \underbrace{-\theta-\pi/2}_{\beta})}$$

$$\tan\theta = \frac{\omega L - \frac{1}{\omega C}}{R}$$

Phase: $\beta = -\theta - \pi/2$



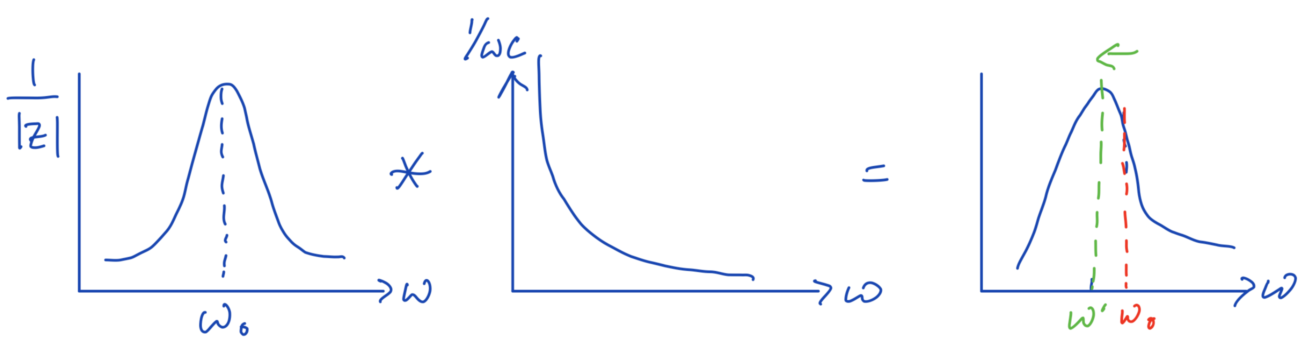
$$\boxed{v_C = \frac{V}{\omega C|Z|} e^{j(\omega t - \theta - \pi/2)}}$$

$$\tan\theta = \frac{\omega L - \dfrac{1}{\omega C}}{R}$$

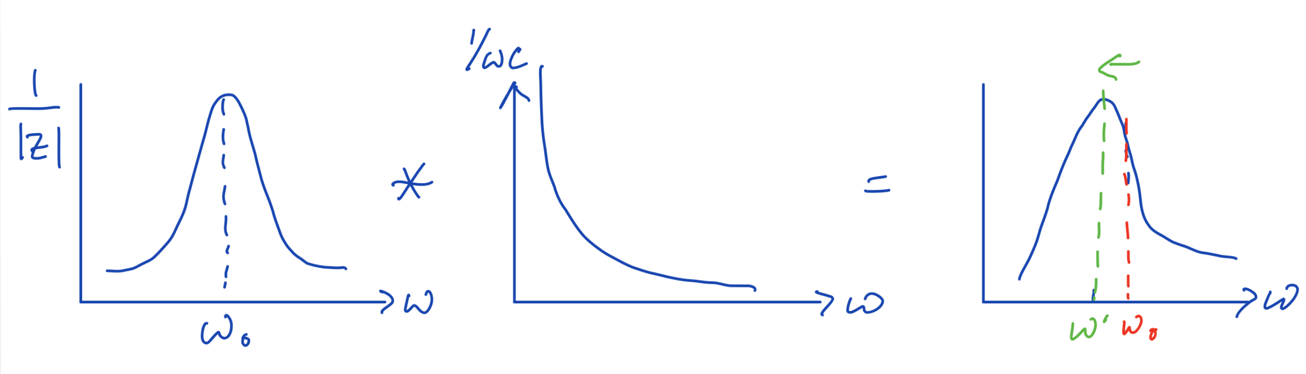
Capacitor resonance:

Recall, for the resistor, $v_R = \dfrac{V}{|Z|} e^{j(\omega t - \theta)}$ has max amplitude when $\dfrac{1}{|Z|}$ is max

Similarly, v_C has max amplitude when $\dfrac{1}{\omega C|Z|}$ is max



Similarly, v_C has max amplitude when $\dfrac{1}{\omega C|Z|}$ is max



To find ω', set $\dfrac{d}{d\omega}\dfrac{1}{\omega C|Z|}=0$ → $\omega'^2 = \omega_0{}^2 - \dfrac{R^2}{2L^2}$

e.g.
$R = 100\Omega$
$C = 0.1\mu\text{F}$
$L = 250\text{ mH}$
$V = 10\text{ V}$

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}} = 1006.6\text{ Hz}$$

$$f' = \frac{\omega'}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}} = 1005.6\text{ Hz}$$

(e) Inductor voltage

Peak Value (numerically):

$$V_{L\max} = I_{\max} X_L = \omega_0 L I_{\max} \quad = 2\pi(1000\text{Hz})(250\text{mH})(0.10\text{A}) = 158\ \text{V}$$

function of time:

$$v_L = i z_L = \frac{v}{z} j\omega L \qquad \text{but} \qquad z = |Z| e^{j\theta}, \quad j\omega L = \omega L e^{j\pi/2}$$

so

$$v_L = V e^{i\omega t} \frac{\omega L}{|Z|} e^{j(-\theta+\pi/2)}$$

$$\boxed{v_L = \frac{V}{|Z|/(\omega L)} e^{j(\omega t\underbrace{-\theta+\pi/2}_{\alpha})}}$$

$$v_L = \frac{V}{|Z|/(\omega L)} e^{\underbrace{j(\omega t - \theta + \pi/2)}_{\alpha}}$$

Amplitude:

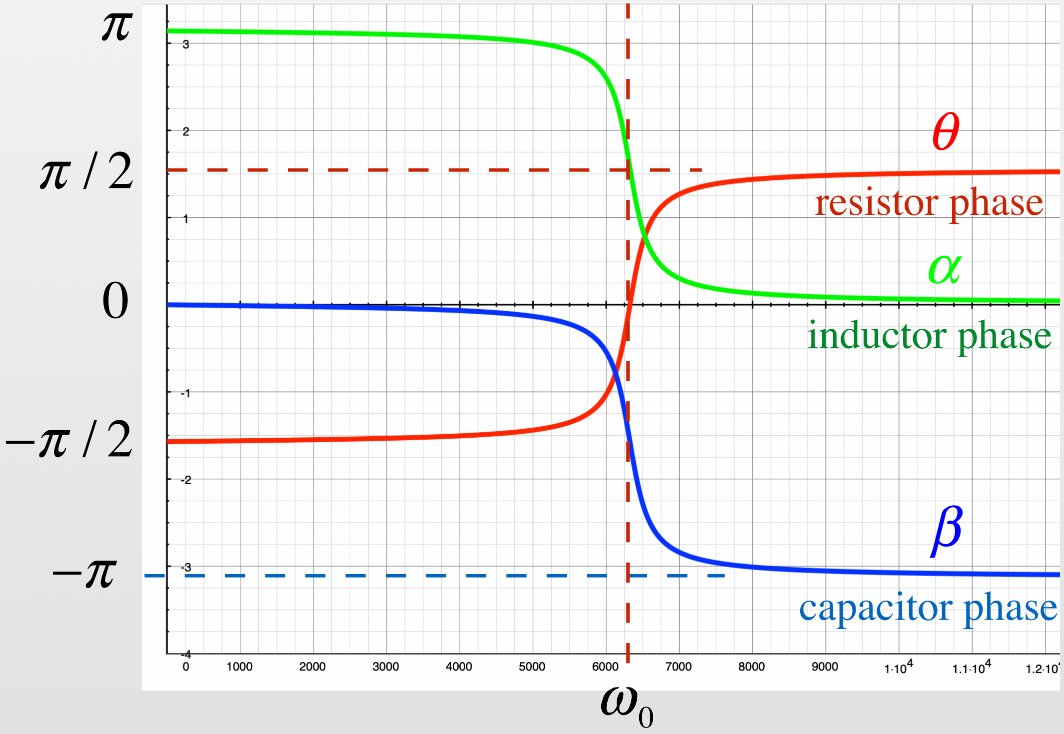
$$V_L = \frac{V}{|Z|/(\omega L)} = \frac{V}{\sqrt{\left(\frac{R}{\omega L}\right)^2 + \left(1 - \frac{1}{\omega^2 LC}\right)^2}}$$

At resonance, $$\frac{V_L}{V} = \frac{\omega_0 L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}} = Q$$

$$\boxed{v_L = \frac{V}{|Z|/(\omega L)} e^{j\underbrace{(\omega t-\theta+\pi/2)}_{\alpha}}}$$

$$\tan\theta = \frac{\omega L - \dfrac{1}{\omega C}}{R}$$

Phase: $\alpha = -\theta + \pi / 2$



$$\boxed{\alpha - \beta = \pi}$$

capacitor and inductor voltage always 180° out of phase

At resonance, the amplitudes are the same, so

$$v_C + v_L = 0$$

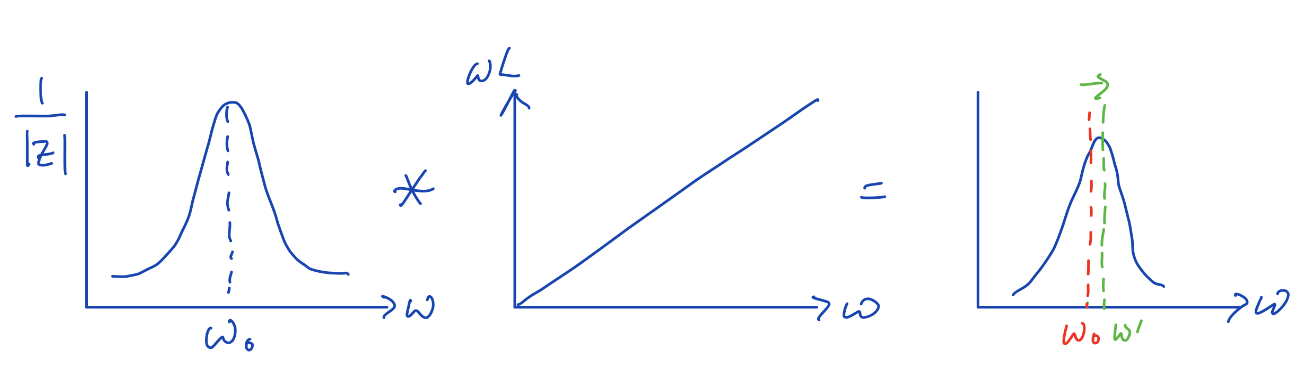
$$\boxed{v_L = \frac{V}{|Z|/(\omega L)} e^{j(\omega t - \theta + \pi/2)}}$$

$$\tan\theta = \frac{\omega L - \dfrac{1}{\omega C}}{R}$$

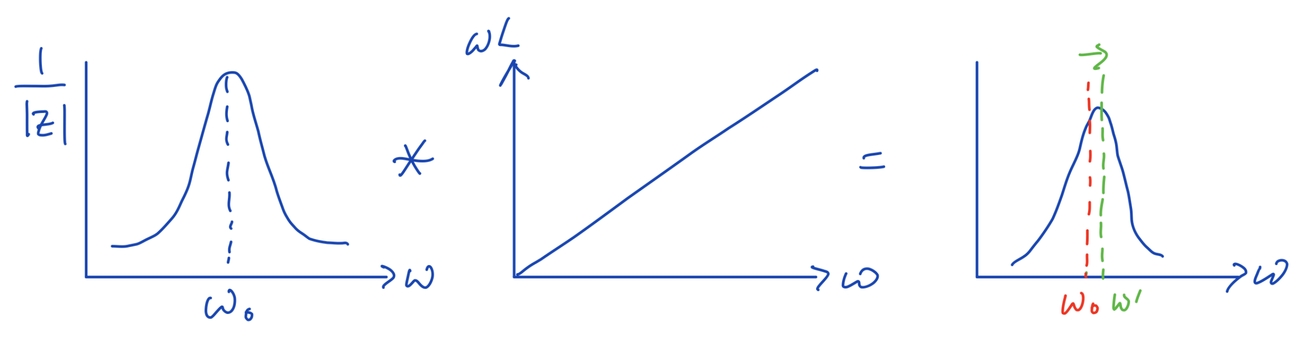
Inductor resonance:

Recall, for the resistor, $v_R = \dfrac{V}{|Z|} e^{j(\omega t - \theta)}$ has max amplitude when $\dfrac{1}{|Z|}$ is max

Similarly, v_L has max amplitude when $\dfrac{1}{|Z|/(\omega L)} = \dfrac{\omega L}{|Z|}$ is max



Similarly, v_L has max amplitude when $\dfrac{1}{|Z|/(\omega L)} = \dfrac{\omega L}{|Z|}$ is max



To find ω', set $\dfrac{d}{d\omega}\dfrac{\omega L}{|Z|} = 0$ $\Rightarrow$ $\omega' = \dfrac{1}{\sqrt{\frac{1}{\omega_0{}^2} - \frac{(RC)^2}{2}}}$

e.g.

$R = 100\Omega$

$C = 0.1\mu\text{F}$

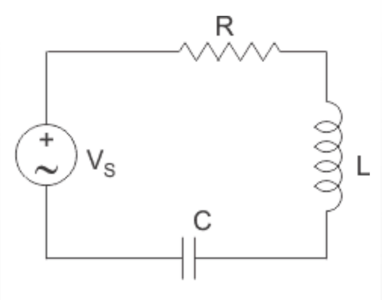
$L = 250\text{ mH}$

$V = 10\text{ V}$

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}} = 1006.6\text{ Hz}$$

$$f' = \frac{\omega'}{2\pi} = \frac{1}{2\pi}\frac{1}{\sqrt{\frac{1}{\omega_0{}^2} - \frac{(RC)^2}{2}}} = 1007.6\text{Hz}$$

$$v = Ve^{j\omega t}$$

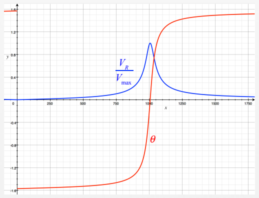


$$i = \frac{V}{|Z|} e^{j(\omega t-\theta)}$$

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

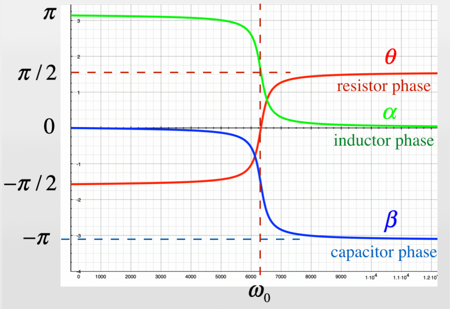
$$\tan\theta = \frac{\omega L - \frac{1}{\omega C}}{R}$$

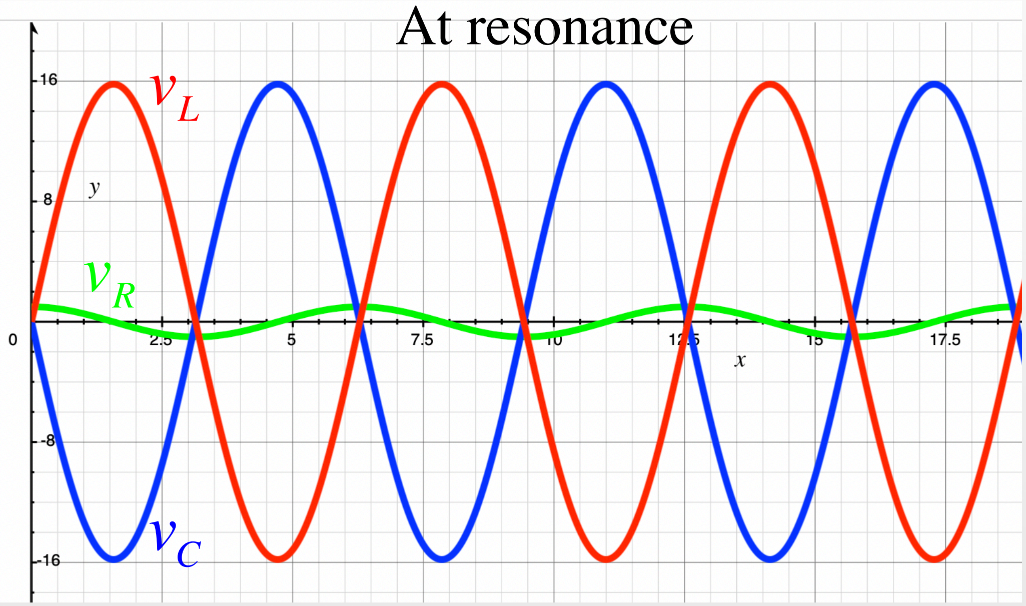
$$v_R = \frac{VR}{|Z|} e^{j(\omega t-\theta)}$$



$$v_C = \frac{V}{\omega C|Z|} e^{j(\omega t-\theta-\pi/2)}$$

$$v_L = \frac{V}{|Z|/(\omega L)} e^{j(\omega t-\theta+\pi/2)}$$

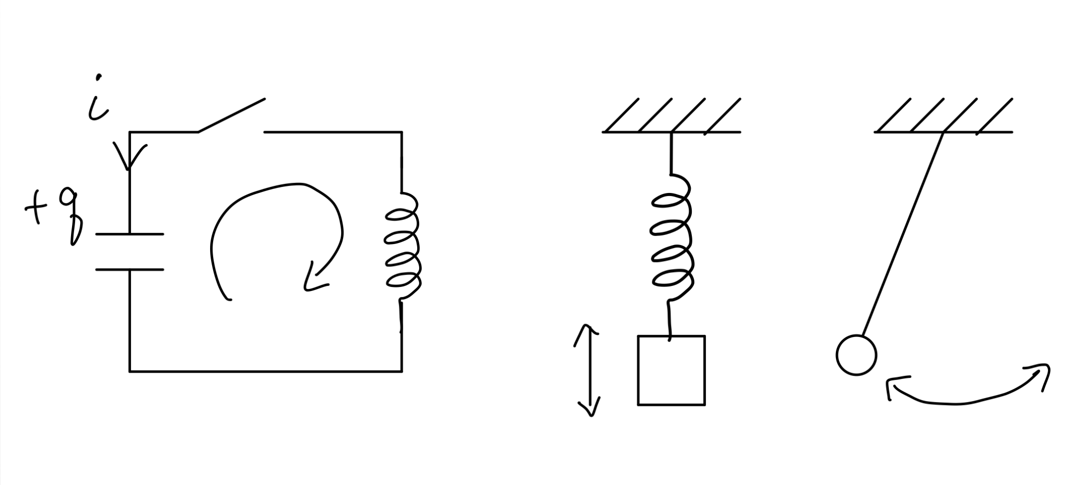




(f) Notch filter (band stop filter)

(g) parallel resonance

8) LC Oscillations



Mass on a spring:

$$F=-kx=m\ddot{x} \Longrightarrow \ddot{x}+\tfrac{k}{m}x=0$$

Solution: $x=X\cos(\omega t+\varphi)$

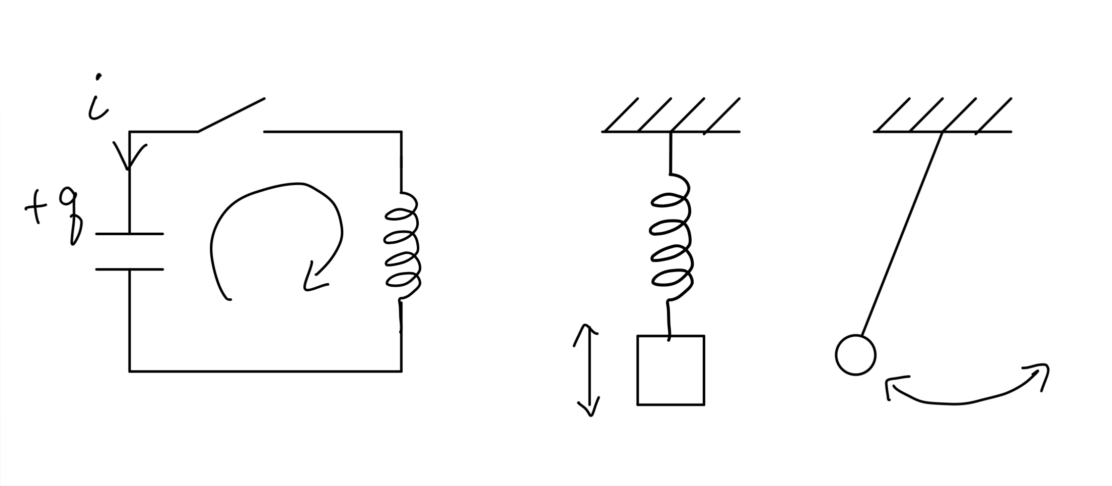
where $\omega=\sqrt{k/m}$

Total energy: $U=\tfrac{1}{2}mv^2+\tfrac{1}{2}kx^2$

If there is no energy dissipation, $\dfrac{dU}{dt}=0$

$$\Longrightarrow m\cancel{v}\frac{dv}{dt}+kx\cancel{\frac{dx}{dt}}=0$$

$$\Longrightarrow \ddot{x}+\tfrac{k}{m}x=0$$



LC oscillator:

$$\tfrac{1}{2}Cv^2=\tfrac{1}{2}C\left(\frac{q}{C}\right)^2=\tfrac{1}{2}q^2/C$$

$$\frac{q}{C}+L\frac{di}{dt}=0 \Rightarrow \ddot{q}+\tfrac{1}{LC}q=0$$

Solution: $q=Q\cos(\omega t+\varphi)$

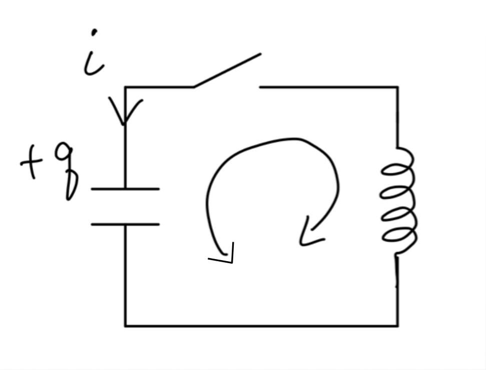
where $\omega=\dfrac{1}{\sqrt{LC}}$

Total energy: $U=\tfrac{1}{2}Li^2+\tfrac{1}{2}q^2/C$

If there is no energy dissipation, $\dfrac{dU}{dt}=0$

$$\Rightarrow L\cancel{i}\frac{di}{dt}+\frac{q}{C}\frac{\cancel{dq}}{\cancel{dt}}=0$$

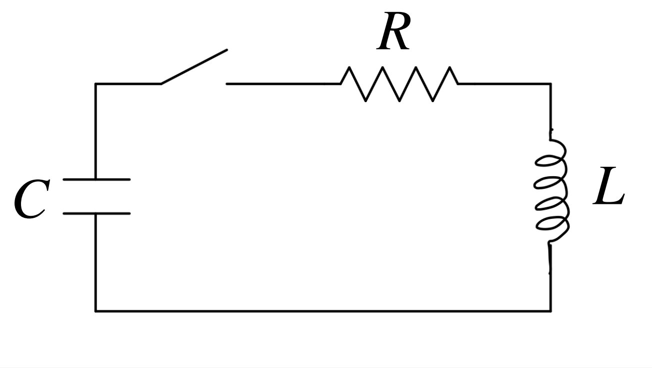
$$\Rightarrow \ddot{q}+\tfrac{1}{LC}q=0$$

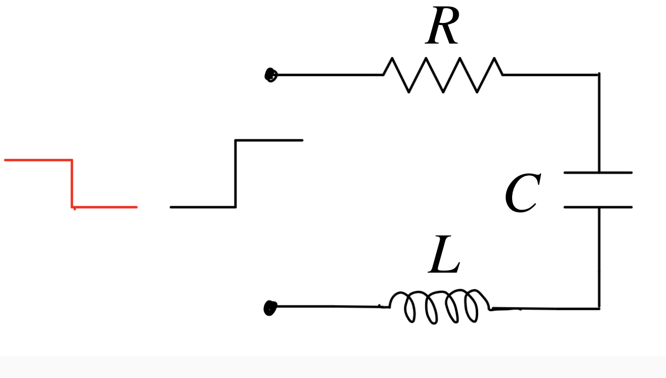


$$q = Q\cos(\omega t + \varphi) \qquad \omega = \frac{1}{\sqrt{LC}}$$

⇨ $$i = \frac{dq}{dt} = -Q\omega\sin(\omega t + \varphi)$$

9)Transient response of LRC circuit *(damped harmonic oscillator)*



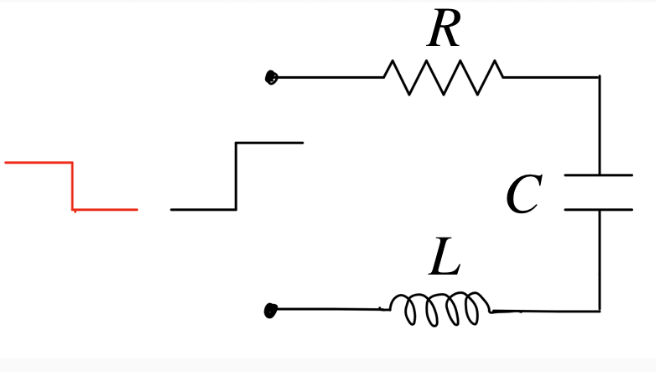


K.L. $\frac{q}{C}+L\frac{di}{dt}+iR=0$

Define $\gamma=\frac{R}{L}$ and use $\omega_0=\frac{1}{\sqrt{LC}}$

$$\ddot{q}+\frac{R}{L}\dot{q}+\frac{q}{LC}=0$$

$$\ddot{q}+\gamma\dot{q}+\omega_0{}^2q=0$$



$$\ddot{q}+\gamma\dot{q}+\omega_0{}^2q=0$$

Solution: $q=Qe^{\lambda t}$ Then $\dot{q}=Q\lambda e^{\lambda t}=\lambda q$

and $\ddot{q}=\lambda^2 q$

so $\lambda^2 q+\lambda\gamma q+\omega_0{}^2 q=0$

$$\lambda^2+\lambda\gamma+\omega_0{}^2=0 \quad \Rightarrow \quad \lambda=-\frac{\gamma}{2}\pm\sqrt{\left(\tfrac{\gamma}{2}\right)^2-\omega_0{}^2}$$

$$\boxed{q = Qe^{\lambda t}}$$

$$\lambda = -\frac{\gamma}{2} \pm \sqrt{\left(\tfrac{\gamma}{2}\right)^2 - \omega_0{}^2}$$

Define attenuation: $\alpha = \frac{\gamma}{2} = \frac{R}{2L}$ (also called neper frequency)

Define damping factor: $\zeta = \frac{\alpha}{\omega_0} = \frac{R}{2}\sqrt{\frac{C}{L}}$ $\left(= \frac{1}{2Q}\right)$

$$\boxed{\lambda = -\alpha \pm \sqrt{\alpha^2 - \omega_0{}^2}}$$

represents 2 particular solutions

$$\boxed{q = Qe^{\lambda t}}$$

$$\boxed{\lambda = -\alpha \pm \sqrt{\alpha^2 - \omega_0{}^2}}$$

<u>Case 1: Underdamped</u> $\alpha < \omega_0\,,\qquad \zeta < 1$

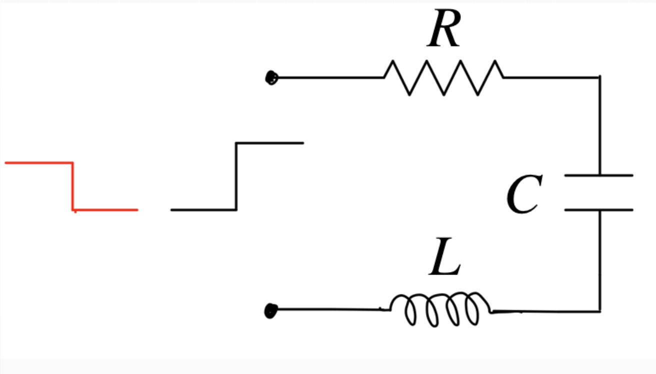
$$\lambda = -\alpha \pm j\omega_1 \qquad \omega_1 = \sqrt{\omega_0{}^2 - \alpha^2} \qquad \left(= \omega_0\sqrt{1-\zeta^2}\right)$$

General solution: $q = e^{-\alpha t}\left(Q_1 e^{j\omega_1 t} + Q_2 e^{-j\omega_1 t}\right)$

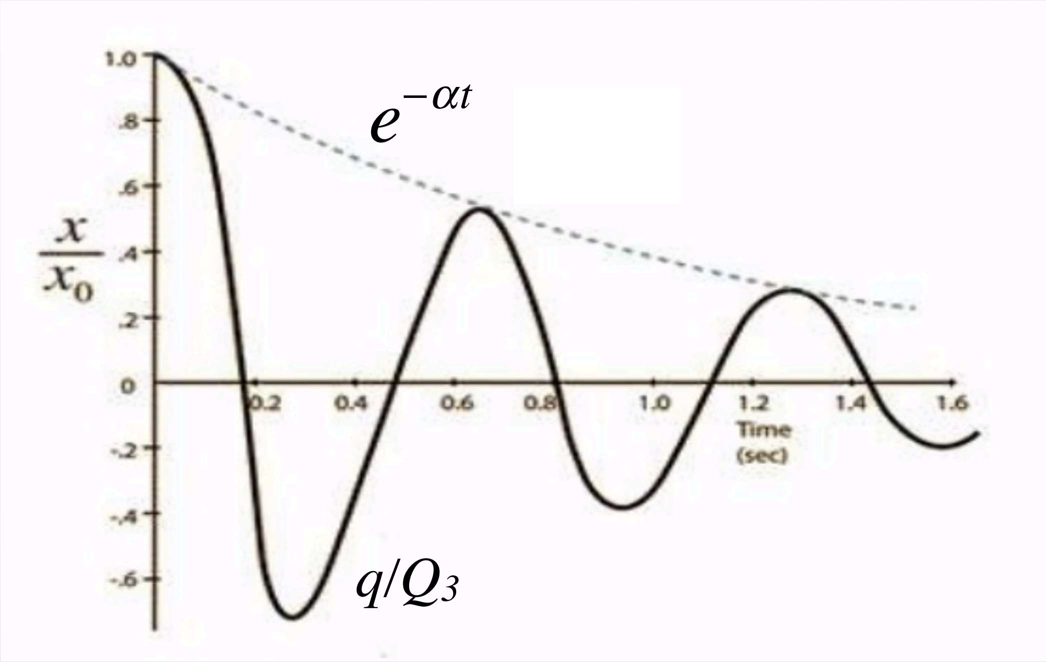
or $q = e^{-\alpha t}\left(A_1 \cos\omega_1 t + A_2 \sin\omega_1 t\right)$

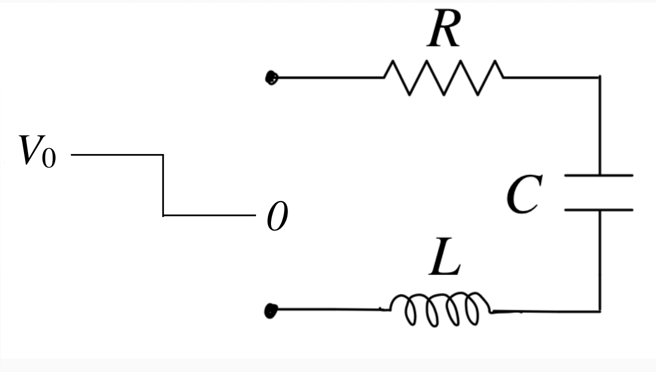
or

$$\boxed{q = \underbrace{Q_3 e^{-\alpha t}}_{\text{exponential decay}} \underbrace{\sin(\omega_1 t + \varphi)}_{\text{oscillation}}}$$



$$q = Q_3 e^{-\alpha t} \sin(\omega_1 t + \varphi)$$





$$\boxed{q = Q_3 e^{-\alpha t}\sin(\omega_1 t + \varphi)}$$

$$\omega_1 = \sqrt{\omega_0{}^2 - \alpha^2}$$

$$\alpha = \frac{R}{2L}$$

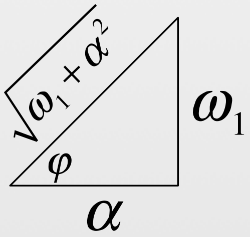
$$i = \frac{dq}{dt} = Q_3\left(-\alpha e^{-\alpha t}\sin(\omega_1 t + \varphi)\right) + e^{-\alpha t}\omega_1\cos(\omega_1 t + \varphi)$$

Boundary conditions give Q_3 and φ

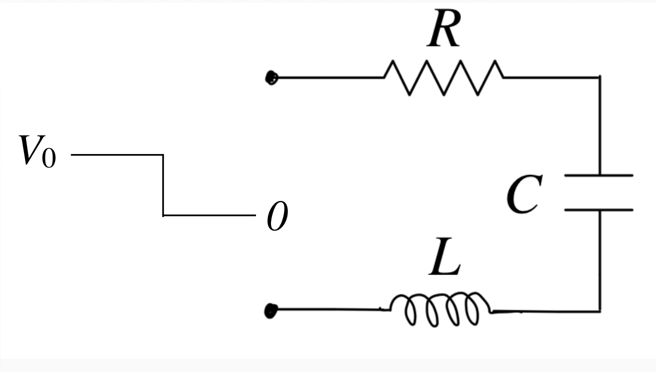
$$t = 0 \rightarrow q = CV_0 \quad \text{and} \quad i = 0$$

Then $CV_0 = Q_3 \sin\varphi$

and $0 = -\alpha\sin\varphi + \omega_1\cos\varphi \quad \rightarrow \quad \boxed{\tan\varphi = \frac{\omega_1}{\alpha}}$



⇨ $Q_3 = \frac{CV_0}{\sin\varphi} = CV_0\frac{\sqrt{\omega_1{}^2 + \alpha^2}}{\omega_1}$ ⇨ $\boxed{Q_3 = CV_0\sqrt{1 + (\alpha/\omega_1)^2}}$



$$q = Q_3 e^{-\alpha t}\sin(\omega_1 t + \varphi)$$

$$\omega_1 = \sqrt{\omega_0{}^2 - \alpha^2}$$

$$\alpha = \frac{R}{2L}$$

$$Q_3 = CV_0\sqrt{1+\left(\alpha/\omega_1\right)^2}$$

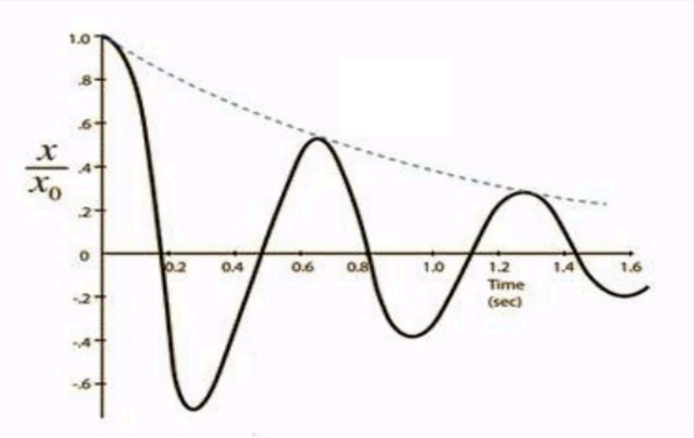
$$\tan\varphi = \frac{\omega_1}{\alpha}$$

For $\alpha \ll \omega_0 \quad (\zeta \ll 1)$, $\quad \omega_1 = \sqrt{\omega_0{}^2 - \alpha^2} \to \omega_0$

$$\tan\varphi \to \infty \ , \ \varphi \to \pi/2$$

$$q \cong CV_0 e^{-\alpha t}\cos\omega_0 t$$

or $$v_C \cong V_0 e^{-\alpha t}\cos\omega_0 t$$



$$\boxed{q = Qe^{\lambda t}} \qquad \boxed{\lambda = -\alpha \pm \sqrt{\alpha^2 - \omega_0{}^2}}$$

Case 2: Overdamped $\quad \alpha > \omega_0 , \quad \zeta > 1$ ⇒ $\lambda = -\alpha \pm \sqrt{\alpha^2 - \omega_0{}^2}$

$$\lambda = -\alpha \pm \beta$$

General solution: $\quad q = Q_1 e^{-(\alpha+\beta)t} + Q_2 e^{-(\alpha-\beta)t}$ $\quad$ Both terms are negative exponential ==> decaying exponential

Boundary conditions: $\quad t = 0 \rightarrow q = CV_0$ and $i = 0$ $\quad$ gives 2 equations for $Q1$ and $Q2$

$$q = \frac{CV_0}{2} e^{-\alpha t}\left[\left(1 - \tfrac{\alpha}{\beta}\right)e^{-\beta t} + \left(1 + \tfrac{\alpha}{\beta}\right)e^{\beta t}\right]$$

For $\alpha \gg \omega_0$, $\beta \rightarrow \alpha$ $\quad \boxed{q = CV_0 e^{-(\alpha-\beta)t}}$ or $\boxed{v_C = V_0 e^{-(\alpha-\beta)t}}$

$$\boxed{q = Qe^{\lambda t}} \qquad \boxed{\lambda = -\alpha \pm \sqrt{\alpha^2 - {\omega_0}^2}}$$

Case 3: Critical damping $\alpha = \omega_0 ,\quad \zeta = 1 \Rightarrow \lambda = -\alpha$ represents only one solution

General solution then includes a linear term (see e.g. Boas for justification):

$$q = (A + Bt)e^{-\alpha t} \Rightarrow i = Be^{-\alpha t} - (A + Bt)\alpha e^{-\alpha t}$$

Boundary conditions: $t = 0 \to q = CV_0$ and $i = 0$ $\Rightarrow$

$$\boxed{\begin{array}{l} CV_0 = A \\ 0 = B - \alpha A \to B = \alpha CV_0 \end{array}}$$

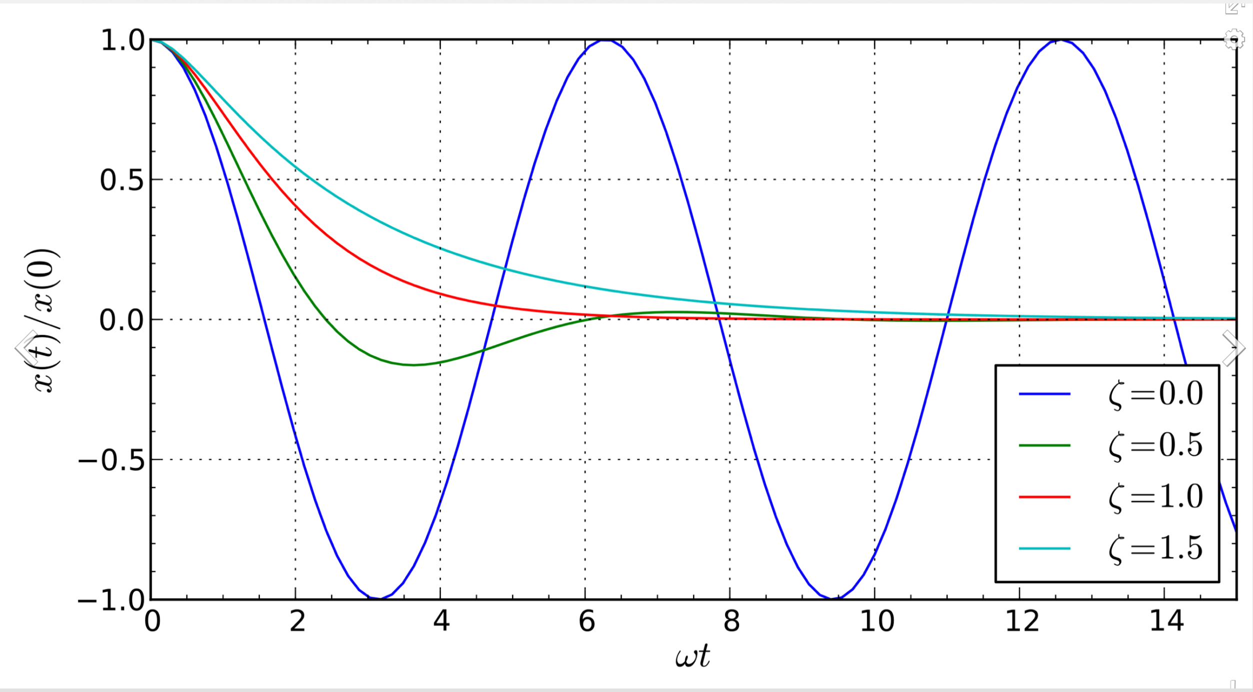
$$\boxed{q = CV_0(1 + \alpha t)e^{-\alpha t}} \quad \text{or} \quad \boxed{v_C = V_0(1 + \alpha t)e^{-\alpha t}}$$

represents the fastest decay
without oscillation

$\alpha < \omega_0, \quad \zeta < 1$ $$\boxed{q = Q_3 e^{-\alpha t}\sin(\omega_1 t + \varphi)}$$ underdamped

$\alpha = \omega_0, \quad \zeta = 1$ $$\boxed{q = CV_0\left(1 + \alpha t\right)e^{-\alpha t}}$$ critically damped

$\alpha > \omega_0, \quad \zeta > 1$ $$\boxed{q = CV_0 e^{-(\alpha-\beta)t}}$$ overdamped



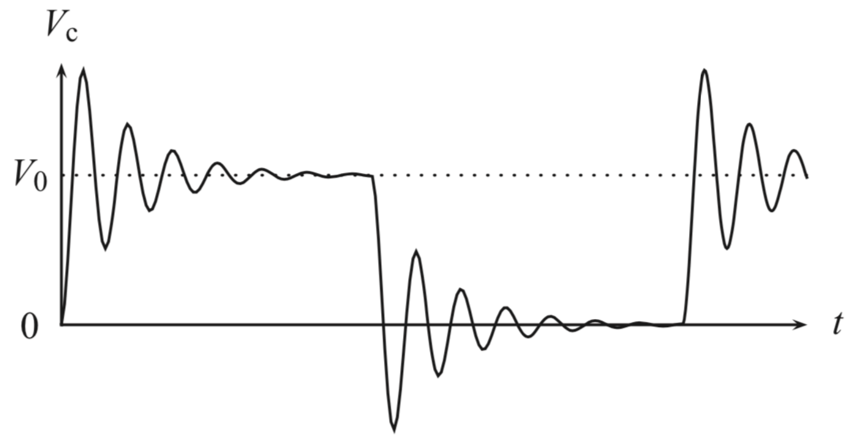


Figure 2.25 Underdamped response of a switched LRC circuit.

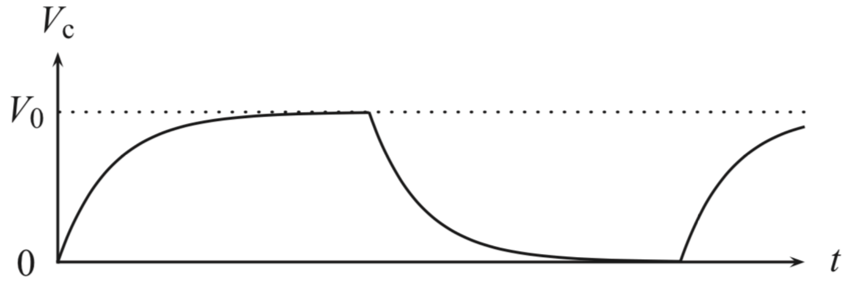


Figure 2.26 Overdamped response of a switched LRC circuit.

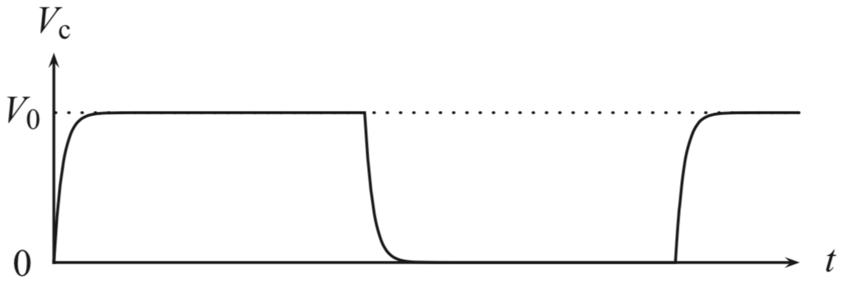


Figure 2.27 Critically damped response of a switched LRC circuit.