UNIVERSITY OF MANITOBA

March 5, 2018 MIDTERM EXAM

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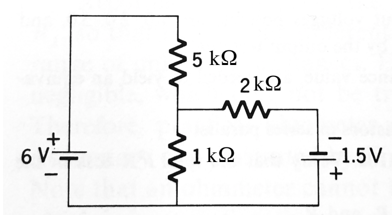
DEPARTMENT & COURSE NO.: PHYS 2610 TIME: 3 hours

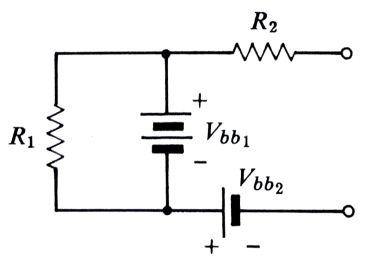
EXAMINATION: Circuit Theory and Introductory Electronics EXAMINER: W Ens

All questions have equal value.

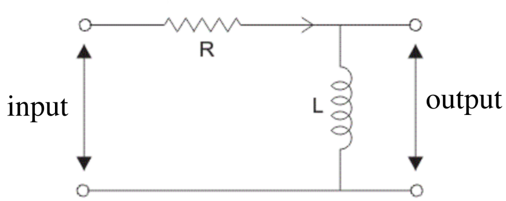
1. For the circuit shown, calculate the current through each resistor and the power delivered by the 6 V battery.



2. Give the Thevenin and Norton equivalent circuits for the circuit below. What load resistance connected to the output terminals would give the largest power transfer?

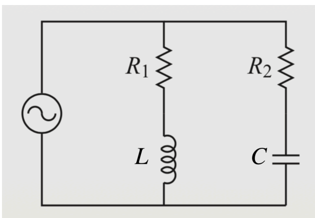


3. In the RL circuit below, determine the output voltage as a function of time if the input is stepped from zero to V_o at time $t = 0$. What is the current at very long times?



4. Show that the circuit of question 3 acts as a differentiating circuit under certain conditions, and give the conditions.

5. Give an expression for the complex impedance connected to the signal generator below. If the input is given by $v = V_0 \sin(\omega t)$, give and expression for the current through R_1.



6. Consider a series RLC circuit with $R = 10\ \Omega$, $L = 100$ mH, and $C = 10\ \mu$F. What is the Q-value of the circuit? What are the two frequencies for which the voltage across the resistor drops by 3 dB from its maximum value?

PHYS 2610: Midterm Formula Sheet 2018

Current: $i = \frac{dq}{dt} = \int \mathbf{J} \cdot \overrightarrow{da}$

Steady state: $\frac{di}{dt} = 0; \oint \mathbf{J} \cdot \overrightarrow{da}$

Ohm's law: $\mathbf{J} = \sigma \mathbf{E} = \frac{\mathbf{E}}{\rho} \Rightarrow v = iR$ with $R = \rho \ell / A$

Current density: $\mathbf{J} = ne\vec{v}_d$

Gauss's law: $\oint \mathbf{E} \cdot \overrightarrow{da} = q_{net}/\varepsilon_0$

Electric potential and potential energy: $V = U/q;\ dU = qdV$

Potential difference and emf: $\int_a^b \mathbf{E} \cdot \overrightarrow{dl} = -(V_b - V_a);\ \oint \mathbf{E} \cdot \overrightarrow{dl} = 0$

Power: $P = vi$

Capacitor: $q = CV,\ U = q^2/(2C)$

Solution to $\frac{dy}{dx} + ax = b$ has the form $y = Ae^{-ax} + b/a$

Faraday's law: $\mathcal{E}_{ind} = \int_a^b \mathbf{E} \cdot \overrightarrow{dl} = -\frac{d}{dt} \int \mathbf{B} \cdot \overrightarrow{da} = -L\frac{di}{dt}$

Inductor: $\mathcal{E} = L\frac{di}{dt}$

Magnetic field of ideal solenoid: $B = \mu_0 nI$

Euler's formula: $e^{j\theta} = \cos\theta + j\sin\theta$

Complex impedance: $Z = R + jX = |Z|e^{j\phi}$; $\tilde{v} = Z\tilde{\imath}$; $v = \mathrm{Re}(\tilde{v}) = V cos\omega t$

Capacitive impedance: $Z_C = -jX_C = \frac{1}{j\omega C}$

Inductive impedance: $Z_L = jX_L = j\omega L$

Series impedance: $Z = \sum Z_i$

Parallel impedance: $\frac{1}{Z} = \sum \frac{1}{Z_i}$

Complex voltage gain: $a = \frac{\tilde{v}_{out}}{\tilde{v}_{in}}$

Gain in dB: $G_{dB} = 20\log \left|\frac{\tilde{v}_2}{\tilde{v}_1}\right|$

Q Factor: $Q = \omega_0 L/R$

Phys 2610 (2018) Midterm Solution

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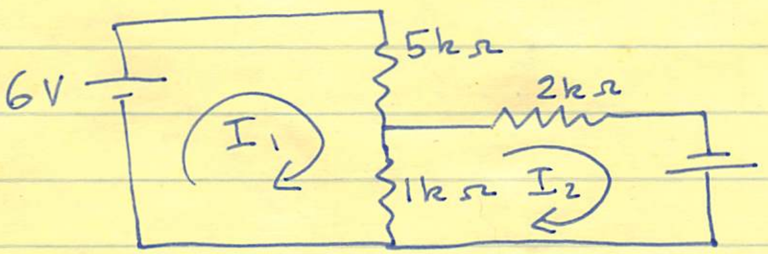


loop 1: $6V - I_1(5k\Omega) - (I_1 - I_2)(1k\Omega) = 0$

$\rightarrow I_1(-6k\Omega) + I_2(1k\Omega) = -6V$ ---- ①

loop 2: $1.5V - (I_2 - I_1)(1k\Omega) - I_2(2k\Omega) = 0$

$\rightarrow I_1(1k\Omega) + I_2(-3k\Omega) = -1.5V$

① + 6② $\rightarrow$ $0 + I_2(1k\Omega - 18k\Omega) = (-6V - 9V)$

$\rightarrow I_2 = \frac{-15V}{-17k\Omega} = 0.88\,mA$

① $\rightarrow I_1 = \frac{-6V - I_2(1k\Omega)}{-6k\Omega} = 1.15\,mA$

Current through 5kΩ: $I_1 = \underline{1.15\,mA}$

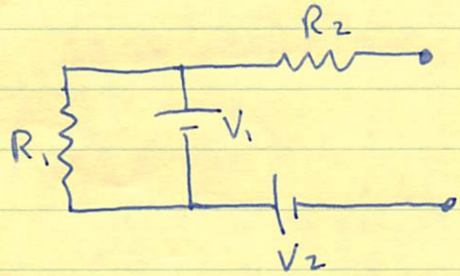
2kΩ: $I_2 = \underline{0.88\,mA}$

1kΩ: $I_1 - I_2 = \underline{0.27\,mA}$

Power delivered by 6V battery: $P = V \cdot I_1 = (6V)(1.15\,mA)$

$= \underline{6.9\,mW}$

(2)



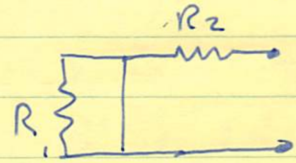
Thevenin

V_{th} = open cct voltage

$\rightarrow$ $\boxed{V_{th} = V_1 + V_2}$

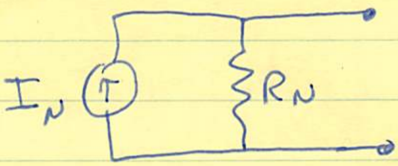
(since no current flows in R_2)

R_{th} is resistance with batteries shorted:



$R_{th} = R_1 // 0 + R_2 \rightarrow \boxed{R_{th} = R_2}$

Norton



$R_N = R_{th} \rightarrow \boxed{R_N = R_2}$

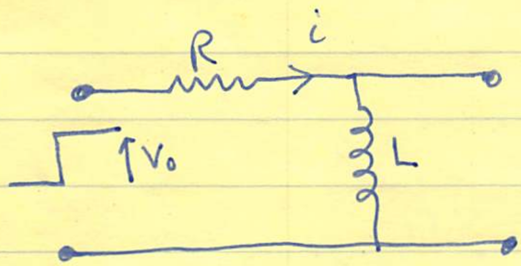
I_N = short circuit current $\rightarrow$ $\boxed{I_N = \dfrac{V_1 + V_2}{R_2}}$

Power Max power transfer $\Rightarrow$ $\boxed{R_L = R_{th} = R_2}$

Recall $P = I^2 R_L = \left(\dfrac{V_{th}}{R_{th} + R_L}\right)^2 R_L = V_{th}^2 \left(\dfrac{R_L}{(R_{th} + R_L)^2}\right)$

and $\dfrac{d}{dR_L} \dfrac{R_L}{(R_{th} + R_L)^2} = 0 \Rightarrow R_L = R_{th}$

(3)



Kirchhoff's rule: $V_0 - iR - L\frac{di}{dt} = 0$

$$\frac{di}{dt} + \frac{R}{L}i = \left(\frac{1}{L}\right)V_0$$

$$\rightarrow \quad i = Ae^{-Rt/L} + \frac{V_0}{R}.$$

Boundary condition: At $t=0$, $i=0$ $\left(L\frac{di}{dt} \text{ is max}\right)$

$$\rightarrow \quad 0 = A + \frac{V_0}{R} \rightarrow A = -\frac{V_0}{R}.$$

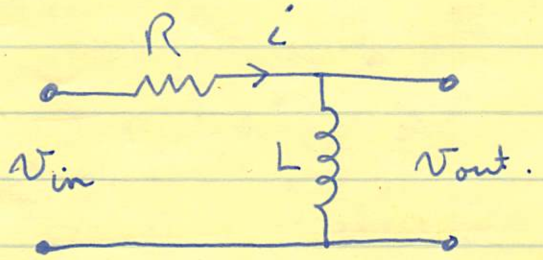
$$\rightarrow \quad i = \frac{V_0}{R}\left(1 - e^{-Rt/L}\right)$$

$\boxed{\text{For } t \to \infty, \quad i \to \frac{V_0}{R}}$

Output Voltage: $v_L = L\frac{di}{dt} = L\,\frac{V_0}{R}\left(-e^{-Rt/L}\right)\cdot\left(\frac{-R}{L}\right)$

$$\boxed{v_L = V_0 e^{-Rt/L}}$$

(4)



$$v_{out} = L \frac{di}{dt}$$

but $i = \dfrac{v_{in} - v_{out}}{R}$, so

$$v_{out} = \frac{L}{R} \frac{d}{dt} (v_{in} - v_{out})$$

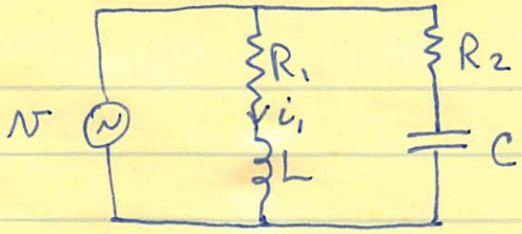
If $\boxed{v_{out} << v_{in}}$, $\boxed{v_{out} = \frac{L}{R} \frac{d}{dt} v_{in}}$

For harmonic voltage, $v_{out} << v_{in}$ corresponds to $\omega L << R$

or

$$\boxed{\frac{\omega L}{R} << 1}$$

5



<u>Impedance</u>: $Z = Z_1 /\!/ Z_2 = (R_1 + j\omega L) /\!/ \left(R_2 + \frac{1}{j\omega C}\right)$

$$Z = \frac{|Z_1| e^{j\theta_1} |Z_2| e^{j\theta_2}}{(R_1+R_2) + j(\omega L - 1/\omega C)} = \boxed{\frac{|Z_1|\,|Z_2|}{|Z_3|} e^{j(\theta_1+\theta_2-\theta_3)} \quad \text{(polar)}}$$

where $|Z_1| = \sqrt{R_1^2 + (\omega L)^2}$, $|Z_2| = \sqrt{R_2^2 + (1/\omega C)^2}$,

$|Z_3| = \sqrt{(R_1+R_2)^2 + (\omega L - 1/\omega C)^2}$, $\tan\theta_1 = \omega L/R_1$,

$\tan\theta_2 = 1/\omega R C$, $\tan\theta_3 = \dfrac{\omega L - 1/\omega C}{R_1 + R_2}$

<u>Current through R_1</u>:

Complex impedance in R_1 branch: $Z_1 = R_1 + j\omega L$

or $Z_1 = |Z_1| e^{j\theta} = \sqrt{R_1^2 + (\omega L)^2}\, e^{j\theta}$; $\tan\theta = \omega L/R$

Then, since $\tilde{i}_1 = \dfrac{\tilde{v}}{Z_1}$, if $v = V_0 \sin\omega t$

$$\boxed{i_1 = \frac{V_0}{\sqrt{R_1^2 + (\omega L)^2}} \sin(\omega t - \theta) \; ; \; \tan\theta = \omega L/R}$$

Note 1: Rectangular representation of Z:

$$Z = \frac{(R + j\omega L)\left(R_2 + \frac{1}{j\omega C}\right)}{(R_1 + R_2) + j(\omega L - 1/\omega C)} = \frac{(R_1 R_2 + L/C) + j(R_2 \omega L - R_1/\omega C)}{(R_1 + R_2) + j(\omega L - 1/\omega C)}$$

$$= \frac{\left[(R_1 R_2 + L/C) + j(R_2 \omega L - R_1/\omega L)\right]\left[(R_1 + R_2) - j(\omega L - 1/\omega C)\right]}{(R_1 + R_2)^2 + (\omega L - 1/\omega C)^2}$$

$$= \left\{ \frac{\left[(R_1 R_2 + L/C)(R_1 + R_2) + (R_2 \omega L - R_1/\omega C)(\omega L - 1/\omega C)\right] + j\left[(R_1 R_2 + L/C)(1/\omega C - \omega L) + \left(R_2 \omega L - \frac{R_1}{\omega C}\right)(R_1 + R_2)\right]}{(R_1 + R_2)^2 + (\omega L - 1/\omega C)^2} \right\}$$

(rect)

Note 2: $v = V_0 \sin \omega t = Re(\tilde{v})$ with $\tilde{v} = V_0 e^{j(\omega t - \pi/2)}$

or $= Im(\tilde{v})$ with $\tilde{v} = V_0 e^{j\omega t}$

Then $\tilde{i} = \frac{\tilde{v}}{Z_1} = \frac{\tilde{v}}{\sqrt{R_1^2 + (\omega L)^2}} e^{-j\theta}$

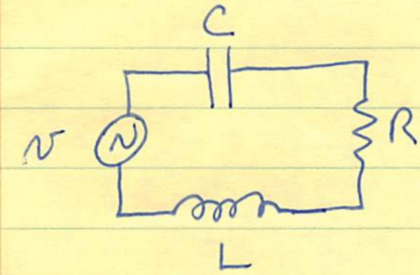
and

$$i = Re\left(\frac{V_0 e^{j(\omega t - \pi/2 - \theta)}}{|Z_1|}\right) \rightarrow \boxed{i = \frac{V_0 \sin(\omega t - \theta)}{|Z_1|}}$$

or

$$i = Im\left(\frac{V_0 e^{j(\omega t - \theta)}}{|Z_1|}\right) \rightarrow \boxed{i = \frac{V_0 \sin(\omega t - \theta)}{|Z_1|}}$$

(6)



$L = 100\,mH$ $\quad v = V_0 e^{j\omega t}$

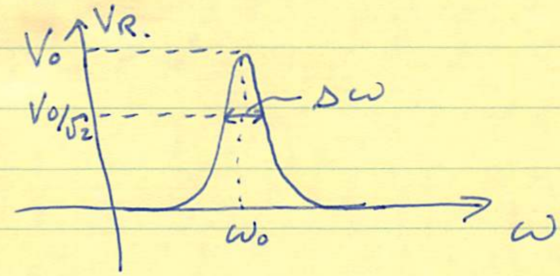
$C = 10\,\mu F$

$R = 10\,\Omega$

Resonant freq: $\omega_0 = 1/\sqrt{LC} = 1000$ rad/s

Q-factor: $Q = \omega_0 L / R = 10$

but $Q = \dfrac{\omega_0}{\Delta\omega}$



Where $\omega_0 \pm \frac{\Delta\omega}{2}$ represent 3dB attenuation points: $\left(\frac{V_R}{V_0} = \frac{1}{\sqrt{2}}\right)$

Then $\Delta\omega = \omega_0 / Q = 100$ rad/s, so

$$\omega_{3dB} = \left(1000 \pm \tfrac{100}{2}\right) \text{ rad/s} = 950, 1050 \text{ rad/s}$$

Also $v_R = iR = \underbrace{\dfrac{R}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} V_0}_{V_R} e^{j(\omega t - \theta)}$

so $\dfrac{V_R}{V_0} = \dfrac{R}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} = \dfrac{1}{\sqrt{1 + (\omega L - 1/\omega C)^2/R^2}}$

$\therefore \dfrac{V_R}{V_0} = \dfrac{1}{\sqrt{2}}$ when $(\omega L - 1/\omega C)^2 = R^2$

$\Rightarrow \omega L - 1/\omega C = \pm R \Rightarrow \omega^2 L \pm \omega R - 1/C = 0$

$\Rightarrow \omega = \dfrac{\pm R \pm \sqrt{R^2 + 4L/C}}{2L} = \pm 950$ rad/s, ± 1050 rad/s

(physical values are positive)