1. GENERAL LABORATORY INSTRUCTIONS

1.1 INTRODUCTION

There are several general goals of the laboratory section of this course which you should think about. Firstly, it must be emphasized that **Physics is an experimental science**; none of the physical and mathematical theories which you learn in this course or anywhere else can claim to have any bearing on reality unless it can stand the test of experimental scrutiny. Therefore, you will need to develop measurement skills and the ability to assess and interpret experimental data as you progress in your training as a professional scientist. In this course, a set of experiments has been selected in order to help you develop these skills while testing some of the physical theories that have been discussed in the lectures. You will be expected to keep a careful record of your activities in the lab, and to analyse and interpret your data. Written reports containing your results and some analysis as specified will be required for each student, for each experiment. The experiments you perform in this course, and the information in this laboratory manual, are considered to be important background preparation for laboratory work in higher level courses in physics.

1.2 BACKGROUND PREPARATION

For each experiment, study the instructions found in this lab manual **before** the weekly lab session. These instructions describe the apparatus and measurement techniques to be used for the experiments and pose specific questions to be answered by your measurements and data analysis. One or more 'pre-lab' questions may be included for each experiment to help you review the relevant theoretical concepts, including a derivation of equations that you will need for subsequent interpretation of the data.

The pre-lab questions will be marked for credit. They are due at the beginning of the lab period on Monday, at which time solutions will be posted.

1.3 LABORATORY PROCEDURES

Each week at the beginning of the laboratory session, the instructor will give a brief discussion of the experiment, with advice on how to perform the measurements, sources of error, etc. Students will work in pairs to perform the measurements, and analyze the data. One report should be prepared, with the primary responsibility (first author) alternating from week to week.

It is intended that you will be able to perform the measurements and at least a major part of the data analysis during the 3-hour laboratory session. *All your work, including original data, derivation of equations, calculations, discussion and conclusions should be recorded directly,*

with a pen, in your laboratory notebook. Don't waste time and risk unnecessary mistakes by recopying data from scraps of loose paper!

During the lab sessions, you should proceed methodically to perform the experiments and record your observations. There are no strict rules to judge whether an observation is 'right' or 'wrong', so do not be afraid to record in your book any observation that seems relevant. If a question occurs to you during the course of the measurements, possibly triggered by an unexpected observation, record the question in your notebook and try to answer it during the lab session – either by performing more measurements, or analysis, or discussing it with the instructor.

Important Guideline:

You should record enough information in your lab notebook that another person referring to it would be able to perform exactly the same experiment with no other written instructions and obtain comparable results.

1.4 NOTEBOOK ORGANIZATION

The notebook should represent a chronological record of your preparations for the lab, activity in the lab, and analysis of the results. It should contain some or all of the following:

i. Title and Date

ii. Statement of Purpose

At the beginning of the laboratory session, write down, **in your own words**, the specific goal or goals of the experiment you are about to perform.

iii. Description of Apparatus and Method

At the beginning of the laboratory session, examine the apparatus – assemble it in the intended configuration for the first measurements, turn it on, test it yourself, and **make sure you understand how it works** before you proceed to perform the measurements. Seek advice from the instructor or T.A. at an early stage if you need help. Refer to the laboratory manual and discuss with your partner how you plan to proceed with the experiment. Summarize this plan in a few sentences in your notebook together with a sketch of the apparatus and circuit diagram to be used – use a ruler and a pencil where appropriate. Refer to both the plan and the sketch as you proceed to record your observations. If a commercial electronic device is to be used, specify the manufacturer's name and the model number, and record the performance rating in your notebook (e.g. for a digital voltmeter, you should record the manufacturer's specification of the absolute accuracy for measurements of V on the scales you intend to use for the experiment).

iv. Observations and Data

For each set of measurements, record your quantitative observations in a data table. Specify both the quantities and the units in which they are measured at the headings of the columns in your data table. Your best estimate of the uncertainty in the measurement can also be recorded with the data, although in most cases in these experiments, it is not necessary. Each data table should have a descriptive title and should be accompanied by a description of how the measurements were obtained, referring to the apparatus diagram and specifying the settings of any meters, signal generators, etc. used in the measurements. The last column in a data table should be titled 'Comments', to be used for noting any changes to the apparatus or its behavior during the course of the measurements. A sample data table is shown below.

Sample Data Table: Test of Ohm's Law

V (volts)	I (Amps)	Comments
1.00±0.01	0.60±0.02	resistor feels hot – does this affect the measurements?
0.50±0.01	0.31±0.02	
0.20±0.01	0.098±0.002	changed Ammeter scale to 0.10 max
0.099±0.001	0.048±0.002	changed Voltmeter scale to 0.10 max

v. Analysis

Perform all your calculations in the lab notebook. If the data are to be compared with a theoretical model as worked out in the pre-lab exercises, state the specific comparison that is to be made and how it is to be tested. If the raw data are to be manipulated or 'reduced' in order to plot a graph to test the theory, write down the equations that will be used in this process. When necessary, work out the appropriate equations to determine the uncertainties in the individual entries of the reduced data set. Set up a table to perform the calculations – ideally on a page in your notebook opposite the graph paper that you will use to plot the results. Pay careful attention to units and significant figures in both the raw and reduced data sets.

General rule: report errors to one significant figure, and report measurements such that the least significant figure is the first uncertain figure according to the error estimate.

If a graph is to be plotted, it should have the following attributes: a descriptive title; clearly labelled axes (with units); points plotted in ink with experimental error bars; best straight line (or other, as appropriate) fit drawn through the points with parameters of the fit indicated on the graph. You may also prepare graphs using the software available on the lab computers; these should be printed at an appropriate size and taped into your lab book.

It is important to complete at least a preliminary reduction of the data during the lab period so that you can test whether your measurements make sense. If the results do not make sense, discuss them with the instructor and repeat the measurements more carefully before drawing conclusions from the experiment.

vi. Results and Discussion

Refer to the goals of the experiment stated in section ii, and summarize the quantitative results in light of these goals. Answer any questions that were specifically posed in the lab manual. Make quantitative comparisons with theory where possible. Comment on the dominant sources of statistical and systematic error in the measurements. Discuss any modifications to the apparatus or experimental procedure that you can think of making in order to improve the quality of the measurements.

vii. Conclusion

Give an overall assessment of the experiment and its general outcome in one or two sentences. (What aspects of the experiment were a success? What aspects of the experiment failed? 'Success' and 'failure' should be quantified in light of the theoretical expectations and the accuracy of the measurements.)

1.5 SUBMISSION OF WORK AND MARKING SCHEMES

One report from each pair of students should be submitted for experiments 1 to 8. The report should contain a brief description of the procedure, including necessary diagrams, the results of the measurements, analysis, including graphs, discussion and conclusions. Your lab notebook may also be inspected by the instructor or TA, and may be used to assist the evaluation. One half of the evaluation will be based on the measurements and analysis and will go to both students, while the evaluation of the full report will go to the first author.

The overall laboratory mark is worth 25% of the course mark for PHYS 2610. The pre-lab exercises will make up 5%, and the reports the remaining 20%.

Submission Deadlines

Lab reports are due on Mondays in the following lab session. Prelab questions should be handed in at the beginning of the respective lab session on Monday.

NOTE: If you are ill and cannot attend a Monday lab session, consult the instructor so that a "make-up" lab can be scheduled.

2. ERROR ANALYSIS AND INTERPRETATION OF DATA

The following notes and exercises provide a brief introduction to several important topics in error analysis and data interpretation. For more details, consult a standard textbook on the subject – the paperback, *Experimentation: An Introduction to Measurement Theory and Experiment Design* by D.C. Baird, is worth investing in, particularly if you expect to be doing more lab work in future science courses. Another standard reference work, written at a somewhat higher level, is: *Data Reduction and Error Analysis for the Physical Sciences*, by P.R. Bevington (also available in paperback). J.R. Taylor's book: 'An Introduction to Error Analysis: The Study of Uncertainties in Physical Measurements' is also a great resource.

2.1 COMBINATION OF ERRORS – SIMPLE CASE

Provided that each element of a set of measurement data is independent of the other elements, then the individual measuring errors should be combined in *quadrature*. For example, if we have two measurements of the voltage drop across different portions of a circuit:

$$V_1 \pm \delta V_1 \quad V_2 \pm \delta V_2$$

then the total voltage drop $V = V_1 + V_2$ should be reported as:

$$V \pm \delta V = (V_1 + V_2) \pm \sqrt{\delta V_1^2 + \delta V_2^2}$$

The combination of errors in quadrature takes into account the fact that it is unlikely that the two independent measurements V_1 and V_2 would each deviate from the 'true' values by the maximum possible amounts δV_1 and δV_2 respectively – hence, this method gives a smaller and more reasonable estimate of the uncertainty in the sum of the two voltage drops, $(V_1 + V_2)$, since by the 'triangle inequality':

$$\sqrt{\delta V_1^2 + \delta V_2^2} \leq \delta V_1 + \delta V_2 \qquad (1)$$

where δV_1 and δV_2 are both positive quantities. The quadrature method for combining errors is rooted in statistical theory; for a rigorous derivation, see J. R. Taylor, "An Introduction to Error Analysis," ch. 5.

Often, it is desirable to make repeated measurements of the same quantity x in order to improve the accuracy of the experiment. This technique reduces the uncertainty in the experimental value of x by averaging over random statistical fluctuations in the individual measurements, which tend to cancel out for a large number of observations. If there are N measurements of the same quantity, then the individual measurements are denoted x_i for i = 1,

N with measurement errors δx_i. The experimental result to be reported is the **mean value** $\bar{x}$, given by

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \tag{2}$$

The error in the mean, $\delta\bar{x}$, can be obtained from the quadrature formula discussed above, if the measurements are all independent (usual case) and the uncertainties δx_i are *correctly assessed* for the conditions of the experiment, in which case:

$$\delta\bar{x} = \frac{1}{N}\sqrt{\sum_i \delta x_i^2} \tag{3}$$

Notice that if all of the individual measurements have exactly the same uncertainty, $\delta x_i = \delta x$, then equation (3) implies that the error in the mean value is proportional to $1/\sqrt{N}$. Thus, according to equation (3), the uncertainty in $\bar{x}$ can be made arbitrarily small simply by performing enough measurements; to reduce the uncertainty by a factor of 2 requires 4 times as many measurements, and so on.

Problem 2.1.1: Prove the above assertion by writing out the combination of errors according to equation (3) for a particular value of N.

2.2 STATISTICAL AND SYSTEMATIC ERRORS

Unfortunately, it turns out that the simple form of equation (3) is **not** the best way to assess the error in the mean value $\bar{x}$. At issue is the question of whether the individual uncertainties δx_i are appropriately assessed at the input to the calculation. It turns out, in general, to be difficult if not impossible to estimate the errors δx_i just by assessing the measuring apparatus alone. To do the job right requires a distinction between two types of uncertainty in any given measurement, namely **statistical** and **systematic** errors. Statistical errors lead to random fluctuations of the individual measurements about their mean value $\bar{x}$, and it is the effects of these random fluctuations which can be reduced by performing a large number of measurements and reporting the mean value. Systematic errors affect all measurements the same way and hence do not make any contribution to fluctuations in the individual measurements x_i; therefore, the effects of systematic errors cannot be reduced by performing a large number of measurements and reporting the mean value. Because of their different natures and implications for the outcome of an experiment, statistical and systematic errors should always be assessed and reported separately.

As an example, suppose it was desired to determine the voltage drop V across a resistor in a circuit to high accuracy by performing a large number of measurements V_i. Statistical errors would be responsible for the fact that all individual measurements V_i did not have exactly the

same value; systematic errors would arise if the voltmeter used were wrong by a constant scale factor – that is, if the voltmeter read systematically too high or too low. Statistical errors could be assessed by analyzing the spread of the distribution of individual measurements V_i; the systematic error could be assessed by doing a separate experiment to compare the voltmeter used to measure the V_i against a known standard. The best way to report errors is to assess the statistical and systematic effects separately and report them as such. In the circuit example, the result should be reported as:

$$\bar{V} \pm \delta V_{stat} \pm \delta V_{sys} \tag{4}$$

Reporting errors in this way conveys the maximum amount of information. The quantity δV_{stat} could be assessed independently from the spread in the individual data values. The quantity δV_{sys} could be assessed independently by looking up the manufacturer's specifications for the voltmeter that was used. If it was desired to improve the accuracy to which $\bar{V}$ was known, one would first compare the statistical and systematic error values. If it turned out that the statistical error was largest, then one should perform additional measurements V_i to reduce that contribution to the total error. On the other hand, if the systematic error was largest, then one should attempt to perform an additional experiment to calibrate the measuring device. If the calibration factor were known, the original data could be corrected in compensation, resulting in a different value of $\bar{V}$ with a smaller systematic uncertainty.

For most of the experiments in this course, it will be sufficient to look up the manufacturer's specifications for absolute accuracy (systematic error) for the devices that you use. Be careful when you read the instruction manuals, as these specifications often differ depending on the absolute value of the input – for example, a digital voltmeter may be rated to an absolute accuracy of ±0.001 volts on the 0.1 - 1 volt scale and to an absolute accuracy of ±0.01 volts on the 1 - 10 volt scale. Changing voltmeter scales part way through an experiment could therefore result in a 'jump' of the data values as the systematic error of the voltmeter changes with the scale setting. If you need to change scales part way through a series of measurements, you should try to take a reading of exactly the same potential difference on two voltmeter scales to check the consistency of the data.

2.3 ASSESSMENT OF THE STATISTICAL ERROR CONTRIBUTION

It turns out that the best way to assess the statistical error in a measured quantity is to perform a reasonably large number of measurements under exactly the same conditions and examine the spread in the data values. We can define a quantity called the **standard deviation** σ of a data set, which quantifies the statistical fluctuations. First of all, we define the deviation of each data value from the mean, as the quantity:

$$\Delta x_i = (x_i - \bar{x}) \tag{5}$$

Then, for large N, the standard deviation is essentially the root mean square (RMS) deviation of the data set from the mean value:

$$\sigma = \sqrt{\sum_{i=1}^{N} \frac{(\Delta x_i)^2}{N-1}} = \sqrt{\sum_{i=1}^{N} \frac{(x_i - \bar{x})^2}{N-1}} \tag{6}$$

The factor *N* - 1 in the denominator of equation (6) is referred to as the number of **degrees of freedom** in the data set. Obviously, this equation breaks down if *N* = 1, in which case we have only one data entry, i = 1, and we report $\bar{x} = x_1$. There is no information with which to assess the fluctuations in the data, so the case N = 1 is said to have 'zero degrees of freedom' – the single data entry in this case has no freedom to take on a different value since there is only one measurement. We are unable to assess the standard deviation σ in this case. As the number of data entries increases, we have increasingly more information with which to assess the statistical fluctuations, and our ability to assess the statistical error δx_{stat} improves. In this case, the standard deviation σ is a reliable measure of the deviation of each individual data entry from the mean value. The standard deviation σ has the interpretation that if one more entry is added to the data set, we would expect that entry to have a value in the range $\bar{x} \pm \sigma$. Thus, we would expect that the individual error estimates δx_i which correctly account for the **statistical** fluctuations in the data set should be assessed as:

$$\delta x_i = \sigma \tag{7}$$

in which case, it follows from the quadrature method that the uncertainty in the mean value $\bar{x}$, arising from the **statistical** fluctuations in individual data entries should be reported as:

$$\delta \bar{x}_{stat} = \sigma_{\bar{x}} = \frac{1}{N} \sqrt{\Sigma \delta x_i^{\,2}} = \frac{\sigma}{\sqrt{N}} \tag{8}$$

The correct way to report the result of N measurements to determine the mean value $\bar{x}$ is then:

$$\bar{x} \pm \frac{\sigma}{\sqrt{N}} \pm \delta x_{sys} \tag{9}$$

where $N \geq 5$ is the minimum number of data entries required to obtain a reasonable estimate of σ. Looking back at equation (3) in section 2.1, it is clear that a test for the reliability of the originally assessed errors δx_i (i.e. the values you would initially record in your data table, before calculating the standard deviation) is to compare the values δx_i with the value of σ defined in equation (6). If the δx_i are much larger than σ after accounting for the overall systematic error as a separate contribution, then the δx_i must have been initially overestimated. Likewise, if the data values fluctuate much more than the δx_i would imply, then the original error values were underestimated.

2.4 THE GAUSSIAN DISTRIBUTION

The amazing thing about statistical errors is that the random fluctuations in data values for repeated measurements will **always** result in the same type of distribution of observed values provided that two conditions are met:

(i) the number of measurements N is large

(ii) the measurements are all done under the same conditions.

This distribution is called the **Gaussian** or **Normal** distribution, and is given by:

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\bar{x})^2}{2\sigma^2}} \tag{10}$$

Equation (10) defines a **normalized** Gaussian distribution, which has the property:

$$\int_{-\infty}^{+\infty} g(x)dx = 1 \tag{11}$$

This distribution has a mean value $\bar{x}$ and a standard deviation σ which is consistent with the definition of equation (6) in the limit $N \to \infty$. A graph of the normalized Gaussian distribution function *g(x)* is shown in figure 1.

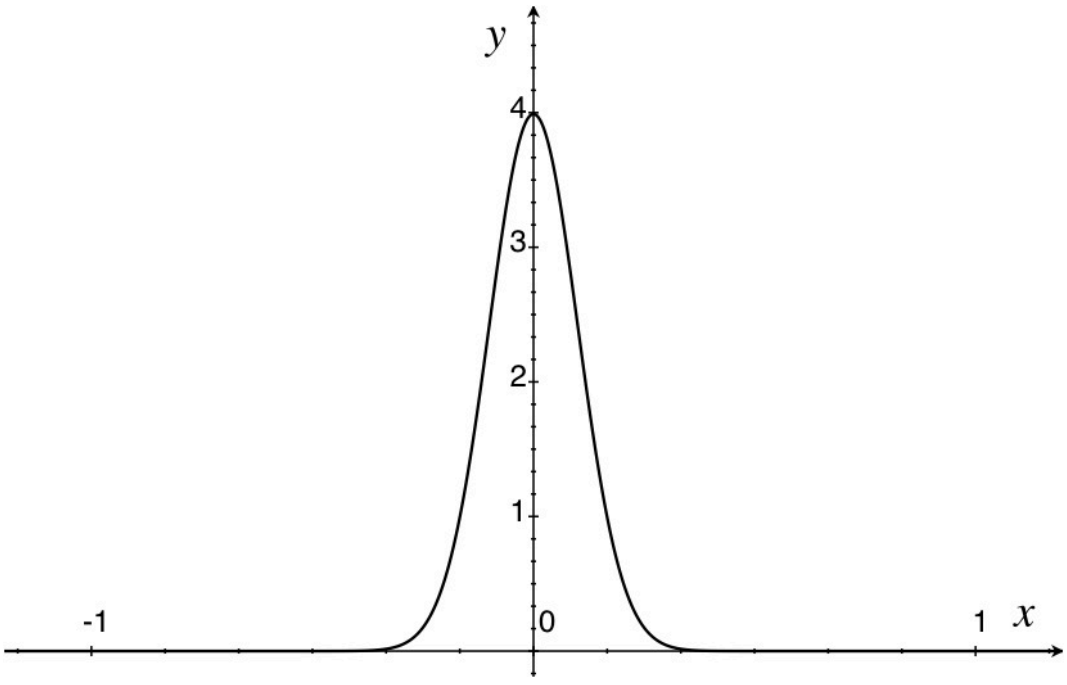

Figure 1: Gaussian distribution *y*=g*(x)* with $\bar{x} = 0,\ \sigma = 0.10$.

The interpretation of the normalized Gaussian distribution *g(x)* is that the probability of observing a particular value of x in the range $(x-\frac{dx}{2})$ to $(x+\frac{dx}{2})$ in a series of repeated measurements is given by *g(x) dx*. In other words, *g(x)* is a *probability density* distribution. Since the probability of observing any value at all as the outcome of a given measurement must be equal to 1, it follows that $\int_{-\infty}^{+\infty} g(x)dx = 1$, as stated in equation (11).

The probability density *g(x)* has its maximum value at $x = \bar{x}$, and its full width at half maximum value (fwhm) is 2.355σ; $\frac{2}{3}$ of the data entries lie within ±σ of $\bar{x}$ and the remaining $\frac{1}{3}$ of the entries are more than one standard deviation away from the mean value. Since *g(x)* is normalized to unity and its width is proportional to σ, it follows that a wider distribution will have a smaller maximum value. For comparison, a graph of two normalized Gaussian distributions $g_1(x)$ and $g_2(x)$ versus x with different standard deviations σ_1 and σ_2 is shown in figure 2. Incidentally, if a set of normalized distributions $g_i(x)$ is plotted as a function of the dimensionless ratio $\frac{(x-\bar{x})}{\sigma_i}$, then all of the distributions lie on a single universal curve, as shown in figure 3.

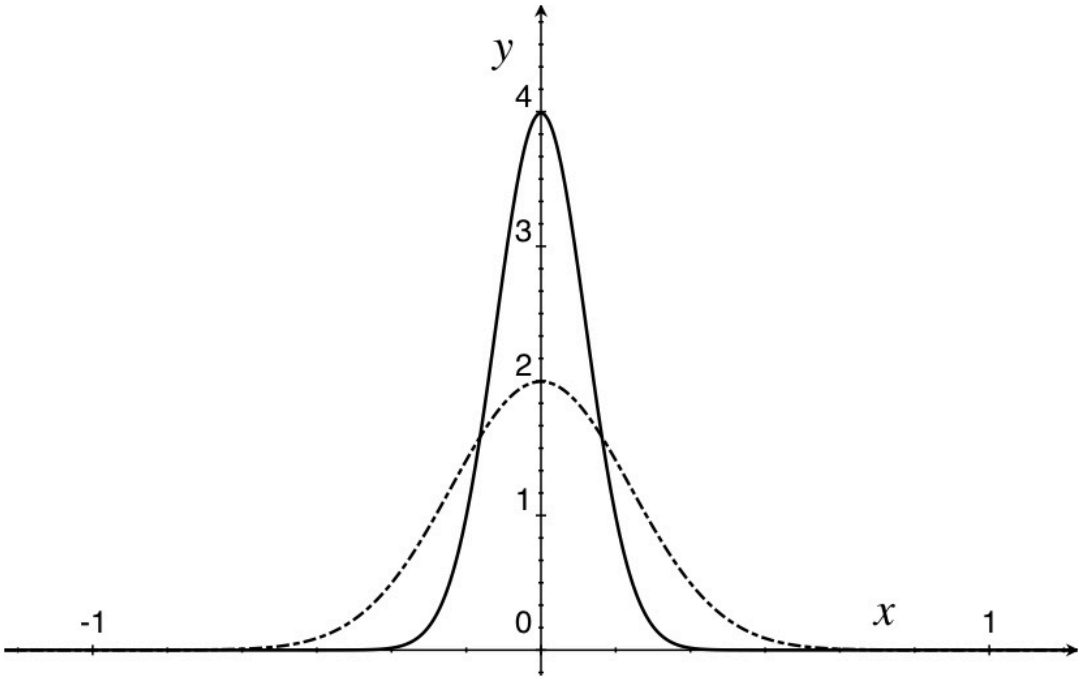

Figure 2: Gaussian distributions with $\bar{x} = 0 : \mathrm{y} = g_1(x)$ the solid curve has $\sigma_1 = 0.10$, and $\mathrm{y}=g_2(x)$., the dashed curve, has $\sigma_2 = 0.20$

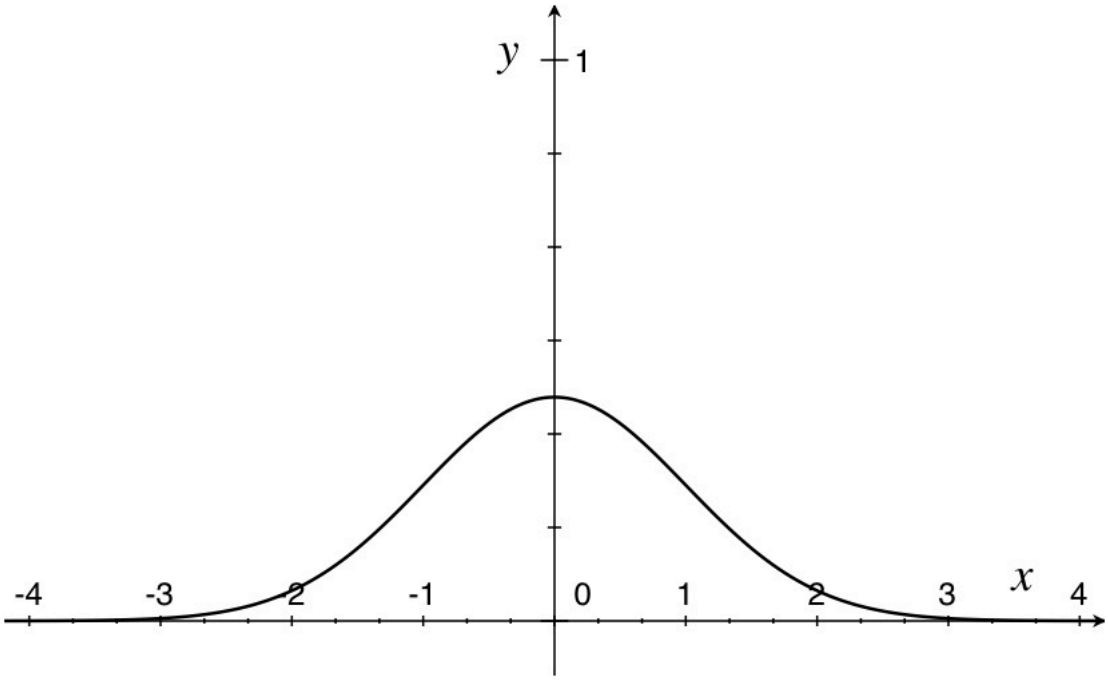

Figure 3: Universal curve: $y = g(\frac{x-\bar{x}}{\sigma})$ versus $(\frac{x-\bar{x}}{\sigma})$, which describes all Gaussian distributions.

To study a particular distribution of data values x_i and make a comparison with the expected Gaussian distribution *g(x)*, it is necessary to sort the data values into 'bins' and draw a histogram of the distribution of data entries. Some care is required to choose the optimum bin size for a given data set – if the bins are too large, then all data values will fall into one or two bins, and the resolution of the histogram will be too low for a meaningful comparison with the expected distribution. On the other hand, if the bins are too small, then there will be relatively few entries in each bin, and the experimental information will be too 'spread out' to make much sense. The optimally selected histogram should have 10-20 bins containing all the data, and consists of a set of bin contents *n*(*j*) for each bin *j*, corresponding to a fixed value x_j at the center of the bin, with a total of $N = \Sigma n(j)$ data entries.

To compare the data histogram to the expected Gaussian shape, we can make use of the interpretation of *g(x)* as a probability density distribution. Since *g(x)dx* gives the probability of obtaining a data value within *dx* of *x* for any given measurement, then it follows that $g(x_j)\Delta x_j$ must be the probability of obtaining a data value lying in the jth bin of the discrete histogram, with bin size Δx_j. The expected number of measurements whose values lie in the jth bin, *n(j),* is the total number of measurements N multiplied by the probability of a measurement occurring in the jth bin. Thus:

$$n(j) = Ng(x_j)\Delta x_j = \frac{N\Delta x_j}{\sigma\sqrt{2\pi}} e^{\frac{-(x_j-\bar{x})^2}{2\sigma^2}} \qquad (12)$$

For a data set of finite size, the histogram entries $n(j)$ will deviate randomly from their expected values given in equation (12); the size of these random deviations, as expected from statistical analysis, is given by:

$$\delta n(j) = \sigma_{n(j)} = \sqrt{n(j)} \tag{13}$$

The random relative deviation between a given histogram bin entry $n(j)$ and its expected value from the Gaussian distribution is therefore given by:

$$\frac{\delta n(j)}{n(j)} = \frac{1}{\sqrt{n(j)}} \tag{14}$$

For a given size of data set, the binning should be optimized so that the relative deviations of the bin entries are reasonably small for most of the bins – otherwise, the real data histogram cannot be expected to look very much like the Gaussian distribution of equation (12) even though the data set itself may be perfectly consistent with that description! These points are illustrated with the set of graphs that follows.

For a finite sized data set, we should take care to distinguish between the true mean $\bar{x}$ and standard deviation σ of the 'parent distribution' and the mean and standard deviation of the actual data set, referred to as the 'sample mean' and 'sample standard deviation'. To be perfectly clear, we should use different symbols to distinguish those quantities calculated from the actual data sample, e.g. $\bar{x}$ and σ_s (equations 2 and 6), from those which define the true Gaussian distribution (equations 10 and 12). The central idea here is that any finite data set consists of a number of measurements N which are randomly scattered about the true or parent distribution. As the number of measurements becomes very large, the statistical fluctuations in the data set will tend to average out, and the actual measured data histogram approaches the true distribution defined in equation (12). In this limit, the sample mean and the mean of the parent distribution will approach exactly the same value, and likewise for the sample and parent standard deviation.

The random discrepancies between $\bar{x}_s$ and $\bar{x}$, and between σ_s and σ scale with $1/\sqrt{N}$; for a finite-sized sample, **both** $\bar{x}_s$ and σ_s should be accurate to within an error of $\pm\sigma_s/\sqrt{N}$ (standard deviation of the respective quantities). For the distributions illustrated in the next few figures, the actual data set (histogrammed, with error bars according to equation (13)) is compared to the parent distribution (solid line, equation (12)). For the sample of 100 data entries, the parent mean $\bar{x}$ was 62.50 as compared with the sample mean $\bar{x}_s$ of 61.72; the parent standard deviation σ was 12.00 as compared with the sample standard deviation of 11.12. These small discrepancies are consistent within the expected uncertainty range of $\pm\sigma_s/\sqrt{N} = \pm 1.2$.

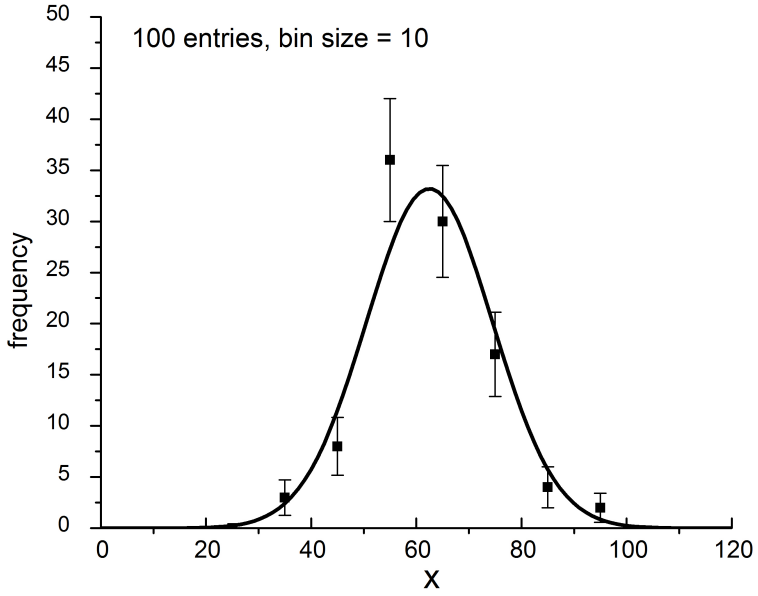

Figure 4: Random data sample (points) from a parent distribution (solid line) with $\bar{x}$ = 62.50, σ = 12.00. For 100 data entries distributed and a bin size of 10 units, the data histogram gives a fairly good representation of the parent Gaussian distribution.

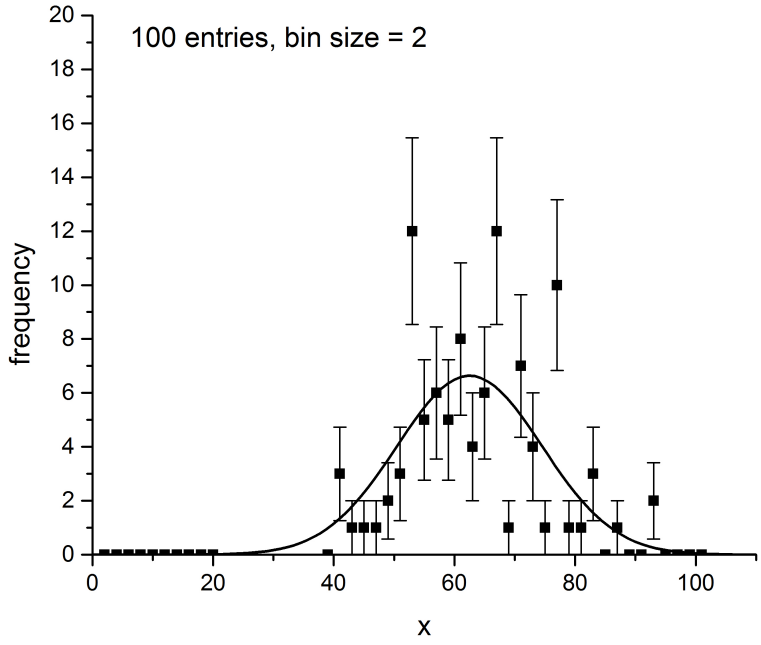

Figure 5: Random data sample (points) from a parent distribution (solid line) as in the previous figure. For 100 data entries and a bin size of 2 units, the statistical fluctuations in the data histogram are too large for it to look very much like the parent distribution.

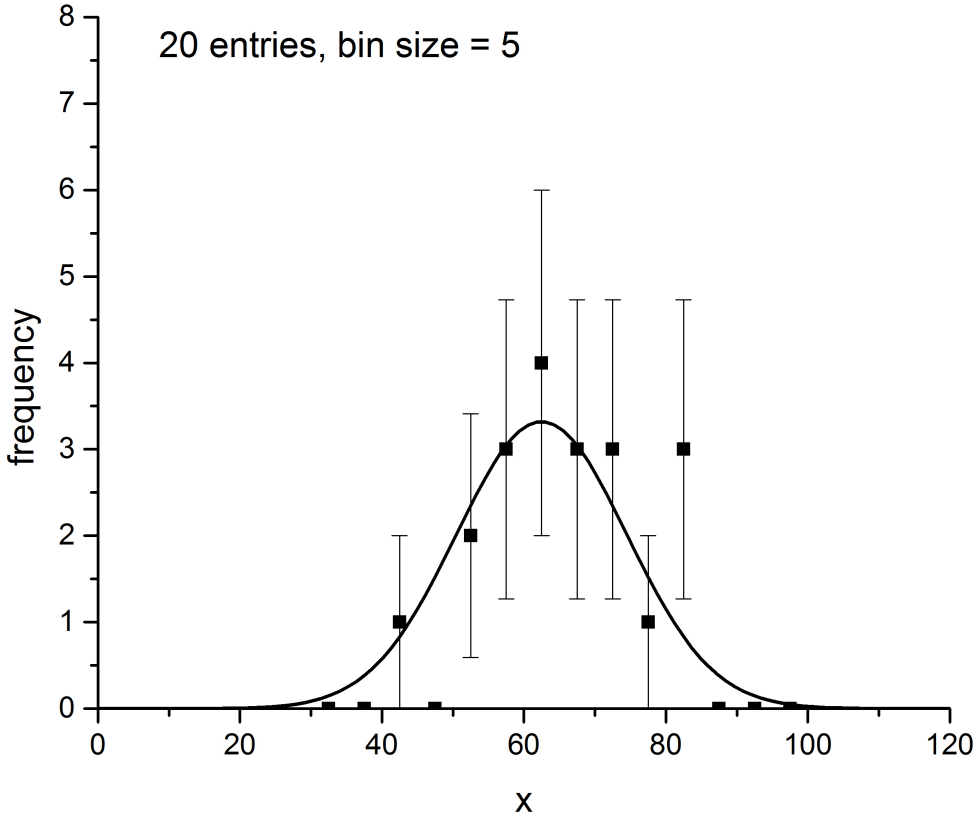

Figure 6: Random data sample (points) from the same parent distribution (solid line) as the previous figures. For the first 20 data entries with a bin size of 5 units, the data histogram has relatively large fluctuations. The sample mean is 60.64 compared with the parent mean of 62.5. There are not enough points to adequately test whether the data distribution fits well to a Gaussian or to some other function.

PROBLEM (2.4.1)

The distribution of student marks on a recent PHYS 1020 term test consisting of 30 questions is summarized in the data set below:

Mark out of 30	Number of Students
7	4
8	1
9	7
10	11
11	15
12	33
13	36
14	34
15	51
16	62
17	55
18	61
19	48
20	49
21	43
22	41
23	30
24	29
25	19
26	11
27	8
28	2
29	5
30	2

a) Plot a histogram of this student mark distribution. Each point should be drawn with an appropriate statistical error bar, as described in section 2.4.
b) Calculate the mean and standard deviation of the mark distribution.
c) Assume that the marks are distributed according to a Gaussian distribution with the mean and standard deviation calculated in part b), and use this information to draw a solid line on the graph corresponding to this Gaussian distribution. (Your final graph should be identical in style to figure 4.) Calculate the fwhm of the Gaussian; then σ = fwhm/2.355. Does this agree with the result from (b)? Comment on the agreement between the data set and the Gaussian distribution.

2.5 PROPAGATION OF ERRORS IN A CALCULATION

Often, a set of basic quantities is measured in order to determine the value of some particular function of these quantities which has a physical significance. If the set of experimentally measured quantities consists of the data values: $((a \pm \delta a),(b \pm \delta b),...)$ and the function to be evaluated is denoted $f(a,b,...)$, then we need to establish the correct procedure to evaluate the uncertainty δf in terms of the individual measurement errors $(\delta a, \delta b,...)$. In the case where the individual *relative* errors $(\frac{\delta a}{a}, \frac{\delta b}{b},...)$ are small (which they must be if you have done a reasonably good job of the measurements!), then differential calculus may be used to evaluate the uncertainty δf. The calculation proceeds in two stages.

(i) evaluate the change δf resulting from a change in the particular data value *a*, given by $\frac{\partial f}{\partial a}\delta a$

(ii) add the contributions of δf from the independent data values in quadrature.

Thus:

$$\delta f = \sqrt{(\frac{\partial f}{\partial a}\delta a)^2 + (\frac{\partial f}{\partial b}\delta b)^2 + ...} \qquad (15)$$

Incidentally, point (ii) above gives the correct approach for combining standard deviations of statistical (Gaussian) error distributions, as it must do, in order to be consistent with our discussions of the previous section.

As a specific example of how to combine measuring errors, suppose that an experimental study of Joule heating in a resistor is performed. An ammeter and digital ohm-meter are used to measure the values $I \pm \delta I$ and $R \pm \delta R$ respectively. The Joule heating power is calculated as $W \pm \delta W$ from these data values, where $W = I^2 R,$ hence:

$$\delta W = \sqrt{\left(\frac{\partial W}{\partial I}\delta I\right)^2 + \left(\frac{\partial W}{\partial R}\delta R\right)^2} = \sqrt{(2IR\delta I)^2 + (I^2\delta R)^2}$$

and,

$$\frac{\delta W}{W} = \frac{I\sqrt{4R^2\delta I^2 + I^2\delta R^2}}{I^2 R} = \sqrt{\left(2\frac{\delta I}{I}\right)^2 + \left(\frac{\delta R}{R}\right)^2}$$

A **crucial assumption** which restricts the applicability of equation (15) is that the measured values (*a, b,* ...) must be independent of each other; if they are not independent, then the chain rule must be used to correctly determine the change in the function *f* resulting from a change in the input values. A safer way to do the calculation is to re-express the function *f* in terms of independent variables only and then use equation (15). For example, if we want to assess the

error in a function $f(a, b, c) = a^2b+c$ and $b = \sin(a)$, we can use the chain rule, noting that there are really only two independent variables, a and c:

$$f = a^2 b + c$$

$$\delta f = \sqrt{\left(\left(2ab + a^2 \frac{\partial b}{\partial a}\right)\delta a\right)^2 + (\delta c)^2}$$

$$= \sqrt{\left(\left(2a\sin(a) + a^2\cos(a)\right)\delta a\right)^2 + (\delta c)^2}$$

which yields the same result for δf as if we had started by reuniting the original function as $f = a^2 \sin a + c$.

PROBLEMS

(2.5.4 - 2.5.7 are taken from D. C. Baird)

2.5.1 Show that if $f = a - b$, and a and b are independent, then $\delta f = \sqrt{\delta a^2 + \delta b^2}$

2.5.2 Show that if $f = ab^n$ and a and b are independent, then $\frac{\delta f}{f} = \sqrt{\left(\frac{\delta a}{a}\right)^2 + \left(\frac{n\delta b}{b}\right)^2}$

2.5.3 Show that if $f = \frac{a}{b}$, and a and b are independent, then $\frac{\delta f}{f} = \sqrt{\left(\frac{\delta a}{a}\right)^2 + \left(\frac{\delta b}{b}\right)^2}$

2.5.4 A value is quoted as 6.74914±0.5%. State it as a value ± an absolute uncertainty, both with the appropriate number of significant figures.

2.5.5 A simple pendulum is used to measure the acceleration of gravity, using the formula: $T = 2\pi\sqrt{\ell/g}$. The period T was measured to be 1.24±0.02 sec, and the length ℓ was measured to be 0.381±0.002 m. Evaluate the resulting experimental value of g with its correct absolute and relative uncertainty.

2.5.6 A diffraction grating is used to measure the wavelength of light using the equation $\lambda = d\sin(\theta)$. The value of θ is measured to be 13°34'±2'. Assuming that the value of d is 1420×10^{-9}m and that its uncertainty can be ignored, what are the absolute and relative uncertainties in the value of λ?

2.5.7 A simple pendulum is used to measure g using the equation $T = 2\pi\sqrt{\ell/g}$. Twenty measurements of T give a mean of 1.82 sec and a standard deviation of 0.06 sec. Ten measurements of ℓ give a mean value of 0.823 m and a standard deviation of 0.014 m. What is the statistical error in the resulting value of g?

Under these conditions, how many more of each type of measurement would be required to reduce the statistical error in g by one half?

2.6 LEAST SQUARES FIT TO A STRAIGHT LINE

Often, analysis is performed on experimental data to test and/or exploit a linear relationship between different physical quantities. A simple example relevant to this course would be an experimental test of Ohm's Law: $V = IR$. The goals of the data analysis in this case would be:

(i) to determine whether the relationship between V and I is linear
(ii) to determine the slope, R, of the best straight line through the points.

These two goals can be addressed by drawing a careful graph of the experimental data, but it will also be necessary to establish a technique for finding the best straight line through the data points that is more rigorous than the 'eyeball fit' method that you will have used in first year science courses. There are several versions of the 'least squares fit' to polynomial functions which are widely used and described in textbooks on data analysis that will do the job of finding the best line under a given set of conditions. In this section, we will develop the equations needed to find the best straight line through a set of data points (x_i, y_i) under the simplifying conditions that:

(i) the uncertainties δx_i in the 'control variable' are small enough to be ignored.
(ii) the uncertainties δy_i in the 'dependent variable' are all approximately equal.

This turns out to be the simplest case possible for introducing the least squares method, and is referred to as an '*unweighted linear least squares fit*'.

The idea is to find the slope and intercept of the best fit line:

$$y = mx + b \tag{16}$$

which passes most closely through a set of $N > 2$ data points (x_i, y_i). If we consider the ith data point (x_i, y_i), the difference in y between this point and the best fit line is the *deviation* of the ith point from the line, given by:

$$\Delta y_i = y_i - (mx_i + b) \tag{17}$$

The deviations Δy_i form a set of N numbers which can be positive or negative; the best line is specified by values of m and b which minimize the sum of absolute deviations $\left|\Delta y_i\right|$ over the set of N data points – this guarantees that the line will be as close as possible to the set of N data points. To perform the minimization, note that:

$$|\Delta y_i| = \sqrt{\Delta y_i^2}$$

and hence, the problem is solved by finding the unique values of m and b which minimize the sum of squared deviations:

$$M = \sum_{i=1}^{N} \Delta y_i^2 = \sum_{i=1}^{N} \left(y_i - (mx_i + b) \right)^2 \tag{18}$$

To perform the minimization, we note that the data values (x_i, y_i) are fixed, and the only variables which M depends on are the slope m and the intercept b. Therefore, we obtain two equations to determine our two unknowns m and b:

$$\frac{\partial M}{\partial m} = 0 \tag{19}$$

$$\frac{\partial M}{\partial b} = 0 \tag{20}$$

The solution to equations (19) and (20) is straightforward but a little tedious. The following exercises will lead you through the deviations:

PROBLEMS

2.6.1 Show that equation (19) can be rewritten as:

$$M \sum_i x_i^2 + b \sum_i x_i - \sum_i x_i y_i = 0 \tag{21}$$

2.6.2 Show that equation (20) can be rewritten as:

$$m \sum_i x_i + bN - \sum_i y_i = 0 \tag{22}$$

2.6.3 Rearrange equations (21) and (22) to show that:

$$m = \frac{N \sum x_i y_i - \sum x_i \sum y_i}{N \sum x_i^2 - \left(\sum x_i \right)^2} \tag{23}$$

and

$$b = \frac{\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i}{N \sum x_i^2 - \left(\sum x_i \right)^2} \tag{24}$$

The uncertainties δm and δb are evaluated through the dependences of m and b on the data values (x_i,y_i) and the measurement uncertainties δy_i. (*Recall that we made the initial simplifying assumption that* $\delta x_i = 0$ *for all* x_i *for this particular type of least squares fit*.) Formally, then:

$$\delta m = \sqrt{\sum_i \left(\frac{\partial m}{\partial y_i} \delta y_i \right)^2} \tag{25}$$

and

$$\delta b = \sqrt{\sum_i \left(\frac{\partial b}{\partial y_i} \delta y_i \right)^2} \tag{26}$$

The correct equations for δm and δb are:

$$\delta m = \sigma_y \sqrt{\frac{N}{N\Sigma x_i^2 - (\Sigma x_i)^2}} \tag{27}$$

and

$$\delta b = \sigma_y \sqrt{\frac{\Sigma x_i^2}{N\Sigma x_i^2 - (\Sigma x_i)^2}} \tag{28}$$

where, as before:

$$\Delta y_i = y_i - (mx_i + b)$$

and the factor

$$\sigma_y = \sqrt{\frac{\Sigma \Delta y_i^2}{N-2}} \tag{29}$$

which enters both equations (27) and (28) is the standard deviation of the distribution of y_i values with respect to the line $y = mx + b$. The factor (N - 2) in the denominator of equation (29) accounts for the fact that two parameters (m, b) have been determined from the data set, reducing the number of degrees of freedom from N to (N - 2).

2.6.4 (adapted from D. C. Baird)

An experiment was performed to measure the impedance Z of a series R - L circuit as a function of frequency f. The expected relationship for the circuit investigated is:

$$Z^2 = R^2 + 4\pi^2 f^2 L^2$$

where R is the resistance measured in ohms (Ω), L is the inductance measured in Henries (H) and f is the frequency in Hz. The impedance Z was measured as a function of f, as shown in the data table below. A graph of Z^2 versus f^2 is expected to yield a straight line whose slope can be used to determine the inductance L:

f (Hz)	**Z** (Ω)
123±4	7.4±0.2
158	8.4
194	9.1
200	9.6
229	10.3
245	10.5
269	11.4
292	11.9
296	12.2

In the following exercises you must show all of your calculations and formulas in order to receive full marks!

a) Reduce these data in order to plot them in the manner indicated on a sheet of graph paper, and draw error bars to show the uncertainties on the data points. (Set up a new data table with additional columns to allow for the quantities you need to calculate, together with their uncertainties, in order to draw the graph.) Make sure the graph is neatly drawn and clearly labelled.

b) Using a clear plastic ruler, draw the best 'eyeball fit' solid line through the points, attempting to minimize the distance between the line and the set of data points by eye. Calculate the slope of this line, and the corresponding value of L.

c) Draw dotted lines on your graph corresponding to the largest and smallest reasonable slopes that would be consistent with the data (in your opinion), and extract from the slopes of these lines an uncertainty range for the value of L. The

range does not have to be symmetric about the optimum value obtained in part b)!

d) From the intercepts for the lines drawn in (b) and (c), obtain the best value for R and an uncertainty range for this value.

e) Perform an unweighted least squares fit to obtain the slope and intercept of the best line through the data points. Use the formulae given in this section of the lab manual in order to obtain the uncertainties in the slope and intercept of the line. Deduce the corresponding values of R and L and their uncertainties.

f) Compare your findings for R and L and their uncertainties for the two methods used to analyze the data, and comment on the results.