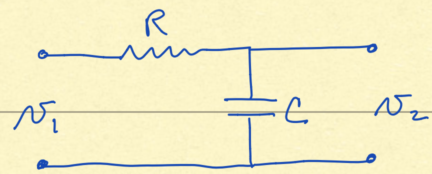
Phys 2610 (2019) Assignment 3 solutions

1. Design a low-pass RC filter that will attenuate a 5.6 kHz sinusoidal signal by 3 dB. Sketch the gain in dB vs the log(frequency) for the circuit.

low pass filter:



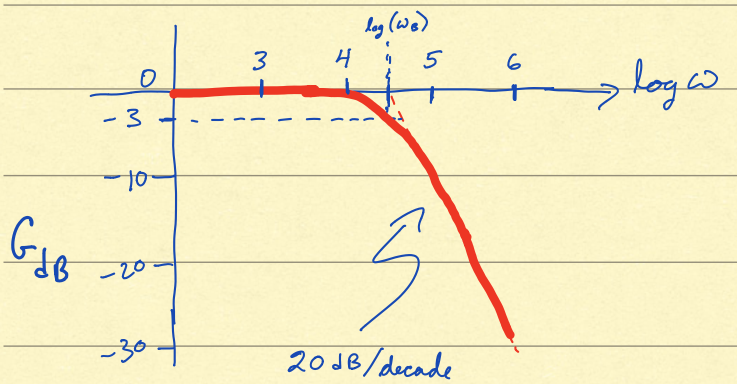
The gain is $a = \frac{v_2}{v_1} = \frac{1}{\omega C z} e^{j\phi}$ where $z = R + 1/j\omega C$

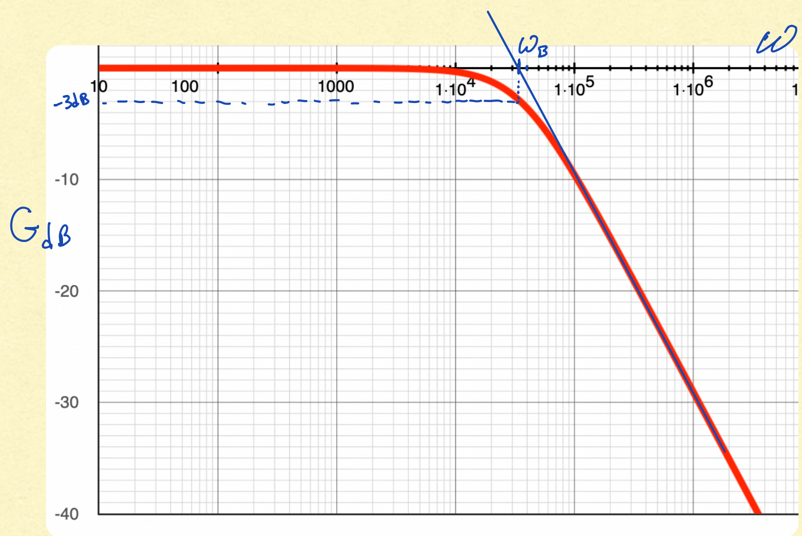
so $|a| = \frac{V_2}{V_1} = \frac{1}{\sqrt{1+(\omega RC)^2}}$ and $G_{dB} = 20 \log |a|$

For 3 dB attenuation, $|a| = \frac{V_2}{V_1} = \frac{1}{\sqrt{2}}$, so $\omega RC = 1$

so for $f = 5.6\,\text{kHz}$, $\omega_B = 2\pi f = 35.2 \times 10^3$ rad/s $\rightarrow$ $RC = \frac{1}{\omega_B} = 28.4\,\mu s$

e.g. $C = 2.2\,nF$, $R = 12.9\,k\Omega$





2. (a) Determine the resonant frequency and the Q-factor for a series LRC circuit with $R = 560\ \Omega$, $L = 100$ mH, and $C = 33$ nF.

(b) Show that the frequency for the peak capacitor voltage is given by $\sqrt{\omega_0{}^2 - \frac{R^2}{2L^2}}$, and evaluate the frequency for the given values.

(c) Find the magnitude and phase of the impedance of the circuit for a frequency of 1.0 kHz.

(a) Resonant freq: $\omega_0 = \frac{1}{\sqrt{LC}} = 17{,}400$ rad/s $\rightarrow f_0 = \frac{\omega_0}{2\pi} = 2.77$ kHz

Q-factor: $Q = \frac{1}{R}\sqrt{\frac{L}{C}} = 3.11$

(b) The amplitude of the capacitor voltage is

$V_c = I X_c = \frac{V}{|Z|}\left(\frac{1}{\omega C}\right)$ where V is the input amplitude, and

$|Z| = |R + j(\omega L - 1/\omega C)| = \sqrt{R^2 + (\omega L - 1/\omega C)^2}$

The freq. for max amplitude satisfies $\frac{d}{d\omega} V_c = 0$

$$\Rightarrow \frac{d}{d\omega} \frac{1}{\sqrt{(\omega RC)^2 + (\omega^2 LC - 1)^2}} = 0$$

$$\Rightarrow \left(-\tfrac{1}{2}\right)\left((\omega RC)^2 + (\omega^2 LC - 1)^2\right)^{-3/2} \cdot \left(2\omega (RC)^2 + 2(\omega^2 LC - 1)\, 2\omega LC\right) = 0$$

$$\Rightarrow 2\omega (RC)^2 + 4(\omega^2 LC - 1)\,\omega LC = 0$$

$$\Rightarrow \frac{2\omega (RC)^2}{4\omega LC} = 1 - \omega^2 LC \Rightarrow \omega^2 LC = 1 - \frac{R^2 C}{2L}$$

$$\Rightarrow \omega^2 = \frac{1}{LC} - \frac{R^2}{2L^2} \Rightarrow \boxed{\omega^2 = \omega_0{}^2 - \frac{R^2}{2L^2}}$$

Using the given component values, $\omega = \sqrt{(17{,}400)^2 - \frac{(560)^2}{2(0.1)^2}}$ rad/s

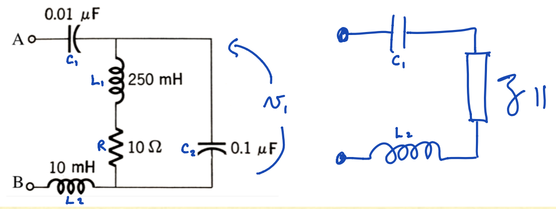
or $\boxed{\omega = 16{,}900 \text{ rad/s}}$

(c) Magnitude of impedance: $|Z| = \sqrt{R^2 + (\omega L - 1/\omega C)^2} = \boxed{4.23\ \text{k}\Omega}$

for $\omega = 2\pi f = 2\pi\,(1000\text{ Hz})$

Phase of impedance: $\theta = \arctan \frac{\omega L - 1/\omega C}{R} = \boxed{-82^\circ \text{ or } -1.44 \text{ rad}}$

3. Calculate the equivalent impedance of the circuit below (between A and B) at a frequencies of 100 Hz and 1000 Hz. Find the rms current in the 10 Ω resistor for a sinusoidal input with an amplitude of 8 V and a frequency of 1000 Hz.



$$Z = Z_{C_1} + Z_{\parallel} + Z_{L_2} \qquad \text{where } Z_{\parallel} = (Z_{L_1} + R) \parallel Z_{C_2}$$

Evaluate component impedance at $f = 100\ \text{Hz},\ 1000\ \text{Hz}$

$$Z_{C_1} = \frac{1}{j\omega C_1} = \frac{-j}{2\pi f (0.01\mu F)} = -j\frac{500k\Omega}{\pi},\ -j\frac{50k\Omega}{\pi}$$

$$Z_{C_2} = \frac{1}{j\omega C_2} = \frac{-j}{2\pi f (0.1\mu F)} = -j\frac{50k\Omega}{\pi},\ -j\frac{5k\Omega}{\pi}$$

$$Z_{L_1} = j\omega L_1 = j2\pi f(250\,mH) = j50\pi\ \Omega,\ j500\pi\ \Omega$$

$$Z_{L_2} = j\omega L_2 = j2\pi f(10\,mH) = j2\pi\ \Omega,\ j20\pi\ \Omega$$

$$Z_R = R = 10\ \Omega$$

Then,

$$Z_{\parallel} = \frac{(R + Z_{L_1})Z_{C_2}}{R + Z_{L_1} + Z_{C_2}} = \frac{(10 + j50\pi)(-j50k/\pi)}{10 + j50\pi - j50k/\pi}\ \Omega,\ \frac{(10 + j500\pi)(-j5k/\pi)}{10 + j500\pi - j5k/\pi}$$ *

$$= (10.2 + j159)\ \Omega,\ (47.7 + j97.5)\ k\Omega$$

and

$$Z = Z_{C_1} + Z_{\parallel} + Z_{L_2} = -j\frac{500k}{\pi} + 10.2 + j159 + j2\pi\ \Omega,\ -j\frac{50k}{\pi} + 47.7k + j97.5k + j20\pi$$

$$Z = \boxed{(10.2 - j159k)\ \Omega},\ \boxed{(47.7 + j81.6)\ k\Omega}$$

100 Hz   1000 Hz

Current through 10 Ω resistor:

$$i_1 = \frac{v_1}{Z_{L_1} + R} \quad \text{with} \quad v_1 = v\frac{Z_{\parallel}}{Z}$$

$$\text{so } i = v\frac{Z_{\parallel}}{Z(Z_{L_1} + R)} = v\frac{(47.7 + j97.5)k}{(47.7 + j81.6)k\,(j500\pi + 10)} = \frac{8Ve^{j\omega t}}{1.37k\Omega\ e^{j\theta}}$$ **

$$\therefore \text{Peak current is } I = \frac{8V}{1.37k\Omega} = 5.84\ mA \Rightarrow \boxed{I_{rms} = \frac{I}{\sqrt{2}} = 4.1\ mA}$$

* example calculation

$$\frac{(10 + j50\pi)(-j\,50k/\pi)}{10 + j50\pi - j\,50k/\pi}\ \Omega \simeq \frac{2.5M - j\,500k/\pi}{10 - j15.76k} \cdot \frac{10 + j15.76k}{10 + j15.76k}\ \Omega$$

$$= \frac{(25M + 2.508\times10^{9}) + j(39.4\times10^{9} - 5M/\pi)}{100 + 2.48\times10^{8}}\ \Omega$$

$$= (10.2 + j159)\ \Omega$$

6%

WolframAlpha | PRO PREMIUM

(10+I*50*pi)*(-I*50000/pi)/(10+I*50*pi-I*50000/pi)

Browse Examples   Surprise Me

Input:

$$(10 + i\times 50\,\pi)\left(-\frac{i\times\frac{50000}{\pi}}{10 + i\times 50\,\pi - i\times\frac{50000}{\pi}}\right)$$

Open code

i is the imaginary unit

Result:

$$-\frac{50\,000\,i\,(10 + 50\,i\,\pi)}{\left(10 - \frac{50000\,i}{\pi} + 50\,i\,\pi\right)\pi}$$

Decimal approximation:

More digits

10.200349188813971631250928244128683417229221405217251260 3... +
158.6389270638075853143755184032532016545919969654497016 83... i

**

WolframAlpha | PRO PREMIUM

(47.7+I*81.6)*(I*500*pi+10)/(47.7+I*97.5)

Browse Examples   Surprise Me

Input:

$$(47.7 + i\times 81.6)\times\frac{i\times 500\,\pi + 10}{47.7 + i\times 97.5}$$

Open code

i is the imaginary unit

Result:

More digits

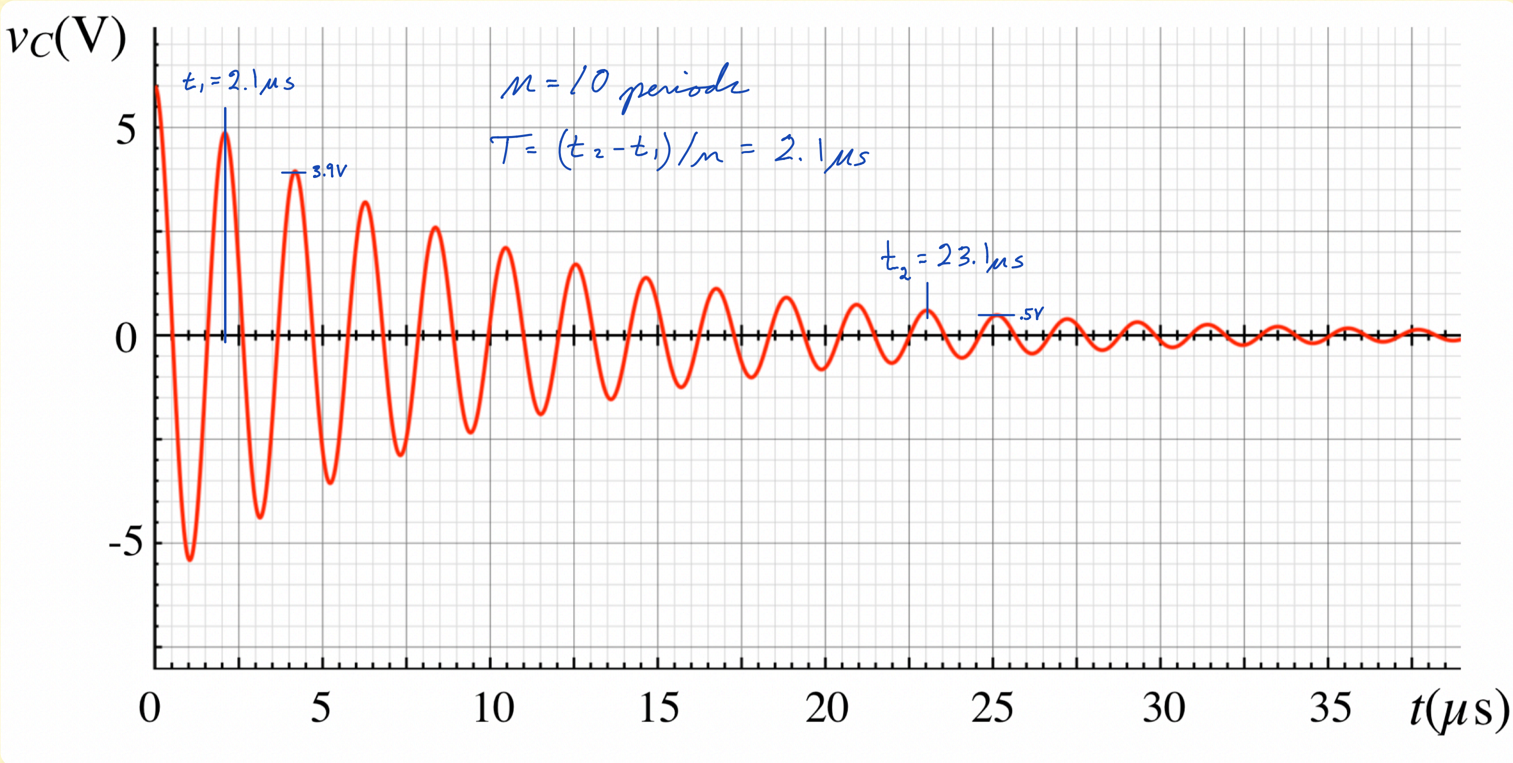
109.803... +
1363.46... i

Polar coordinates:

$r = 1367.88$ (radius), $\theta = 85.3958°$ (angle)

4. A series RLC circuit is subjected to an applied pulse. The resulting voltage waveform across the capacitor is shown in the figure. The horizontal scale is in μs and the vertical scale is in volts. The general solution for the transient response of the capacitor voltage, with small damping is: $v_C = V_C e^{-\alpha t}\sin(\omega t + \varphi)$.

(a) Using the graph below, determine the values for the constants V_c, α, ω, and φ.



(a) Frequency, $\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{(t_2 - t_1)/m}$ $\Rightarrow$ $\boxed{\omega = 3.0 \text{ Mrad/s}}$

Since v_c is max for $t = t_1 = T$, it is also max for $t = mT$ $\underset{(\text{for } m=0)}{\Rightarrow}$ $\sin\varphi = 1$

$\Rightarrow$ $\boxed{\varphi = \pi/2}$

Since v_c is a max at $t = 0$, $v_c(0) = \boxed{V_c = 6V}$

Finally, for $t = 2T$, $v_c = V_c e^{-\alpha 2T} = 3.9V$ $\Rightarrow$ $\left(\alpha = \ln\frac{V_c}{3.9V}\right)\left(\frac{1}{2T}\right)$

$\Rightarrow$ $\boxed{\alpha = 100 \times 10^3 \ s^{-1}}$

(b) Find component values for R, L, and C that could produce such a response.

Since $\omega = \sqrt{\omega_0^2 - \alpha^2}$, $\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{\omega^2 + \alpha^2} \Rightarrow LC = \frac{1}{(\omega^2 + \alpha^2)}$

Also, $\alpha = R/2L$

So, the value of L defines R & C.

e.g. $L = 0.5\,mH \Rightarrow R = 2L\alpha = 100\,\Omega$

and $\Rightarrow C = \frac{1}{L(\omega^2 + \alpha^2)} = 222\,pF$

(c) What is the resonant frequency $\omega_0 = 1/\sqrt{LC}$? Is the condition for small damping satisfied?

It's clear from part (a) that $\alpha \ll \omega$, so that $\omega_0 \cong \omega = 3.0 \times 10^6$ rad/s

This is consistent with $\omega_0 = 1/\sqrt{LC} = 3.0 \times 10^6$ rad/s.

The condition can also be expressed as $\zeta = \frac{\alpha}{\omega_0} = \frac{R}{2}\sqrt{\frac{C}{L}} \ll 1$

Using the above components $\zeta = 0.03$