Chapter 28

The Theory of Relativity

- Borne of experiment
- Resolves problem of action-at-a-distance
- Explains magnetism

Part I: Introduction

- The principle of relativity
- Does E&M violate the principle?
- The need for ether
- Einstein’s postulates of special relativity

1) Principle of Relativity

(a) Galilean relativity

...have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still... from Dialogue Concerning the Two Chief World Systems, 1632.

Galileo's Principle of Relativity

The mechanical laws of physics are the same for every inertial observer. By observing the outcome of mechanical experiments, one cannot distinguish a state of rest from a state of constant velocity. Absolute rest cannot be defined.

(b) Inertial reference frame

- A reference frame in which the law of inertia holds
  \[ \vec{F} = 0 \Rightarrow \vec{a} = 0 \Rightarrow \vec{v} = \text{const} \]
- Any frame moving with uniform velocity wrt an inertial frame is an inertial frame
- Accelerated frame is not an inertial frame

(c) Event

- Defined by spatial coordinates: x, y, z
  and by time: t
- x, y, z depend on ref frame but classically, distance is absolute
- t depends on ref frame (start time) but classically, time intervals are absolute
- i.e. space and time are separately absolute
c) Galilean Transformation

\[
x' = x - vt
\]
\[
y' = y
\]
\[
z' = z
\]
\[
t' = t
\]
Coordinates are not the same, but can be easily converted.

\[
(x', y', z', t') = (x, y, z, t)
\]

\[
(x', y', z', t') = (x - vt, y, z, t)
\]

d) Velocity addition

\[
x' = x - vt \rightarrow \quad u' = u - v
\]
\[
y' = y
\]
\[
z' = z
\]
\[
t' = t
\]

Velocity is not the same but can easily be converted.

2) E & M and Relativity

a) Matter waves (sound, water...)?

- Preferred reference frame exists
  (That of the medium)
- Relativity intact: medium is detectable

b) Magnetism

Coulomb repulsion

Coulomb repulsion + magnetic attraction

Physics depends on reference frame

\[
F > 0 \text{ in lab}
\]
\[
F = 0 \text{ in particle’s rest frame}
\]

How is \( v \) defined?

Magnetism seems to require a preferred reference frame.
c) The speed of light
- a fundamental and testable prediction

Galilean transformation:
\[ u' = u - v \]
\[ c' = c - v \]

Maxwell’s equations:
\[ c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \] (ref frame not specified)

Either • a preferred frame exists (medium), or
• one of the above is wrong

b) Properties of the ether

• penetrates all matter without friction
• zero mass, unaffected by gravity
• perfectly elastic and extremely rigid
• odorless etc
• no longitudinal waves

3) Luminiferous Ether

a) The theory

Electromagnetism results from stresses and strains of a medium:

The ether.

Maxwell’s equations apply in the rest frame of the ether
(speed of light is c only in this ref frame)

b) Properties of the ether

The luminiferous ether, that is the only substance we are confident of in dynamics… One thing we are sure of, and that is the reality and substantiality of the luminiferous ether.

Lord Kelvin (1891)

The beginner will find it best to accept the ether theory, at least as a working hypothesis… Even if the extreme relativists are right and that there is no ether, it is likely that the change will involve no serious readjustments so far as explanations of the ordinary phenomena are concerned.

1928 College text

We have learned too that radiant heat energy is believed to be transmitted by a medium called the ether. At the present time, some scientists believe that other ether waves produce various other effects… It is possible then that light waves are ether waves.

1939 High School text
c) Detection of the ether

Time to travel $D_A$ or $D_B$ depends on orientation wrt ether wind.

In at least one season we must experience an ether wind with speed $v$.

$\nu = 3 \times 10^4 \text{ m/s}$

$\nu = 3 \times 10^8 \text{ m/s}$

A 0.4 fringe shift was expected on rotation; none was observed, in any season.

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d) Effects of the ether

- Aberration ruled out the earth dragging the ether with it
- Motion through ether causes contraction, time dilation etc
- Changes conspire to make it impossible to detect ether
- Led to Lorentz transformation
- Predictions identical to Einstein’s relativity
- Still has popular following

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4) Postulates of Special Relativity

ON THE ELECTRODYNAMICS OF MOVING BODIES

By A. EINSTEIN

It is known that Maxwell’s electrodynamics—as usually understood at the present time—when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on

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We will raise this conjecture (the purport of which will hereafter be called the “Principle of Relativity”) to the status of a postulate, and also introduce another postulate, which is only apparently irreconcilable with the former, namely, that light is always propagated in empty space with a definite velocity $c$ which is independent of the state of motion of the emitting body.

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These two postulates suffice for the attainment of a simple and consistent theory of the electrodynamics of moving bodies based on Maxwell’s theory for stationary bodies. The introduction of a “luminiferous ether” will prove to be superfluous inasmuch as the view here to be developed will not require an “absolutely stationary space” provided with special properties, nor assign a velocity-vector to a point of the empty space in which electromagnetic processes take place.
4) Postulates of Special Relativity

1) The laws of physics take the same form in all inertial frames
   - Galilean transformation wrong (approximation)
2) Speed of light independent of velocity of the source or observer
   - Immediate consequence: time must stop if your ride the crest of a light wave

Part II - Relativistic Kinematics

Einstein spent most of year struggling with the invariance of light speed and the violation of ordinary velocity addition. Then it came to him:

“Time cannot be absolutely defined, and there is an inseparable relation between time and signal velocity. With this new concept, I could resolve all difficulties... Within 5 weeks, the Special Theory of Relativity was completed.”

1) Simultaneity

Now we must bear carefully in mind that a mathematical description of this kind has no physical meaning unless we are quite clear as to what we understand by “time.” We have to take into account that all our judgments in which time plays a part are always judgments of simultaneous events. If, for instance, I say, “That train arrives here at 7 o’clock,” I mean something like this: “The pointing of the small hand of my watch to 7 and the arrival of the train are simultaneous events.”

There is one single and invariable time, which flows in two movements in an identical and simultaneous manner... Thus, in regard to movements which take place simultaneously, there is one and the same time, whether or not the movements are equal in rapidity... The time is absolutely the same for both.

Aristotle

What then is time? If no one asks me, I know: If I wish to explain it to one that asketh, I know not.

St. Augustine

a) Definition

For 2 events at the same location, the definition is obvious

For 2 events separated in space, they are simultaneous if the signal reaches an observer at the midpoint at the same time

Is it independent of reference frame?
**b) Relativity of simultaneity**

Two flares fired simultaneously by the station, observed by the station.

Light waves reach the rocket observer at different times.

Light waves reach station observer simultaneously.

**Observed by rocket**

Rocket observer sees Flare A first, so in the rocket frame, it is fired first. (Speed of signal is the same.)

**Simultaneity is relative**

**2. Time dilation**

**a) Light clock**

![Light clock diagram]

**Craft** | **Speed** | **\( \gamma - 1 \)**
---|---|---
Craft | km/s | m/s
Car | 100 | 28 | 4.E-15
Commercial jet | 885 | 246 | 3.E-13
Concord (mach 2) | 2448 | 680 | 3.E-12
SR-71 | 3550 | 980 | 3.E-12
Space shuttle | 16,200 | 4500 | 1.E-08
Cassini | 16,200 | 4500 | 1.E-08
TV electrons (c = 0.28c) | 1.04 | 0.866 | 2.00
TV electrons (c = 0.90c) | 2.29 | 0.90 | 3.20E+00
TV electrons (c = 0.95c) | 3.20E+00 | 0.95 | 7.09E+00

**b) Time dilation**

Time interval measured in the ship:

\[
\Delta t = \frac{2D}{c}
\]

Time interval measured on earth:

\[
\Delta t_{\text{obs}} = \frac{2D}{c} \left[ 1 - \frac{v^2}{c^2} \right]
\]

\[
\Delta t = \frac{\Delta t_{\text{obs}}}{\gamma} = \gamma \Delta t_{\text{obs}}
\]

\[
\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

**c) Proper time, \( \Delta t_0 \)**

Time measured in rest frame of clock.

Time interval between 2 events at the same location

Shortest time between events

Time elapsed for observer in any other frame is longer

Hence, time dilation

Moving clocks run slow
There was a young lady named Bright,  
Who traveled much faster than light.  
She started one day  
In the relative way,  
And returned on the previous night.

Punch, 1923  
Reginald Buller  
Professor of Botany  
University of Manitoba

Example: Lifetime of a muon

Mean lifetime in rest frame: $\Delta t_0 = 2.2 \text{ ms}$  
Produced by cosmic rays 5000 m above sea level.  
Speed $v = 0.998c$  
Find the mean distance it would travel ignoring time dilation.  
Find the muon lifetime in the earth’s reference frame.  
Find the mean distance the muon travels.

Example:

Find time to travel to a star 3.5 light years away at speed $0.866c$ according to the astronaut, and according to an earthbound observer.

1 light year = distance light travels in one year  
$= c \cdot (1 \text{ year}) = 3 \times 10^8 \cdot 365.25 \cdot 24 \cdot 60 \cdot 60 = 9.45 \times 10^{15} \text{ m}$

The answer can be determined by applying time dilation in any inertial frame. The rocket is not an inertial frame.

<table>
<thead>
<tr>
<th>frames</th>
<th>Earth clock</th>
<th>Inbound</th>
<th>Turn-around</th>
<th>Return</th>
</tr>
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<tr>
<td>Earth frame</td>
<td>0</td>
<td>4</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>Rocket clock</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td></td>
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<tr>
<td>Outbound frame</td>
<td>Earth clock</td>
<td>0</td>
<td>1</td>
<td>1 + 1 = 8</td>
</tr>
<tr>
<td>Rocket clock</td>
<td>0</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inbound frame</td>
<td>Earth clock</td>
<td>0</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>Rocket clock</td>
<td>0</td>
<td>2</td>
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<td></td>
</tr>
<tr>
<td>Rocket frame</td>
<td>Earth clock</td>
<td>0</td>
<td>1 + 7 = 8</td>
<td>8</td>
</tr>
</tbody>
</table>

$\gamma = 0.990c$  
$\gamma = 7.0$

The travelling twin is unambiguously younger.
3. Length contraction

a) The effect

In the rocket example, the relative speeds are equal, but time intervals are different.

$$=>$$ Distances must be different.

### b) Proper length, $\Delta L_0$

Length of object in its rest frame.

Length measured in any other frame is shorter.

Hence, length contraction.

Example: Problem 28.16

Rockets A and B flying parallel with relative speed $0.940c$.

According to A, the rockets have equal length. What is the ratio of lengths according to B?
The mathematical education of the young physicist [Albert Einstein] was not very solid, which I am in a good position to evaluate since he obtained it from me in Zurich some time ago.

From henceforth, space by itself, and time by itself, have vanished into the merest shadows and only a kind of blend of the two exists in its own right.

Hermann Minkowski, 1864 - 1909

5. Velocity addition

For the observer on the ground,

\[ v_{AB} = \frac{v_{AC} + v_{CB}}{1 + \frac{v_{AC}v_{CB}}{c^2}} \]

\[ v_{AB} = \text{velocity of A relative to B} \]

Confirmation for a light beam

For the observer on the ground,

\[ v_{AC} = c \]
\[ v_{CB} = v \]
\[ v_{BA} = c + v \]
\[ 1 + \frac{v}{c} \]
\[ (c + v)/c = c \]

Example

Rocket A moves at .866c away from earth.
Rocket B moves at .866c toward earth (in the same line).

Find speed of rocket B relative to A.

\[ v_{BA} = \frac{v_{AC} + v_{CB}}{1 + \frac{v_{AC}v_{CB}}{c^2}} \]

\[ v_{BA} = !.866c \]
\[ !.866c \]
\[ 1 + .866^2 \]
\[ .9897c \]

6. Cosmic Speed limit (1)

• If \( v > c \), an observer would be seen to overtake light, violating postulate 2.

• \[ y = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow \text{If } v > c, y \text{ is imaginary.} \]

• \[ y_{AB} = \frac{v_{AC} + v_{CB}}{1 + \frac{v_{AC}v_{CB}}{c^2}} \Rightarrow \text{If } v < c \text{ in one frame, it is less than } c \text{ in all frames.} \]

Part III - Relativistic Dynamics
1. Mass and Energy
(from Maxwell’s equations)

Light with energy $E$, has momentum: $p = \frac{E}{c}$

At $t = 0$, light pulse is emitted
By cons. of momentum, $Mv = \frac{E}{c}$

At time $t = L/c$, light pulse is absorbed

The enclosure moves a distance $x = vt = \frac{EL}{Mc^2}$

The CM of an isolated system doesn’t move, so $Mx = mL$ = 0

At time $t = L/c$, light pulse is absorbed

$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$

2. Relativistic Momentum

a) $p = mv$ not conserved in all reference frames

Inelastic collision in CM frame (S)
$p_f = 0, p_i = 0$

In rest frame of right mass, classically
$p_i = 2mv_0, p_f = 2mv_0$

In rest frame of right mass, relativistically
$p_i < 2mv_0, p_f = 2mv_0$

b) Requirements for relativistic momentum
- Conserved in all inertial frames
- Reduce to $mv$ for $v \ll c$

c) Expression
Derived using glancing collision, with $v_y \ll c$ but $v_x$ near $c$

$p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$

$d) Relativistic mass (no longer recommended)$

Could preserve the classical form, by defining rest mass as $m_0$, and relativistic mass as $m = \gamma m_0$

Modern terminology defines mass $m$ as the invariant property of an object (rest mass), and redefines momentum as $p = \gamma mv$

Example: Problem 28.21 revised

Skaters, 100 kg and 50 kg push, each other. The 100 kg skater recoils at 2.5 m/s relative to the ice. Find the recoil velocity of the 50 kg skater classically, and relativistically if $c = 3$ m/s.

3. Relativistic Energy

a) Kinetic energy

Recall in 1d, $\Delta KE = F\Delta x$, and $F = \frac{dp}{\Delta t}$

Using relativistic momentum, $\Delta KE = \frac{\Delta (m\gamma v)}{\Delta t} \Delta x$

From this, it can be shown (integrating): $KE = m\gamma v^2 + \text{const.}$

For $v = 0, \gamma = 1$, $KE = 0$, so $0 = mc^2 + \text{const.}$

Then $\text{const.} = -mc^2$, so $KE = m\gamma v^2 + mc^2$

$KE = mc^2(\gamma - 1)$
\[ KE = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} = 1 + \frac{1}{2} \frac{v^2}{c^2} + \frac{3}{8} \left( \frac{v^2}{c^2} \right)^2 + \ldots \]

For \( v \ll c \), \( \gamma \rightarrow 1 + \frac{1}{2} \frac{v^2}{c^2} \)

Then \( KE = mc^2(\gamma - 1) = \frac{1}{2} m v^2 \)

\[ KE = mc^2(\gamma - 1) \quad mc^2 \gamma = KE + mc^2 \]

\[ E = KE + E_0 \]

\[ E = m \gamma c^2 \]

**c) Rest Energy**

Since \( mc^2 \) is independent of velocity, it is called the rest energy:

\[ E_0 = mc^2 \]

\**d) Total energy**

The total energy is the sum of rest and kinetic energy:

\[ E = KE + E_0 \]

\[ E = m \gamma c^2 \]

\**e) Conservation of total energy; mass-energy equivalence**

Total energy, \( E = mc^2 \), is always conserved (like momentum).

Since \( E = KE + m c^2 \)

and KE is not always conserved, other forms of energy (heat, potential energy) must have mass.

**Numerical examples:**

(i) Rest energy of a baseball
(ii) Rest energy of an electron
(iii) Rest energy of a proton
(iv) Total and KE of a .85c electron
(v) Velocity of a 1 TeV electron
(vi) Mass change of the sun

**Example: Problem 28.29**

How close would two stationary electrons have to be placed to double their mass?
4. Energy-momentum relation

Velocity near c is often an inconvenient quantity (.9999…)

Eliminating \( v \) from \( E = m\gamma c^2 \) and \( p = m\gamma v \), gives

\[
E^2 = p^2c^2 + m^2c^4
\]

Example: Problem 28.28

Total energy 5.0E15 J, kinetic energy 2.0E15J. Find relativistic momentum.

5. Zero mass particles

a) \( v < c \)

If \( m = 0 \), and \( v < c \), then \( p = m\gamma v = 0 \) and \( E = m\gamma c^2 = 0 \)

\( \Rightarrow \) no particle.

b) \( v = c \)

If \( v = c \), then \( p = m\gamma v \) is undefined

\( E = m\gamma c^2 \) is undefined

but from \( E^2 = p^2c^2 + m^2c^4 \)

\( E = pc \)

(predicted by Maxwell’s equations for light)

6. Cosmic speed limit (2)

\( E = m\gamma c^3 \)

For an object with finite mass, as \( v \to c \), \( E \to \infty \)

Therefore, infinite work is required to accelerate a mass to the speed of light.

Part IV: Brief look at general relativity

1) The need for general relativity

(a) GR is first and foremost a theory of gravity

Coulomb’s law + SR \( \to \) Maxwell’s eqns

removes action-at-a-distance from C’s law

Newton’s gravity + relativity theory \( \to \) ?

needed to remove action-at-a-distance from gravity
(b) Generalize principle of relativity to non-inertial frames

“The weakness in the principle of inertia is this, that it involves an argument in a circle: a mass moves without acceleration if it is sufficiently far from other bodies; we know it is sufficiently far from other bodies only by the fact that it moves without acceleration.” - Einstein

Einstein set out to accomplish the 2 goals in 1905, but they turned out to be equivalent.

It took 10 years (compared to 5 weeks for SR).

No other major scientific achievement is so much the result of one person.

As an older friend I must advise you against it for in the first place you will not succeed, and even if you succeed no one will believe you. - Max Planck, 1913

2) The principle of equivalence

The happiest thought of my life...

The gravitational field has only a relative existence... Because for an observer freely falling from the roof of a house - at least in his immediate surroundings - there exists no gravitational field. -- Einstein

Gravitation: A very special force.

\[ F = m_i a \]

\[ F = \frac{Gm_i M}{r^2} \]

If \( m_i = m_g \),

\[ \frac{F}{a} = \frac{Fr^2}{GM} \rightarrow a = \frac{GM}{r^2} = g \]

Acceleration independent of mass

It is, however, clear that science is fully justified in assigning such a numerical equality only after this numerical equality is reduced to an equality of the real nature of the two concepts.” -- Einstein

The principle of equivalence

A uniform gravitational field is completely equivalent to a uniformly accelerated reference frame.

- No local experiment can distinguish them
- Concept of inertial frame no longer useful
- Gravity is geometrical
3) The principle of general relativity

- All reference frames are equally suitable for the formulation of the laws of physics
- “We shall be true to the principle of relativity in its broadest sense if we give such a form to laws that they are valid in every 4-d system of coordinates”

4) Curved space

- Shortest path is a curve; like a geodesic
- Curved geometry emulates force:
  - Consider 2 people walking north from the equator.
  - They begin parallel, and follow straight lines, yet meet at the pole.
  - If north axis is time, they accelerate towards each other as though attracted by a force

5) Gravity and light

Light is bent in an accelerating frame

--> Light is bent by gravity

Verified by Eddington during eclipse of 1919; Einstein became instant celebrity
• **The Einstein Cross**
  4 images of quasar
  formed by gravitational
  lensing

6) Other consequences of general relativity
• **Time slows down in gravitational field**
• **Precession of perihelion of Mercury by 0.01° per century**
• **Black holes**
  – Recall escape velocity $v_{esc} = \sqrt{\frac{2GM}{R}}$
  – If $v_{esc}=c$, light cannot escape
  – Schwarzschild radius $R_s = \frac{2GM}{c^2}$
    • Earth: 9 mm
    • Sun: 3 km

7) Cosmic speed limit (3)
• **Gravity waves**
  – Not yet detected
  – Speed of gravity = c (GR)
    • Preliminary measurements: ~ c
• **SR applies to inertial frames. One could argue it has no authority over accelerating frames**
• **Super-luminal speed?**
  – Stars in a rotating reference frame
  – Trace of moving light beam
  – Intersection of scissors
  – Relative speed of 2 objects in 3rd frame
• **GR statement: No signal can be transferred faster than c.**
• **Open doors?**
  – worm holes
  – warp drives