The Relativity of Length

If Δt_0 is the proper time interval between events in one reference frame, then in any other frame the time Δt between these events is

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-\beta^2}} = \gamma \Delta t_0 \qquad \text{(time dilation)}$$

where v is the relative speed between these frames ($\beta = v/c$).

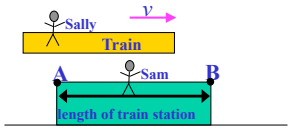
A direct consequence of this is "length contraction"

$$L = L_0\sqrt{1-\beta^2} = \frac{L_0}{\gamma} \qquad \text{(length contraction)}$$

The length L_0 of an object in the rest frame of the object is its **proper length** or **rest length**. The length L measured in any other reference frame that is in motion parallel to the length is always less than the proper length.

Proof of Length Contraction

Sam is sitting on a bench at the train station. Using a tape measure, Sam determines the length of the station in his frame, which is the proper length L_0. Sally is sitting on a train that passes through the station, and measures the length of the station using a stopwatch. What is the length L of the train station that Sally measures?



Event 1: Sally passes point A
Event 2: Sally passes point B

According to Sam, Sally moves through the station (passing between point A and then point B, which are **different places** in Sam's frame) in a time interval $\Delta t = L_0/v$:

$$\therefore L_0 = v\Delta t \qquad \text{(Sam)}$$

For Sally, the platform moves past her. She passes points A and B at the **same place** in her reference frame, in a proper time interval Δt_0:

$$\therefore L = v\Delta t_0 \qquad \text{(Sally)}$$

$$\therefore \frac{L}{L_0} = \frac{v\Delta t_0}{v\Delta t} = \frac{1}{\gamma} \quad \text{or} \quad L = \frac{L_0}{\gamma}$$

Time Dilation and Length Contraction

Time dilation: Moving clocks run **slow** (compared to the proper time interval Δt_0)

$$\Delta t = \gamma \Delta t_0 \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \geq 1$$

Proper time Δt_0: occurs in frame where events take place at the same spatial location.

Length contraction: Moving objects are **contracted** along the direction of motion (compared to the proper length L_0)

$$L = \frac{L_0}{\gamma}$$

Proper length L_0: measured in frame where the object is at rest.

Example

Suppose you are at a railroad crossing, watching a train go by.

(a) Both you and a passenger in the train are looking at a clock on the train. Which of you measures the proper time interval?

(b) Who measures the proper length of the train car?

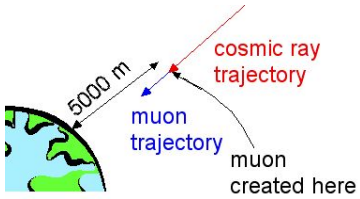
(c) Who measures the proper distance between the railroad ties?

A. you, passenger, you
B. passenger, passenger, you
C. passenger, passenger, passenger
D. passenger, you, you
E. you, passenger, passenger

Example: Lifetime of the muon

A muon is an elementary particle which decays into other elementary particles in 2.2×10^{-6} s, if measured when the muon is at rest in laboratory.

Muons are created high (5000 m) in the upper atmosphere by fast-moving cosmic particles. Speed of newly-created muon is 0.9904 c



$$v = 0.9904\,c$$
$$\therefore \gamma = \frac{1}{\sqrt{1-(v/c)^2}} = 7.23$$

Example: Lifetime of the muon

(a) Find the lifetime of the muon in the earth's reference frame.

(b) Find the distance the muon travels according to an observer on earth.

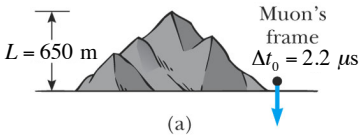
(c) Using length contraction, explain this result in the frame of the muon.

The Muon Experiment

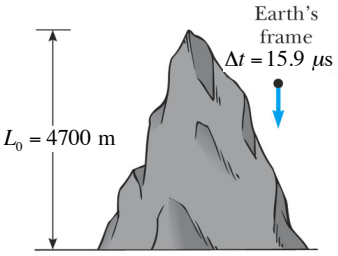
$$L = \frac{L_0}{\gamma};\quad \frac{L}{\Delta t_0} = v$$

$$v = 0.9904\,c$$
$$\therefore \gamma = \frac{1}{\sqrt{1-(v/c)^2}} = 7.23$$

$$\Delta t = \gamma \Delta t_0;\quad \frac{L_0}{\Delta t} = v$$



(a)



(b)

Example

Vega is 26.0 light years (ly) away. Sally leaves for Vega at $v = 0.990\,c$. Her twin Sam stays behind on earth.

(a) According to Sam's clock, when does Sally reach Vega? 26.3 yr

(b) If Sally sends a radio signal back from Vega, what does Sam's clock read when it arrives? 52.3 yr

(c) What does Sally's clock read when she reaches Vega? 3.7 yr

(d) If Sally returns to earth, what does her clock read on arrival, and what does Sam's clock read? 7.4 yr, 52.6 yr