9-7 Elastic Collisions in One Dimension

For a target at rest (lab frame), $v_{2i} = 0$, so we have

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f} \qquad \text{(linear momentum)}$$

$$\tfrac{1}{2} m_1 v_{1i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2 \qquad \text{(kinetic energy)}$$

Find
$$\boxed{v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}, \quad v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}}$$

- Equal masses, $m_1 = m_2$: $v_{1f} = 0$, $v_{2f} = v_{1i}$
  First object stops, exchange of velocities (*e.g.* billiard balls)
- Massive target, $m_2 >> m_1$: $v_{1f} = -v_{1i}$
  First object bounces backwards, but speed unchanged.
- Massive projectile, $m_1 >> m_2$: $v_{1f} \approx v_{1i}$, $v_{2f} \approx 2v_{1i}$
  First object keeps going; target goes forward at twice its speed
- For last 2 cases, massive object hardly affected, while Δv of lightest object is $2v_{1i}$ in magnitude

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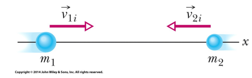
- For a target that is also moving, we have the general result:

Here is the generic setup for an elastic collision with a moving target.



$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$\tfrac{1}{2} m_1 v_{1i}^2 + \tfrac{1}{2} m_2 v_{2i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2$$

- After some algebra, we find:

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i}$$

- Special case:
  - Equal masses: $v_{1f} = v_{2i}$, $v_{2f} = v_{1i}$ (exchange of velocities)

(We recover the previous result for $v_{2i} = 0$.)

- In the centre-of-mass frame S′

$$P_i' = P_f' = 0$$

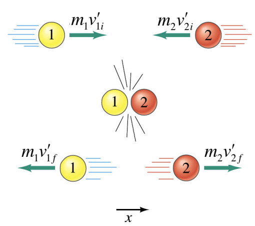
$$\therefore m_1 v_1' = -m_2 v_2' \quad \text{at all times}$$

- Completely **inelastic**:

$$v_{1f}' = v_{2f}' = 0$$

- Completely **elastic**:

$$v_{1f}' = -v_{1i}', \quad v_{2f}' = -v_{2i}'$$



Frame S′ moves at velocity u relative to lab frame S.

So $v_{1i}' = v_{1i} - u$, $v_{2i}' = v_{2i} - u$, etc.

What is u? $P_i' = m_1 v_{1i}' + m_2 v_{2i}' = m_1(v_{1i} - u) + m_2(v_{2i} - u) = 0$

$$\therefore u = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2} = v_{\text{com}}$$

i.e. u is just the velocity of the centre-of-mass (as expected)

$$\therefore v_{1f}' = -v_{1i}' \Rightarrow (v_{1f} - u) = -(v_{1i} - u)$$

$$v_{2f}' = -v_{2i}' \Rightarrow (v_{2f} - u) = -(v_{2i} - u)$$

Substituting for u, and rearranging, we find:

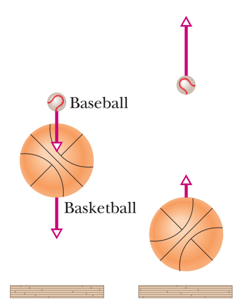
$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i}$$

This is the same as the previous result, but we have gained additional insight by using the centre-of-mass frame!

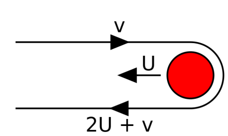
Example

Analyze the motion of two balls, one massive and one light, dropped as shown in the figure with a slight separation between the balls. Assume the collisions are all elastic.

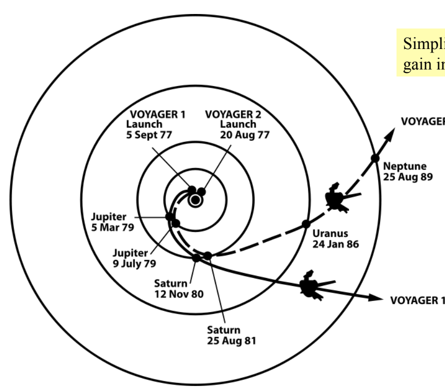


(*a*) Before (*b*) After

The gravity-assist slingshot



Simplified analysis showing maximum gain in speed for "head-on" collision



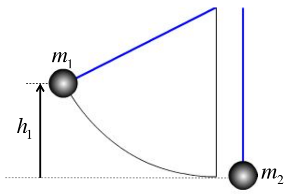
The trajectories that enabled NASA's twin Voyager spacecraft to tour the four gas giant planets and achieve enough velocity to escape the Solar System

Example

Consider a Newton's cradle with two balls, but now take them to be of different masses. One bob has mass 30 g and begins at a height 8.0 cm above the bottom of its circular arc. The other bob of mass 75 g is initially at rest.

(a) What is the maximum height of bob 1 after the collision (assuming it is elastic)?

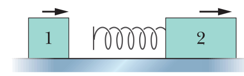
(b) What is the maximum height of bob 2 after the collision?



Problem 9.59

Block 1 has mass 2.0 kg and a velocity of 10 m/s. Block 2 has mass 5.0 kg and a velocity of 3.0 m/s. The surface is frictionless. A spring of spring constant $k = 1120$ N/m is fixed to block 2. Find the maximum compression of the spring during the collision phase.

Hint: At this point both blocks move as one at a common velocity v_f. The collision is completely inelastic at this point. The lost kinetic energy is stored in the potential energy of the spring, which we can use to calculate the compression.



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