9.1 The Center of Mass

- The **centre of mass** (*com*) of a system of particles:

> The center of mass of a system of particles is the point that moves as though (1) all of the system's mass were concentrated there and (2) all external forces were applied there.

- For two particles separated by a distance d, where the origin is chosen at the position of particle 1:

$$x_{com} = \frac{m_2}{m_1 + m_2} d$$

- For two particles, for an arbitrary choice of origin:

$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

9.1 The Center of Mass

- For n particles, we can generalize the equation, where $M = m_1 + m_2 + \ldots + m_n$:

$$x_{com} = \frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{M} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i$$

- In three dimensions, we find the centre of mass along each axis separately:

$$x_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i, \quad y_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i, \quad z_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i$$

9.1 The Center of Mass

- More concisely, we can write in terms of vectors:

$$\vec{r}_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i$$

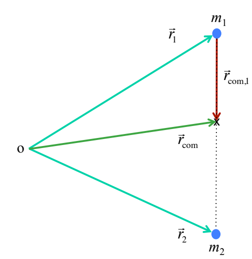
$$\vec{v}_{com} = \frac{d\vec{r}_{com}}{dt} = \frac{1}{M}\sum_{i=1}^{n} m_i \frac{d\vec{r}_i}{dt} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{v}_i$$

$$\vec{a}_{com} = \frac{d\vec{v}_{com}}{dt} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{a}_i$$

- We found that: $\vec{F}_{ext} = M\vec{a}_{com}$
  implying that the total external force acts as if all the mass of the object was at it's centre of mass.
- Therefore, if $F_{ext} = 0$, v_{com} is **constant**.

Example

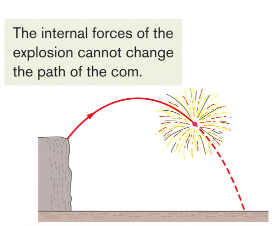
Show that the *com* of a two-particle system lies along a straight line between the two particles a fractional distance $m_2/(m_1+m_2)$ from object 1.



9-2 Newton's Second Law for a System of Particles

Examples: Using the centre of mass motion equation:

- Billiard collision: forces are only internal, $F_{ext} = 0$ so $a = 0$.
- Baseball bat: $a = -g$, so *com* follows gravitational trajectory.
- Exploding rocket: explosion forces are internal, so only the gravitational force acts on the system, and the *com* follows a gravitational trajectory as long as air resistance can be ignored for the fragments.



Problem 9.12

Two skaters, one of mass 65 kg and the other of mass 40 kg, stand 10 m apart on a frictionless ice surface. They pull themselves along a rope until they meet. What is the position where they meet?

Problem

A person of mass 50 kg stands at the one end of a boat of mass 30 kg and length 5 m. Assume there is no friction between the boat and the water, and that the boat is pointed towards shore.

If the person walks a distance d_P along the boat, how far does the boat move from shore?

Show that you get the expected result when d_P = 5 m.

9.1 The Center of Mass

- For solid bodies, we take the limit of an infinite sum of infinitely small particles → integration!
- We limit ourselves to objects of uniform density, $\rho = \frac{M}{V}$.
- Therefore the mass dm in a volume element dV is $dm = \rho\, dV$.

$$\vec{r}_{\text{com}} = \frac{1}{M}\int \vec{r}\, dm = \frac{1}{V}\int \vec{r}\, dV = \int \vec{r}\, dx\, dy\, dz$$

- Note that this is a three-dimensional integral for each component of $\vec{r}_{\text{com}}$.
- You can bypass one or more of these 3 integrals if the object has symmetry.