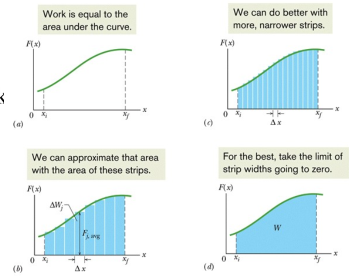
Work Done by a General Variable Force

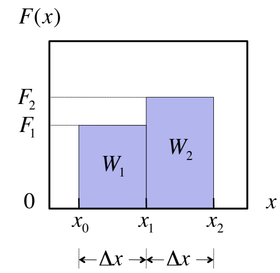
- We take a one-dimensional example
- We need to integrate the work equation (which normally applies only for a constant force) over the change in position
- We can show this process by an approximation with rectangles under the curve



Consider a piecewise constant force in 1D with 2 terms:

Total work done is equal to area under $F(x)$ vs. x curve:

$$W = W_1 + W_2 = F_1\Delta x + F_2\Delta x$$



In each panel:

$$W_1 = \Delta K_1 = \tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_0^2$$

$$W_2 = \Delta K_2 = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$

Therefore

$$W = \Delta K_1 + \Delta K_2 = (\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}\cancel{mv_1^2}) + (\tfrac{1}{2}\cancel{mv_1^2} - \tfrac{1}{2}mv_0^2)$$
$$= \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_0^2 = \Delta K$$

i.e. W equals change in kinetic energy over the whole interval!

Extend this to N rectangles: $W = \sum_{j=1}^{N} F_j \Delta x; \quad \Delta x = \dfrac{x_f - x_i}{N}$

Now let $N \to \infty$, or equivalently, $\Delta x \to 0$

$$W = \lim_{\Delta x \to 0}\left(\textstyle\sum_j F_j \Delta x\right)$$

This is the definition of the definite integral between x_i and x_f:

$$W = \int_{x_i}^{x_f} F(x)\,dx$$

The change in kinetic energy over the whole interval is

$$\Delta K = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2$$

The work-kinetic energy theorem still applies:

$$\Delta K = W$$

We can also obtain this result algebraically:

Start with $W = \int_{x_i}^{x_f} F(x)\,dx = \int_{x_i}^{x_f} ma\,dx$

Now $a\,dx = \dfrac{dv}{dt}dx$, and by the chain rule $\dfrac{dv}{dt} = \dfrac{dv}{dx}\dfrac{dx}{dt} = v\dfrac{dv}{dx}$

$$\therefore a\,dx = v\frac{dv}{dx}dx = v\,dv$$

We can therefore make a change in variable from x to v:

$$W = m\int_{v_i}^{v_f} v\,dv = \tfrac{1}{2}mv^2\Big|_{v_i}^{v_f} = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2$$

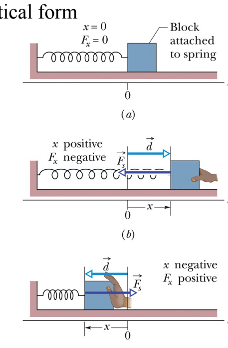
Gravity

For gravity, $F_g = -mg$, taking y as positive upward.

$$\therefore W_g = \int_{y_i}^{y_f} F_g\,dy = -mg\int_{y_i}^{y_f} dy$$
$$= -mg(y_f - y_i)$$
$$= -mg\Delta y$$

Work done by a spring force

- A **spring force** is the *variable force* from a spring
  - A spring force has a particular mathematical form
  - Many forces in nature have this form
- Figure (a) shows the spring in its **relaxed state**: since it is neither compressed nor extended, no force is applied
- If we stretch or extend the spring it resists, and exerts a *restoring force* that attempts to return the spring to its relaxed state
- The relaxed state is called **equilibrium**.



- The spring force is given by **Hooke's law**:

$$\vec{F}_s = -k\vec{d}$$

- The negative sign represents that the force always opposes the displacement from equilibrium
- The **spring constant** k is a is a measure of the stiffness of the spring. Units are N/m (Newtons per metre).
- This is a variable force (function of position) and it exhibits a linear relationship between F and d
- For a spring along the x-axis we can write:

$$F_s = -kx$$

- We can find the work by integrating:

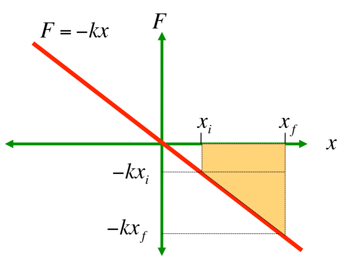
$$W_s = \int_{x_i}^{x_f} F_s\,dx = \int_{x_i}^{x_f} (-kx)\,dx$$

- The final result: $W_s = -\left(\tfrac{1}{2}kx_f^2 - \tfrac{1}{2}kx_i^2\right)$
- The work can be positive or negative.

$$x_f^2 > x_i^2 \Rightarrow W_s < 0 \Rightarrow \Delta K < 0 \Rightarrow v_f^2 < v_i^2$$
$$x_f^2 < x_i^2 \Rightarrow W_s > 0 \Rightarrow \Delta K > 0 \Rightarrow v_f^2 > v_i^2$$

Graphical visualization of work done **by a spring** in compressing from x_i to x_f:

W = area under F vs. x curve
= area of rectangle
+ area of triangle

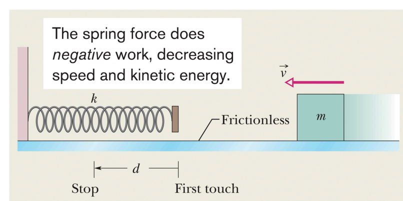


$$W = (x_f - x_i)(-kx_i) + \tfrac{1}{2}(x_f - x_i)(-k)(x_f - x_i) = -\tfrac{1}{2}k(x_f^2 - x_i^2)$$

W is negative in this case (since spring is being compressed)

Example

A block of mass 5.7 kg has a speed of 1.2 m/s. It encounters an ideal spring of spring constant k = 1500 N/m. By how much does the block compress the spring before coming to rest?



Example

An ideal spring has a spring constant k and an equilibrium length L. Suppose we hang an object of mass m from this spring in a vertical orientation.

(a) Find the new equilibrium length.
(b) Show that Hooke's Law is obeyed for displacements from the new equilibrium position.